# MODIFIED NEW THIRD-ORDER ITERATIVE METHOD FOR NON-LINEAR EQUATIONS

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***ABSTRACT:*** *In this paper, we establish a modified new third-order iterative method for solving nonlinear equations extracted from fixed point method by adopting the technique as discussed in [1]. The proposed method is then applied to solve some problems in order to assess its validity and accuracy.*

***AMS Subject Classification:*** *65H05.*

**Key words:** *Fixed point method, nonlinear equations, convergence analysis.*

## 1. INTRODUCTION

During the last many years, the numerical techniques for solving nonlinear equations have been successfully applied (see for example [1-2] and the references there in).

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method. In order to use fixed point iteration method, we need the following information:

1. We need to know that there is a solution to the equation.
2. We need to know approximately where the solution is (i.e. an approximation to the solution).

It is well known that the fixed point iteration method has the first order convergence.

In this paper, a modified new third-order iterative method extracted from fixed point method by adopting the technique as discussed in [1] is proposed to solve nonlinear equations. The proposed method is then applied to solve some problems in order to assess its validity and accuracy.

We need the following results.

In the fixed point iteration method for solving the nonlinear equation $f(x)=0,$ the equation is usually rewritten as

$$x=g(x), \tag{1.1}$$

where

(i) there exists $[a,b]$ such that $g(x)\in C^1[a,b]$ for all $x\in[a,b],$

(ii) there exists $[a,b]$ such that $\left|g'(x)\right|\le L<1$ for all $x\in(a,b).$

Considering the following iteration scheme:

$$x_{n+1}=g(x_n), \quad n=0,1,2,... \tag{1.2}$$

and start with a suitable initial approximation $x_0$, we build up a sequence of approximations, say $\{x_n\}$, for the solution of the nonlinear equation, say $\alpha$. The scheme will converge to the root $\alpha$, provided that

(i) the initial approximation $x_0$ is chosen in the interval $[a,b]$,

(ii) $g$ has a continuous derivative on $(a,b)$,

(iii) $\left|g'(x)\right|<1$ for all $x\in[a,b]$,

(iv) $a\le g(x)\le b$ for all $x\in[a,b]$ (see [3]).

The order of convergence for the sequence of approximations derived from an iterative method is defined in the literature, as

***Definition 1.*** Let $\{x_n\}$ converge to $\alpha.$ If there exist an integer constant $p$, and real positive constant $C$ such that

$$\lim_{n\to\infty}\left|\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}\right|=C,$$

Then $p$ is called *order* and $C$ the constant of convergence.

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of $g(x_n)$

$$g(x_n)=g(x)+\frac{g'(x)}{1!}(x_n-x)+\frac{g''(x)}{2!}(x_n-x)^2+\cdots+\frac{g^{(k)}(x)}{k!}(x_n-x)^k+\cdots \tag{1.3}$$

Using (1.1) and (1.2) in (1.3) we have

$$x_{n+1}-x=g'(x)(x_n-x)+\frac{g''(x)}{2!}(x_n-x)^2+\cdots+\frac{g^{(k)}(x)}{k!}(x_n-x)^k+\cdots$$

and we can state the following result [1]:

***Theorem1.*** Suppose $g(x)\in C^p[a,b]$. If $g^{(k)}(x)=0$, for $k=1,2,...,p-1$ and $g^{(p)}(x)\neq 0,$ then the sequence

$\{x_n\}$ is of order $p$ .

## 2. MODIFIED NEW THIRD-ORDER ITERATIVE METHOD

Consider the nonlinear equation

$$f(x)=0, \quad x\in \mathbb{R} \qquad (2.1)$$

We assume that $\alpha$ is simple zero of $f(x)$ and $\gamma$ is an initial guess sufficiently close to $\alpha$. The equation (2.1) is usually rewritten as

$$x=g\left(x\right). \qquad (2.2)$$

As $\alpha$ is the simple zero of $f(x)$ so $\alpha=g(\alpha-\gamma+\gamma),$ expand using Taylers expension about $\gamma$ , we get

$$\alpha=g(\gamma)+(\alpha-\gamma)g'(\gamma)+\frac{(\alpha-\gamma)^2}{2!}g''(\gamma)+...$$

[4] First order approximation is

$$\alpha=g(\gamma)+(\alpha-\gamma)g'(\gamma) \qquad (2.3)$$

which gives us New Iteration Method, i.e

$$\alpha=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}.$$

***Algorithm 1.*** {[4] New Iterative Method}

For a given $x_0$ , calculate the approximation solution $x_{n+1}$ , by the iteration scheme

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}.$$

The secend order approximation is,

$$\alpha=g(\gamma)+(\alpha-\gamma)g'(\gamma)+\frac{(\alpha-\gamma)^2}{2!}g''(\gamma)$$

we get by simplification,

$$\alpha=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}+\frac{(\alpha-\gamma)^2}{2(1-g'(\gamma))}g''(\gamma) \qquad (2.4)$$

get value of $(\alpha-\gamma)$ from the equation (2.3) and put in the equation (2.4) yields the result

$$\alpha=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}+\frac{(g(\gamma)-\gamma)^2}{2(1-g'(\gamma))^3}g''(\gamma)$$

This formulation allows us to suggest the following iterative method for solving nonlinear equation (2.1).

***Algorithm 2.*** For a given $x_0$ , calculate the approximation solution $x_{n+1}$ , by the iteration scheme

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}+\frac{(g(x_n)-x_n)^2}{2(1-g'(x_n))^3}g''(x_n),$$

$$g'(x_n)\neq 1.$$

## *3. CONVERGENCE ANALYSIS*

Now we discuss the convergence analysis of Algorithm 2.

***Theorem 2.*** Let $f:D\subset R\to R$ for an open interval $D$ and consider that the nonlinear equation $f(x)=0$ (or $g(x)=x$) has a simple root $\alpha\in D,$ where $g:D\subset R\to R$ sufficiently smooth in the neighborhood of the root $\alpha;$ then the order of convergence of Algorithm 2 is at least 3.

**Proof .** The proposed iterative method is given by $x_{n+1}=\dfrac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)\%}+\dfrac{(g(x_n)-x_n)^2}{2(1-g'(x_n))^3}g''(x_n),$

$$g'(x_n)\neq 1$$

For

$$G(x)=\frac{g(x)-xg'(x)}{1-g'(x)}+\frac{(g(x)-x)^2}{2(1-g'(x))^3}g''(x)$$

$$G'(x)=\frac{(g(x)-x)^2}{2(1-g'(x))^3}g'''(x)+\frac{3(g(x)-x)^2}{2(1-g'(x))^4}(g''(x))^2$$

$$G''(x)=-\frac{g(x)-x}{(1-g'(x))^3}[3(g''(x))^2+(1-g'(x))g'''(x)]$$
$$+\frac{(g(x)-x)^2}{(1-g'(x))^5}[6(g''(x))^3+\frac{(1-g'(x))^2}{2}g''''(x)+$$
$$\frac{9}{2}(1-g'(x))g''(x)g'''(x)]$$

$$G'''(x)=[\frac{1}{(1-g'(x))^2}-\frac{3(g(x)-x)g''(x)}{(1-g'(x))^4}]$$
$$[3(g''(x))^2+(1-g'(x))g'''(x)]-\frac{g(x)-x}{(1-g'(x))^3}$$
$$[5g''(x)g'''(x)+(1-g'(x))g''''(x)]+[-\frac{g(x)-x}{(1-g'(x))^3}$$
$$+\frac{5(g(x)-x)^2g''(x)}{(1-g'(x))^4}][6(g''(x))^3+\frac{(1-g'(x))^2}{2}g''''(x)$$
$$+\frac{9}{2}(1-g'(x))g''(x)g'''(x)]+\frac{(g(x)-x)^2}{(1-g'(x))^5}[6(g''(x))^3+$$
$$\frac{(1-g'(x))^2}{2}g''''(x)+\frac{9}{2}(1-g'(x))g''(x)g'''(x)]'$$

Now, it can be easily seen that for $g(\alpha)=\alpha$ we obtain

$$G(\alpha)=\alpha,$$
$$G'(\alpha)=0,$$
$$G''(\alpha)=0,$$

and

$$G'''(\alpha)=\frac{3(g''(\alpha))^2+(1-g'(\alpha))g'''(\alpha)}{(1-g'(\alpha))^2}\neq 0.$$

Hence Algorithm (2) has 3rd order convergence.

***Example 1.*** Consider the equation $\ln x+x=0.$ We have $g(x)=e^{-x}$ , $g'(x)=-e^{-x}$ and $g''(x)=e^{-x}.$ Take

$x_0 = 0.2,$ then the comparison of the Fixed Point Method (FPM), New Iterative Method (NIM) and Modified New Iterative Method (MNIM) is shown in Table 3 correct up to five decimal places.

| $n$ | FPM | NIM | MNIM |
|---|---|---|---|
| 1 | 0.81873 | 0.54020 | 0.56625 |
| 2 | 0.44099 | 0.56701 | 0.56714 |
| 3 | 0.64340 | 0.56714 | |
| 4 | 0.5255 | | |
| 5 | 0.59126 | | |
| 6 | 0.55363 | | |
| 7 | 0.57486 | | |
| 8 | 0.56278 | | |
| 9 | 0.56962 | | |
| 10 | 0.56574 | | |
| 11 | 0.56794 | | |
| 12 | 0.56669 | | |
| 13 | 0.5674 | | |
| 14 | 0.56700 | | |
| 15 | 0.56722 | | |
| 16 | 0.56710 | | |
| 17 | 0.56717 | | |
| 18 | 0.56713 | | |
| 19 | 0.56715 | | |
| 20 | 0.56714 | | |
| 21 | 0.56715 | | |

***Example 2.*** Consider the equation $x^3 + 4x^2 + 8x + 8 = 0.$ We have $g(x) = -(1 + (1/2)x^2 + (1/8)x^3)$, $g'(x) = -x - (3/8)x^2$ and $g''(x) = -1 - (3/4)x.$ The exact solution of this equation is $-2.$ Take $x_0 = -1.6;$ then the comparison of the three methods is shown in the following table.

| $n$ | FPM | NIM | MNIM |
|---|---|---|---|
| 1 | $-1.768$ | $-2.0667$ | $-2.0064$ |
| 2 | $-1.8721$ | $-2.0022$ | $-2.0$ |
| 3 | $-1.9322$ | $-2.0$ | |
| 4 | $-1.9650$ | | |
| 5 | $-1.9822$ | | |
| 6 | $-1.991$ | | |
| 7 | $-1.9955$ | | |
| 8 | $-1.9977$ | | |
| 9 | $-1.9988$ | | |
| 10 | $-1.9994$ | | |
| 11 | $-1.9997$ | | |
| 12 | $-1.9998$ | | |
| 13 | $-1.9999$ | | |
| 14 | $-1.9999$ | | |
| 15 | $-1.9999$ | | |

***Example 3.*** Consider the equation $x + \ln(x-2) = 0.$ We have $g(x) = 2 + e^{-x}$ , $g'(x) = -e^{-x}$ and $g''(x) = e^{-x}.$ The numerical solution of this equation is $2.2100(4D)$ . Take $x_0 = 1.1,$ then the comparison of the three methods is given in the following table up to four decimal places.

| $n$ | FPM | NIM | MNIM |
|---|---|---|---|
| 1 | 2.3329 | 2.0250 | 2.1318 |
| 2 | 2.097 | 2.1195 | 2.12 |
| 3 | 2.1228 | 2.12 | |
| 4 | 2.1197 | | |
| 5 | 2.1201 | | |
| 6 | 2.12 | | |

***Example 4.*** Consider the equation $x^3 + 4x^2 + 8x + 8 = 0.$ We have $g(x) = 0.2 + 1.8x^2 - 2x^3 + x^4 - 0.2x^5,$ $g'(x) = 3.6x - 6x^2 + 4x^3 - x^4$ and $g''(x) = 3.6 - 12x + 12x^2 - 4x^3.$ The numerical solution of this equation is $0.34595(5D)$ . Take $x_0 = 0.28,$ then the comparison of the three method is shown in the following table.

| n | FPM | NIM | MNIM |
|---|---|---|---|
| 1 | 0.30302 | 0.34046 | 0.3457 |
| 2 | 0.31755 | 0.34592 | 0.34595 |
| 3 | 0.32699 | 0.34595 | |
| 4 | 0.33322 | | |
| 5 | 0.33737 | | |
| 6 | 0.34016 | | |
| 7 | 0.34203 | | |
| 8 | 0.34330 | | |
| 9 | 0.34416 | | |
| 10 | 0.34474 | | |
| 11 | 0.34513 | | |
| 12 | 0.34540 | | |
| 13 | 0.34558 | | |
| 14 | 0.34557 | | |
| 15 | 0.34578 | | |
| 16 | 0.34584 | | |
| 17 | 0.34588 | | |
| 18 | 0.3459 | | |
| 19 | 0.34592 | | |
| 20 | 0.34593 | | |
| 21 | 0.34594 | | |

## 4. CONCLUSIONS

A modified new third-order iterative method for solving nonlinear equations is introduced. By using some examples the efficiency of the method is also discussed. The method is performing very well in comparison to the fixed point method and the method discussed in [1], [2], [3], [4], [5], [6]. It is worth to mention that our method is so simple to apply in comparison to the method discussed in [1]. However we suggest that one can take $x_0 = \dfrac{a+b}{2}$.

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