On $\mathbb{Z}_{3k}-$Magic Labeling of Certain Families of Graphs

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ABSTRACT: *Let $G=(V,E)$ be a finite, simple and undirected graph and A be a non-trivial abelian group with respect to addition. If there exist a map $\lambda: E(G)\to A\setminus\{0\}$ such that the induced map $\lambda^{+}: V(G)\to A$, defined by $\lambda^{+}(x)=\sum_{y\in N(x)}\lambda(xy)$, where $N(x)$ is neighborhood of vertex x, is constant, then the graph G is said to be $A-$magic graph. In this paper we prove that generalized prism, generalized Antiprism, Fan and Friendship graphs are $\mathbb{Z}_{3k}-$magic for $k\ge 1$.*

Keywords *: Induced map, $A-$magic graph, $\mathbb{Z}_3-$magic graph, $\mathbb{Z}_{3k}-$magic graph, magic labeling.*

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INTRODUCTION

A labeling of a graph G is a map that carries graph elements to integers (usually non-negative integers). A labeling ϕ is called vertex or edge labeling if the domain is a vertex or edge set. The concept of magic labeling was given by Kotzig and Rosa [11]. Motivated by this concept, Bača and Holländer [2] define the prime magic labeling of complete bipartite graphs $K_{n,n}$. Bača [1] define the consecutive magic labeling of generalized petersen graphs, Bača *et.al* [4] define the magic total labeling of generalized petersen graphs and Javed [10] define the super edge magic total labeling on w-tree. So in last four decades, various labelings of graphs such as vertex-magic labeling, edge-magic labeling, graceful labeling and prime labeling have been studied. For further details see [3,9,20].

The concept of an A-magic graph is due to J. Sedlack [16,17] who defined it to be a graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices.

In [18,19], R. P Stanley introduced the $\mathbb{Z}-$magic graphs, where he pointed out that the theory of magic labeling could be studied in the general context of linear homogenous diophantine equation. Moreover the construction of magic graphs, generalization of magic graphs and characterization of regular magic graphs are studied in [6], [7] and [8] respectively. Further in [15], R. M. Low and S. M. Lee give the necessary and sufficient condition for a graph to be $\mathbb{Z}_2-$magic i.e A graph G is $\mathbb{Z}_2-$magic if and only if all the vertices of G having same degree.

Motivated by the papers [5, 12, 13,14] we discuss $\mathbb{Z}_3-$magicness of certain families of graphs. For this first we need to know the necessary condition for a graph G to be $\mathbb{Z}_3-$magic. In [15], R. M. Low and S. M. Lee provide the necessary condition for a graph G to be $\mathbb{Z}_3-$magic in the following theorem.

Theorem 1.1 *[15] Let G be $\mathbb{Z}_3-$magic, with p vertices and q edges. Let the induced map λ^{+} induced the constant label x on the vertices of G and $|E_i|$ denote the number of edges labeled i. Then, $px\equiv q+|E_i|,(mod 3)$.*

Moreover to prove the main results, we use the following corollary.

Corollary 1.2 *[15] Let G be $\mathbb{Z}_k-$magic graph, with $k\,|\,n$. Then, G is a $\mathbb{Z}_n-$magic graph.*

Generalized Prism

The generalized prism can be defined as the cartesian product $C_m\times P_n$ of a cycle on m vertices and a path on n vertices. Let $V(C_m\times P_n)=\{v_i^j : i\in[1,m],\ j\in[1,n]\}$ is the vertex set and

$$E(C_m\times P_n)=\{(v_i^j v_{i+1}^j): i\in[1,m],\ j\in[1,n]\}$$
$$\cup\{(v_i^j v_i^{j+1}): i\in[1,m],\ j\in[1,n-1]\}$$

is the edge set of $C_m\times P_n$. Also the indices being taken modulo m,n.

Theorem 2. 3 *For $n\ge 2$, $m\ge 3$, the Generalized Prism admits $\mathbb{Z}_{3k}-$magic labeling.*

Proof. To prove the above statement first we define a map $\mathrm{f}: E(C_m\times P_n)\to \mathbb{Z}_3\setminus\{0\}$ in the following way:

$$\mathrm{f}\ (v_i^j v_i^{j+1})=2,\ \text{for}\ i\in[1,m],\ j\in[1,n-1]$$

$$\mathrm{f}\ (v_i^j v_{i+1}^j)=\begin{cases}2, & \text{if}\ i\in[1,m],\ j=1,n\\ 1, & \text{if}\ i\in[1,m],\ j\in[2,n-1]\end{cases}$$

Now we define the induced map $f^{+}: V(C_m \times P_n) \to Z_3$ as follows:

$$\begin{aligned}
f^{+}(v_1^1) &= f(v_1^1 v_2^1) + f(v_m^1 v_1^1) + f(v_1^1 v_1^2) \\
&= 2+2+2 \equiv 0(mod 3) \\
f^{+}(v_m^1) &= f(v_m^1 v_1^1) + f(v_{m-1}^1 v_m^1) + f(v_m^1 v_m^2) \\
&= 2+2+2 \equiv 0(mod 3) \\
f^{+}(v_1^n) &= f(v_1^n v_2^n) + f(v_m^n v_1^n) + f(v_m^{n-1} v_m^n) \\
&= 2+2+2 \equiv 0(mod 3) \\
f^{+}(v_m^n) &= f(v_m^n v_1^n) + f(v_{m-1}^n v_m^n) + f(v_m^{n-1} v_m^n) \\
&= 2+2+2 \equiv 0(mod 3) \\
f^{+}(v_i^1) &= f(v_i^1 v_{i+1}^1) + f(v_{i-1}^1 v_i^1) + f(v_i^2 v_i^2) \\
&= 2+2+2 \equiv 0(mod 3), \ for\ i \in [2, m-1] \\
f^{+}(v_i^n) &= f(v_i^n v_{i+1}^n) + f(v_{i-1}^n v_i^n) + f(v_i^n v_i^{n-1}) \\
&= 2+2+2 \equiv 0(mod 3), \ for\ i \in [2, m-1] \\
f^{+}(v_1^j) &= f(v_1^j v_2^j) + f(v_m^j v_1^j) + f(v_1^j v_1^{j+1}) + f(v_1^{j-1} v_1^j) \\
&= 1+1+2+2 \equiv 0(mod 3), \ for\ j \in [2, n-1] \\
f^{+}(v_m^j) &= f(v_m^j v_1^j) + f(v_{m-1}^j v_m^j) + f(v_m^j v_m^{j+1}) + f(v_m^{j-1} v_m^j) \\
&= 1+1+2+2 \equiv 0(mod 3), \ for\ j \in [2, n-1] \\
f^{+}(v_i^j) &= f(v_i^j v_{i+1}^j) + f(v_{i-1}^j v_i^j) + f(v_i^j v_i^{j+1}) + f(v_i^{j-1} v_i^j) \\
&= 1+1+2+2 \equiv 0(mod 3), \\
& for\ i \in [2, m-1],\ j \in [2, n-1].
\end{aligned}$$

Hence $f^{+}(v_i^j) \equiv 0(mod 3)$ for all $i \in [1, m],\ j \in [1, n]$. So Generalized prism is Z_3-magic. Now by corollary (1), we conclude that Generalized prism is $Z_{3k}-$magic for $k \ge 1$. This conclude the proof. W

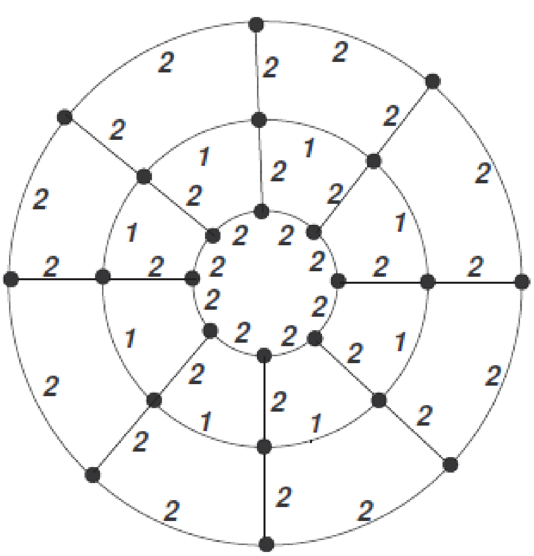

Figure 1: An illustration of $C_8 \times P_3$ labeling.

Generalized Antiprism A_n^m

The generalized Antiprism A_n^m can be obtained from the generalized prism by adding some more edges. So the vertex set and the edge set of Generalized Antiprism A_n^m are defined under modulo n, m in the following way:

$$V(A_n^m) = \{x_i^j : i \in [1, n],\ j \in [1, m]\}$$

$$\begin{aligned}
E(A_n^m) &= \{(x_i^j x_{i+1}^j) : i \in [1, n],\ j \in [1, m]\} \cup \{(x_i^j x_i^{j+1}) : i \in [1, n],\ j \in [1, m-1]\} \\
&\cup \{(x_i^j x_{i+1}^{j+1}) : i \in [1, n],\ j \in [1, m-1]\} \cup \{(x_i^j x_i^{j-1}) : i \in [1, n],\ j \in [2, m]\} \\
&\cup \{(x_i^j x_{i-1}^{j-1}) : i \in [2, n],\ j \in [2, m]\} \cup \{(x_1^j x_m^j - 1) : j \in [2, m]\}
\end{aligned}$$

Theorem 3.4 *For $m \ge 2$, $n \ge 3$, the Generalized Antiprism admits $Z_{3k}-$magic labeling.*

Proof. To prove the above statement first we define a map $h : E(A_n^m) \to Z_3 \setminus \{0\}$ in the following way:

$$h(x_i^j x_{i+1}^j) = 2, \quad for \quad i \in [1, n],\ j = 1, m$$

$$h(x_i^j x_i^{j+1}) = h(x_i^j x_{i+1}^{j+1}) = h(x_i^j x_i^{j-1}) = h(x_i^j x_{i-1}^{j-1}) = h(x_1^j x_m^{j-1}) = 1.$$

Now we define the induced map $h^{+} : V(A_n^m) \to Z_3$ as follows:

$$\begin{aligned}
h^{+}(x_1^1) &= h(x_1^1 x_2^1) + h(x_n^1 x_1^1) + h(x_1^1 x_1^2) + h(x_1^1 x_2^2) \\
&= 2+2+1+1 \equiv 0(mod 3) \\
h^{+}(x_1^m) &= h(x_1^m x_2^m) + h(x_n^m x_1^m) + h(x_1^m x_1^{m-1}) + h(x_1^m x_n^{m-1}) \\
&= 2+2+1+1 \equiv 0(mod 3) \\
h^{+}(x_i^1) &= h(x_i^1 x_{i+1}^1) + h(x_{i-1}^1 x_i^1) + h(x_i^1 x_i^2) + h(x_i^1 x_{i+1}^2) \\
&= 2+2+1+1 \equiv 0(mod 3), \ for\ i \in [2, n] \\
h^{+}(x_i^m) &= h(x_i^m x_{i+1}^m) + h(x_{i-1}^m x_i^m) + h(x_i^m x_i^{m-1}) + h(x_i^m x_{i-1}^{m-1}) \\
&= 2+2+1+1 \equiv 0(mod 3), \ for\ i \in [2, n] \\
h^{+}(x_1^j) &= h(x_1^j x_2^j) + h(x_n^j x_1^j) + h(x_1^j x_1^{j+1}) \\
&+ h(x_1^j x_2^{j+1}) + h(x_1^j x_1^{j-1}) + h(x_1^j x_n^{j-1}) \\
&= 1+1+1+1+1+1 \equiv 0(mod 3), \ for\ j \in [2, m-1] \\
h^{+}(x_i^j) &= h(x_i^j x_{i+1}^j) + h(x_{i-1}^j x_i^j) + h(x_i^j x_i^{j+1}) \\
&+ h(x_i^j x_{i+1}^{j+1}) + h(x_i^j x_i^{j-1}) + h(x_i^j x_{i-1}^{j-1}) \\
&= 1+1+1+1+1+1 \equiv 0(mod 3), \\
& for\ j \in [2, m-1], i \in [2, n]
\end{aligned}$$

Clearly $h^{+}(x_i^j) \equiv 0(mod 3)$ for all $i \in [1, n],\ j \in [1, m]$. So Generalized Antiprism is Z_3-magic. Now by corollary (1), we conclude that Generalized Antiprism is $Z_{3k}-$magic for $k \ge 1$. This conclude the proof. W

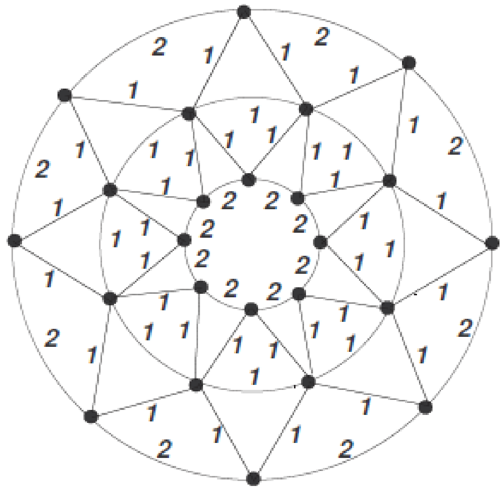

Figure 2. An illustration of A_8^3 labeling

Fan graph F_n

A *fan* is a graph obtained by joining all vertices of path P_n to a further vertex c, called the center. So the vertex set and the edge set of *fan* graph are defined under modulo n in the following way:

$$V(F_n) = \{c, x_i : i \in [1,n]\}$$

$$E(F_n) = \{(x_i x_{i+1}) : i \in [1,n]\} \cup \{(cx_i) : i \in [1,n]\}$$

Theorem 4. 5 *For $n \geq 4$, the fan graph is Z_{3k} – magic.*

Proof. ***Case(i)*** when $n \equiv 0(mod 6)$

First we define a map $\xi_1 : E(F_n) \to Z_3 \setminus \{0\}$ in the following way:

$$\xi_1(cx_1) = \xi_1(cx_n) = \xi_1(x_1x_2) = \xi_1(x_{n-1}x_n) = 2,$$

$$\xi_1(x_i x_{i+1}) = 1 \text{ for } i \in [2, n-2]$$

$$\xi_1(cx_i) = \begin{cases} 2, & \text{for } 3 \leq i \leq n-2 \\ 1, & \text{for } i = 2, n-1 \end{cases}$$

Now we define the induced map $\xi_1^+ : V(F_n) \to Z_3$ as follows:

$$\xi_1^+(c) = \sum_{i=3}^{n-2} \xi_1(cx_i) + \xi_1(cx_1) + \xi_1(cx_n) + \xi_1(cx_2) + \xi_1(cx_{n-1})$$

$$= 2(n-4) + 2 + 2 + 1 + 1$$

$$= 2n - 2 \equiv 1(mod 3)$$

$$\xi_1^+(x_1) = \xi_1^+(x_n) = 2 + 2 \equiv 1(mod 3)$$

$$\xi_1^+(x_2) = \xi_1^+(x_{n-1}) = 2 + 1 + 1 \equiv 1(mod 3)$$

$$\xi_1^+(x_i) = \xi_1(x_{i-1}x_i) + \xi_1(x_i x_{i+1}) + \xi_1(cx_i)$$

$$= 1 + 1 + 2 \equiv 1(mod 3), \text{ for } i \in [3, n-2]$$

Case(ii) when $n \equiv 3(mod 6)$

First we define a map ξ_2, which is induced by ξ_1 in the following way:

$$\xi_2(cx_i) \equiv 2\xi_1(cx_i)(mod 3), \text{ for } i \in [1,n]$$

$$\xi_2(x_i x_{i+1}) \equiv 2\xi_1(x_i x_{i+1})(mod 3), \text{ for } i \in [1, n-1]$$

Now we define the map ξ_2^+, which is induced by ξ_1^+ in the following way:

$$\xi_2^+(c) = 2\xi_1^+(c) \equiv 2(mod 3),$$

$$\xi_2^+(x_i) = 2\xi_1^+(x_i) \equiv 2(mod 3), \text{ for } i \in [1,n]$$

Case(iii) when $n \equiv 2(mod 6)$

First we define a map $\xi_3 : E(F_n) \to Z_3 \setminus \{0\}$ in the following way:

$$\xi_3(cx_i) = \begin{cases} 1, & \text{for } i = 1, n \\ 2, & \text{for } 2 \leq i \leq n-1 \end{cases}$$

$$\xi_3(x_i x_{i+1}) = \begin{cases} 1, & \text{for odd } i; 1 \leq i \leq n-1 \\ 2, & \text{for even } i; 1 \leq i \leq n-1 \end{cases}$$

Now we define the induced map $\xi_3^+ : V(F_n) \to Z_3$ as follows:

$$\xi_3^+(x_1) = \xi_3^+(x_n) = 1 + 1 \equiv 2(mod 3)$$

$$\xi_3^+(x_i) = \xi_3(x_{i-1}x_i) + \xi_3(x_i x_{i+1}) + \xi_3(cx_i)$$

$$= 1 + 2 + 2 \equiv 2(mod 3), \text{ for } i \in [2, n-1]$$

$$\xi_3^+(c) = \sum_{i=2}^{n-1} \xi_3(cx_i) + \xi_3(cx_1) + \xi_3(cx_n)$$

$$= 2(n-2) + 1 + 1 = 2n - 2 \equiv 2(mod 3)$$

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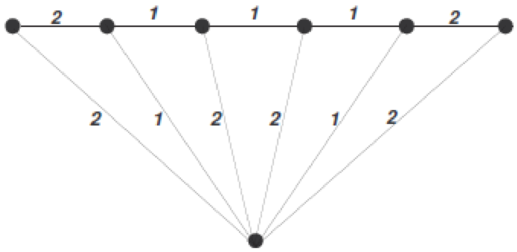

Figure 3. An illustration of F_6 labeling

Case(iv) when $n \equiv 1 (mod 6)$

First we define a map $\xi_4 : E(F_n) \to Z_3 \setminus \{0\}$ in the following way:

$$\xi_4(cx_1) = \xi_4(cx_3) = \xi_4(cx_n) = \xi_4(x_1x_2) = 1$$

$$\xi_4(x_2x_3) = \xi_4(cx_2) = 2, \xi_4(cx_i) = 2, \quad \text{for } i \in [4, n-1]$$

$$\xi_4(x_ix_{i+1}) = \begin{cases} 2, & \text{for odd } i ; 3 \le i \le n-1 \\ 1, & \text{for even } i ; 3 \le i \le n-1 \end{cases}$$

Now we define the induced map $\xi_4^+ : V(F_n) \to Z_3$ as follows:

$$\xi_4^+(c) = \sum_{i=4}^{n-1} \xi_4(cx_i) + \xi_4(cx_1) + \xi_4(cx_2) + \xi_4(cx_3) + \xi_4(cx_n)$$

$$= 2(n-4) + 1 + 2 + 1 + 1 = 2n - 3 \equiv 2(mod 3)$$

$$\xi_4^+(x_1) = \xi_4^+(x_n) = 1 + 1 \equiv 2(mod 3)$$

$$\xi_4^+(x_i) = 2 + 2 + 1 \equiv 2(mod 3), \text{ for } i \in [2, n-1]$$

Case(v) when $n \equiv 4 (mod 6)$

First we define a map $\xi_5 : E(F_n) \to Z_3 \setminus \{0\}$ in the following way:

$$\xi_5(cx_1) = \xi_5(cx_n) = 2, \xi_5(x_1x_2) = 1$$

$$\xi_5(cx_i) = \xi_5(x_ix_{i+1}) = 1 \text{ for } i \in [2, n-1]$$

Now we define the induced map $\xi_5^+ : V(F_n) \to Z_3$ as follows:

$$\xi_5^+(c) = \sum_{i=2}^{n-1} \xi_5(cx_i) + \xi_5(cx_1) + \xi_5(cx_n)$$

$$= n - 2 + 2 + 2 = n + 2 \equiv 0(mod 3)$$

$$\xi_5^+(x_1) = \xi_5^+(x_n) = 1 + 2 \equiv 0(mod 3)$$

$$\xi_5^+(x_i) = 1 + 1 + 1 \equiv 0(mod 3), \text{ for } i \in [2, n-1]$$

Case(vi) when $n \equiv 5 (mod 6)$

First we define a map $\xi_6 : E(F_n) \to Z_3 \setminus \{0\}$ in the following way:

$$\xi_6(cx_1) = \xi_6(cx_3) = \xi_6(cx_4) = \xi_6(cx_5) = \xi_6(cx_n) = 2$$

$$\xi_6(x_1x_2) = \xi_6(x_{n-1}x_n) = 2, \ \xi_6(x_ix_{i+1}) = 1, \text{ for } i \in [2, 5]$$

$$\xi_6(cx_2) = \xi_6(cx_i) = 1, \text{ for } i \in [6, n-1]$$

$$\xi_6(x_ix_{i+1}) = \begin{cases} 1, & \text{for odd } i ; 6 \le i \le n-2 \\ 2, & \text{for even } i ; 6 \le i \le n-2 \end{cases}$$

Now we define the induced map $\xi_6^+ : V(F_n) \to Z_3$ as follows:

$$\xi_6^+(c) = \sum_{i=6}^{n-1} \xi_6(cx_i) + \sum_{i=3}^{5} \xi_6(cx_i) + \xi_6(cx_1) + \xi_6(cx_2) + \xi_(cx_n)$$

$$= (n-6) + 6 + 2 + 1 + 2 = n + 5 \equiv 1(mod 3)$$

$$\xi_6^+(x_1) = \xi_6^+(x_n) = 2 + 2 \equiv 1(mod 3)$$

$$\xi_6^+(x_i) = 2 + 1 + 1 \equiv 1(mod 3), \text{ for } i \in [2, n-1]$$

It is easy to see that in all cases the induced maps are constant. So *fan* graph is Z_3 − magic. Now by corollary (1), we conclude that *fan* graph is Z_{3k} − magic for $k \ge 1$. This conclude the proof. W

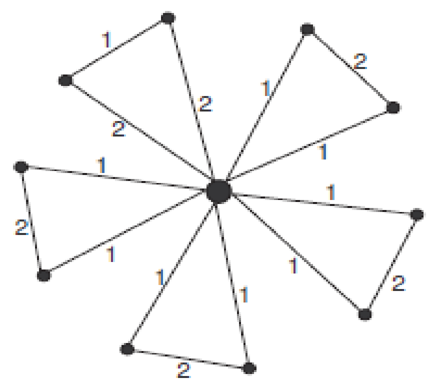

Figure 5. An illustration of F_5 labeling.

Friendship graph T_n

The *friendship graph* T_n is a set of n triangles having a common central vertex and otherwise disjoint. Let c denote the central vertex. For the i th triangle, let x_i and y_i denote the other two vertices. So the edge set of friendship graph is $\{cx_i, cy_i, x_iy_i : i \in [1, n]\}$.

Theorem 5.6 *For* $n \ge 3$, *the friendship graph is* Z_{3k} − *magic.*

Proof. ***Case(i)*** when $n \equiv 0, 3 (mod 6)$

First we define a map $\psi_1 : E(T_n) \to Z_3 \setminus \{0\}$ in the following way:

$\psi_1(cx_i) = \psi_1(cy_i) = 1,\ \psi_1(x_i y_i) = 2$, for $i \in [1,n]$

Now we define the induced map $\psi_1^+ : V(T_n) \to Z_3$ as follows:

$$\psi_1^+(c) = \sum_{i=1}^{n}[\psi_1(cx_i) + \psi_1(cy_i)] = 2n \equiv 0(mod 3)$$

$$\psi_1^+(x_i) = \psi_1(cx_i) + \psi_1(x_i y_i) = 1 + 2 \equiv 0(mod 3)$$

$$\psi_1^+(y_i) = \psi_1(cy_i) + \psi_1(x_i y_i) = 1 + 2 \equiv 0(mod 3)$$

Case(ii) when $n \equiv 1, 4(mod 6)$

First we define a map $\psi_2 : E(T_n) \to Z_3 \setminus \{0\}$ in the following way:

$\psi_2(cx_i) = \psi_2(cy_i) = \psi_2(x_i y_i) = 1$, for $i \in [1,n]$

Now we define the induced map $\psi_2^+ : V(T_n) \to Z_3$ as follows:

$$\psi_2^+(c) = \sum_{i=1}^{n}[\psi_2(cx_i) + \psi_2(cy_i)] = 2n \equiv 2(mod 3)$$

$$\psi_2^+(x_i) = \psi_2(cx_i) + \psi_2(x_i y_i) = 1 + 1 \equiv 2(mod 3)$$

$$\psi_2^+(y_i) = \psi_2(cy_i) + \psi_2(x_i y_i) = 1 + 1 \equiv 2(mod 3)$$

Case(iii) when $n \equiv 2(mod 6)$

$$\psi_3(x_i x_{i+1}) = \begin{cases} 2,\ for\ ;\ 1 \le i \le \frac{n}{2} \\ 1,\ for\ ;\ \frac{n}{2}+1 \le i \le n \end{cases}$$

$$\psi_3(cx_i) = (cy_i) = \begin{cases} 2,\ for\ ;\ 1 \le i \le \frac{n}{2} \\ 1,\ for\ ;\ \frac{n}{2}+1 \le i \le n \end{cases}$$

Now we define the induced map $\psi_3^+ : V(T_n) \to Z_3$ as follows:

$$\psi_3^+(c) = \sum_{i=1}^{\frac{n}{2}}[\psi_3(cx_i) + \psi_3(cy_i)] + \sum_{i=\frac{n}{2}+1}^{n}[\psi_3(cx_i) + \psi_3(cy_i)]$$

$$= n + 2n \equiv 0(mod 3)$$

$$\psi^+{}_3(x_i) = \begin{cases} \psi_3(cx_i) + \psi_3(x_i y_i) = 1 + 2 \equiv 0(mod 3),\ for\ 1 \le i \le \frac{n}{2} \\ \psi_3(cx_i) + \psi_3(x_i y_i) = 1 + 2 \equiv 0(mod 3),\ for\ \frac{n}{2}+1 \le i \le n \end{cases}$$

$$\psi^+{}_3(y_i) = \begin{cases} \psi_3(cy_i) + \psi_3(x_i y_i) = 1 + 2 \equiv 0(mod 3),\ for\ 1 \le i \le \frac{n}{2} \\ \psi_3(cy_i) + \psi_3(x_i y_i) = 1 + 2 \equiv 0(mod 3),\ for\ \frac{n}{2}+1 \le i \le n \end{cases}$$

Case(iv) when $n \equiv 5(mod 6)$

First we define a map $\psi_4 : E(T_n) \to Z_3 \setminus \{0\}$ in the following way:

$\psi_4(cx_i) = \psi_4(cy_i) = \psi_4(x_n y_n) = 1$, for $i \in [1, n-1]$

$\psi_4(cx_n) = \psi_4(cy_n) = \psi_4(x_i y_i) = 2$, for $i \in [1, n-1]$

Now we define the induced map $\psi_4^+ : V(T_n) \to Z_3$ as follows:

$$\psi_4^+(c) = \sum_{i=1}^{n-1}[\psi_4(cx_i) + \psi_4(cy_i)] + \psi_4(cx_n) + \psi_4(cy_n)$$

$$= 2(n-1) + 2 + 2 \equiv 0(mod 3)$$

$$\psi_4^+(x_i) = \psi_4(cx_i) + \psi_4(x_i y_i)$$

$$= 1 + 2 \equiv 0(mod 3),\ for\ i \in [1, n-1]$$

$$\psi_4^+(x_n) = \psi_4(cx_n) + \psi_4(x_n y_n) = 2 + 1 \equiv 0(mod 3)$$

$$\psi_4^+(y_i) = \psi_4(cy_i) + \psi_4(x_i y_i)$$

$$= 1 + 2 \equiv 0(mod 3),\ for\ i \in [1, n-1]$$

$$\psi_4^+(y_n) = \psi_4(cy_n) + \psi_4(x_n y_n) = 2 + 1 \equiv 0(mod 3)$$

It is not difficult to check that in all cases the induced maps are constant. So *fan* graph is Z_3 – magic. Now by corollary (1), we conclude that *friendship graph* T_n is Z_{3k} – magic for $k \ge 1$. This conclude the proof. W

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