# UPPER BOUND OF THIRD HANKEL DETERMINANT FOR FUNCTIONS RELATED WITH SYMMETRICAL POINTS

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**ABSTRACT.** In the this paper, we consider a subclass $SL(s,t)$ of analytic functions related with symmetrical points as well as right half of the lemniscates of Bernoulli, $(x^2+y^2)^2-2(x^2-y^2)=0.$ An upper bound of the Hankel determinant $H_3(1)$ is determinant for this class.

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## 1. INTRODUCTION

Let $A$ denote the class of functions $f(z)$ which are analytic in the region $U=\{z\in \mathbb{C}: |z|<1\}$: and normalized by the conditions $f(0)=0$ and $f'(0)-1=0.$ Therefore, for $f(z)\in A,$ one has

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,\ z\in U. \tag{1.1}$$

Also let $S\subset A$ be a family of univalent functions and having the normalized form (1.1). For two functions $f(z)$ and $g(z)$ analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z)\prec g(z)$, if there is an analytic function $w(z)$ with $|w(z)|\le |z|$ such that $f(z)=g(w(z))$. If $g(z)$ is univalent, then $f(z)\prec g(z)$, if and only if

$$f(0)=g(0) \text{ and } f(U)\subseteq g(U).$$

We now define a subclass $SL(s,t)$ of analytic functions as;

$$SL(s,t)=\left\{ f(z)\in A: \left| \left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right)^2-1\right|<1\right\}.$$

The geometrical interpretation of a function $f(z)\in SL(s,t)$ is that $(s-t)zf'(z)/(f(sz)-f(tz))$ lies in the region bounded by right half of the limniscate of Bernouli given by the relation $|w^2-1|<1.$ It can easily be seen that $f(z)\in SL(s,t)$ if

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\prec \sqrt{1+z},\ z\in U. \tag{1.2}$$

For $s=1$ and $t=0,$ the class $SL(s,t)$ reduce to the class $SL$ introduced by Sokol and Stankiewicz [1] and further studied by different authers in [2-6]. The $qth$ Hankel determinant $H_q(n),\ q\ge 1, n\ge 1,$ for a function $f(z)\in A$ is studied by Noor and Thomas [7] as:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

In literature many authors have studied the determinants $H_q(n).$ For example, Arif et al. [8-11] studied the $qth$ Hankel determinant for some subclasses of analytic functions. Hankel determinant of exponential polynomials is obtained Ehrenborg in [12]. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [13]. It is well-known that the Fekete-Szego functional $|a_3-a_2^2|$ is $H_2(1)$. Fekete-Szego then further generalized the estimate $|a_3-\lambda a_2^2|$ with $\lambda$ real and $f(z)\in S.$ Moreover, we also know that the functional $|a_2a_4-a_3^2|$ is equivalent to $H_2(2)$. The sharp upper bound of the second Hankel determinant for the familiar classes of starlike and convex functions wre studied by Janteng, Halim and Darus [14], that is, for $f(z)\in S^*$ and $f(z)\in C$, they obtained $|a_2a_4-a_3^2|\le 1$ and $8|a_2a_4-a_3^2|\le 1$ respectively. Babalola in [15] considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of bounded-turning, starlike and convex functions. Later in 2013, Raza and Malik [16] investigated the upper bound of $H_3(1)$ for the class $SL$ and they obtained that $576\left|H_3(1)\right| \le 43.$ In the present investigation, we study the upper bound of $H_3(1)$ for a subclass $SL(s,t)$ of analytic functions by using Toeplitz determinants.

Let $P$ denotes the class of analytic function $p(z)$ of the form

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n, \quad z \in \mathbb{U} \tag{1.3}$$

For our main result we need the following lemmas.

**Lemma 1.1** [17]. If $p(z) \in P$ and of the form (1.3), then

$$\left|p_2 - v p_1^2\right| \le \begin{cases} -4v+2 & (v \le 0) \\ 2 & (0 \le v \le 1) \\ 4v-2 & (v \ge 1). \end{cases}$$

When $v < 1$ or $v > 0,$ equality holds if and only if $p(z)$ is $\left(1+z\right)/\left(1-z\right)$ or one of its rotations. If $0 < v < 1,$ then equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0,$ equality holds if and only if for $0 \le \xi \le 1,$ $z \in \mathbb{U},$

$$p\left(z\right) = \left(\frac{1+\xi}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\xi}{2}\right)\left(\frac{1-z}{1+z}\right),$$

or one of its rotaions. While for $v = 1,$ equality holds if and only if $p\left(z\right)$ is the reciprocal of one of the functions such that equality holds in the case of $v = 0.$ Although the above upper bound is sharpe, it can be improved as follows when $0 < v < 1;$

$$\left|p_2 - v p_1^2\right| + v\left|p_1\right|^2 \le 2, \quad 0 < v \le \frac{1}{2}$$

and

$$\left|p_2 - v p_1^2\right| + (1-v)\left|p_1\right|^2 \le 2, \quad \frac{1}{2} < v < 1.$$

**Lemma 1.2** [18]. If $p(z) \in P$ and of the form (1.3), then

$$\left|p_n\right| \le 2, \text{ for } n \ge 1.$$

**Lemma 1.3** [17]. If $p(z) \in P$ and of the form (1.3), then

$$\left|p_2 - v p_1^2\right| \le 2\max\left\{1; \left|2v-1\right|\right\}, \text{for } v \in \mathbb{C},$$

nd this result is sharp for the functions given by

$$p(z) = (1+z^2)/(1-z^2) \quad \& \quad p(z) = (1+z)/(1-z).$$

**Lemma 1.4** [19]. If $p(z) \in P$ and of the form (1.3), then

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$

$$4p_3 = p_1^3 + 2\left(4-p_1^2\right)p_1 x - \left(4-p_1^2\right)p_1 x^2 + 2\left(4-p_1^2\right)\left(1-\left|x\right|^2\right)z$$

for some $x,$ $z$ such that $\left|x\right| \le 1, \left|z\right| \le 1.$

## 2. MAIN RESULTS

**Theorem 2.1.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} -\dfrac{2\lambda\left(3-A_3\right) - 3A_2^2 + 8A_2 - 4}{8\left(2-A_2\right)^2\left(3-A_3\right)}, \text{if } \lambda < \dfrac{\left(7A_2-10\right)\left(2-A_2\right)}{2\left(3-A_3\right)}, \\ \dfrac{1}{2(3-A_3)}, \text{ if } \dfrac{\left(7A_2-10\right)\left(2-A_2\right)}{2\left(3-A_3\right)} \le \lambda \le \dfrac{\left(6-A_2\right)\left(2-A_2\right)}{2\left(3-A_3\right)}, \\ \dfrac{2\lambda\left(3-A_3\right) + 3A_2^2 - 8A_2 + 4}{8\left(2-A_2\right)^2\left(3-A_3\right)}, \text{if } \lambda \ge \dfrac{\left(6-A_2\right)\left(2-A_2\right)}{2\left(3-A_3\right)}. \end{cases}$$

Furthermore

$$\left|a_3 - \lambda a_2^2\right| + \left(\lambda - \frac{(7A_2-10)(2-A_2)}{2(3-A_3)}\right)\left|a_2\right|^2 \le \frac{1}{2\left(3-A_3\right)}$$

for

$$\frac{7A_2^2 - 24A_2 + 20}{2\left(A_3-3\right)} < \lambda \le \frac{3A_2^2 - 8A_2 + 4}{2\left(A_3-3\right)}$$

and

$$\left|a_3 - \lambda a_2^2\right| + \left(\frac{(6-A_2)(2-A_2)}{2(3-A_3)} - \lambda\right)\left|a_2\right|^2 \le \frac{1}{2\left(3-A_3\right)}$$

for

$$\frac{3A_2^2 - 8A_2 + 4}{2\left(A_3-3\right)} < \lambda \le \frac{A_2^2 - 8A_2 + 12}{2\left(3-A_3\right)},$$

where

$$A_n = t^{n-1} + t^{n-2}s^1 + t^{n-3}s^2 + \cdots + s^{n-1}.$$

These results are sharp.

**Proof.** If $f(z) \in SL(s,t),$ then by using (1.2), it follows that

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \prec \Phi(z), \quad z \in \mathbb{U}, \tag{2.1}$$

where $\Phi(z) = \sqrt{1+z}.$ Let us define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots.$$

It is clear that $p(z) \in P.$ This implies that

$$\omega(z) = \frac{p(z)-1}{p(z)+1}.$$

From (2.1), we have

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \prec \Phi(w(z)),$$

with

$$\Phi(w(z)) = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

Now

$$\sqrt{\frac{2p(z)}{p(z)+1}} = 1 + \frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right)z^2 + \cdots.$$

Similarly

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = 1+(2-A_2)a_2 z + \left((3-A_3)a_3-(2-A_2)a_2^2A_2\right)z^2 + \cdots .$$

Equating the coefficients of $z, z^2, z^3$ and $z^4$, we obtain

$$a_2 = \frac{1}{4\left(2-A_2\right)}p_1 \tag{2.2}$$

$$a_3 = \frac{1}{4\left(3-A_3\right)}\left\{p_2+\frac{7A_2-10}{8\left(2-A_2\right)}p_1^2\right\} \tag{2.3}$$

$$a_4 = \frac{1}{4\left(4-A_2\right)}\left[p_3+\left\{\frac{3A_2+2A_3-2A_2A_3}{\left(2-A_2\right)\left(3-A_3\right)}\right\}\frac{p_1p_2}{4}+\left\{\frac{-2A_2^2}{\left(2-A_2\right)^2}+\frac{\left(7A_2-10\right)\left(3A_2+2A_3-2A_2A_3\right)}{\left(2-A_2\right)^2\left(3-A_3\right)}+13\right\}\frac{p_1^3}{32}\right] \tag{2.4}$$

Using (2.2) and (2.3), we have

$$\left|a_3-\lambda a_2^2\right| = \frac{1}{4(3-A_3)}\left|p_2-\left\{\lambda\frac{(3-A_3)}{4(2-A_2)^2}-\frac{7A_2-10}{8(2-A_2)}\right\}p_1^2\right|.$$

Hence, using Lemma 1.1, we obtain the required result. The results are sharp for the functions $h_i(z), i=1,2,3,4,$ such that

$$\frac{\left(s-t\right)zh_1'(z)}{h_1(z)} = \sqrt{1+z} \quad \text{if} \quad \lambda < -\frac{3}{4} \text{ or } \lambda > \frac{5}{4},$$

$$\frac{\left(s-t\right)zh_2'(z)}{h_2(z)} = \sqrt{1+z^2} \quad \text{if} \quad -\frac{3}{4} < \lambda < \frac{5}{4}$$

$$\frac{\left(s-t\right)zh_3'(z)}{h_3(z)} = \sqrt{1+\phi(z)} \quad \text{if} \quad \lambda = -\frac{3}{4}$$

$$\frac{\left(s-t\right)zh_4'(z)}{h_4(z)} = \sqrt{1-\phi(z)} \quad \text{if} \quad \lambda = \frac{5}{4},$$

where $\phi(z) = \frac{z(z+\mu)}{1+\mu z}, \quad 0 \le \mu \le 1.$

**Theorem 2.2.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_3-\lambda a_2^2\right| \le \frac{1}{2\left(3-A_3\right)}\max\left(1, \frac{\left|2\lambda\left(3-A_3\right)+3A_2^2-8A_2+4\right|}{4\left(2-A_2\right)^2}\right),$$

where $\lambda$ is a complex number.

**Proof.** Since

$$\left|a_3-\lambda a_2^2\right| = \frac{1}{4\left(3-A_3\right)}\left|p_2-\left\{\lambda\frac{\left(3-A_3\right)}{4\left(2-A_2\right)^2}-\frac{7A_2-10}{8\left(2-A_2\right)}\right\}p_1^2\right|,$$

therefore, using Lemma 1.3, we get the required result. This result is sharp for

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \sqrt{1+z}$$

or

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \sqrt{1+z^2}$$

For $\lambda = 1$, we obtain the following Corollary

**Corollary 2.3.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_3-a_2^2\right| \le \frac{1}{2\left(3-A_3\right)}.$$

**Theorem 2.4.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_2a_4-a_3^2\right| \le \frac{1}{4\left(3-A_3\right)^2}.$$

This result is sharp.

**Proof.** Using (2.2), (2.3) and (2.4), we have

$$a_2a_4-a_3^2 = p_3-\frac{1}{16\left(3-A_3\right)^2}\left\{p_2+\frac{7A_2-10}{8\left(2-A_2\right)}p_1^2\right\}^2 + \left(\frac{p_1}{16\left(2-A_2\right)\left(4-A_4\right)}\right)\left[\left\{\frac{3A_2+2A_3-2A_2A_3}{\left(2-A_2\right)\left(3-A_3\right)}-5\right\}\frac{p_1p_2}{4}+\left\{\frac{-2A_2^2}{\left(2-A_2\right)^2}+\frac{\left(7A_2-10\right)\left(3A_2+2A_3-2A_2A_3\right)}{\left(2-A_2\right)^2\left(3-A_3\right)}+13\right\}\frac{p_1^3}{32}\right]$$

or, equivalently

$$\left|a_2a_4-a_3^2\right| = \left|Ap_1p_3-B_1p_2^2+Cp_1p_2+Dp_1^4\right|$$

with

$$A = \frac{1}{16\left(2-A_2\right)\left(4-A_4\right)}, \quad B = \frac{1}{16\left(3-A_3\right)^2},$$

$$C = \frac{A}{4}\left(\frac{3A_2+2A_3-2A_2A_3}{\left(2-A_2\right)\left(3-A_3\right)}-5\right)-\frac{B(7A_2-10)}{4\left(2-A_2\right)},$$

$$D = 4A\left(\frac{(3A_2+2A_3-2A_2A_3)(7A_2-10)}{128(2-A_2)^2(3-A_3)}+\frac{(A_2-2)}{64\left(2-A_2\right)^3}A_2^2+\frac{13}{128}\right) - \frac{\left(7A_2-10\right)^2}{1024\left(2-A_2\right)^2\left(3-A_3\right)^2}.$$

Putting the value of $p_2$ and $p_3$ from Lemma 1.4 and suppose that $p > 0$ with $p_1 = p \in [0,2]$, we obtain

$$\left|a_2a_4-a_3^2\right| = \left|\begin{array}{l}\left(A-B+2C+4D\right)\frac{p^4}{4} \\ +\frac{1}{2}Ap\left(4-p^2\right)\left(1-|x|^2\right)z \\ +\frac{1}{2}\left(A-B+C\right)p^2\left(4-p^2\right)x \\ -\frac{1}{4}\left(Ap^2+B\left(4-p^2\right)\right)\left(4-p^2\right)x^2\end{array}\right|$$

Using triangle inequality and taking $|x| = \rho$ and $|z| < 1$, we have

$$\left|a_2a_4 - a_3^2\right| \le |A-B+2C+4D| \frac{p^4}{4} + |A-B+C| p^2 \left(4-p^2\right) \frac{\rho}{2}$$

$$\frac{Ap}{2}\left(4-p^2\right)\left(1-\rho^2\right) + \left(Ap^2 + B\left(4-p^2\right)\right)\left(4-p^2\right)\frac{\rho^2}{2}$$

$$= F\left(c, \rho\right),$$

Differentiating with respect to $\rho$, we get

$$\frac{\partial F(p,\rho)}{\partial \rho} = \frac{1}{2}|A-B+C| p^2 \left(4-p^2\right) - Ap\left(4-p^2\right)\rho +$$

$$\left(Ap^2 + B\left(4-p^2\right)\right)\left(4-p^2\right)\frac{\rho}{2}$$

It is clear that $\partial F(p,\rho)/\partial\rho > 0$, which shows that $F\left(p,\rho\right)$ is an increasing function on the interval $\left[0,1\right]$. This implies that maximum value occurs at $\rho = 1$. Therefore $F(p,\rho) = F(p,1) = N(p)$ (say).

$$N(p) = |A-B+2C+4D| \frac{p^4}{4} + \frac{1}{2}|A-B+C| p^2 \left(4-p^2\right)$$

$$+\frac{1}{4}\left(Ap^2 + B\left(4-p^2\right)\right)\left(4-p^2\right)$$

Differentiating again with respect to $\rho$, we get

$$N'(p) = \left(|A-B+2C+4D| - 2|A-B+C| - 4A + 4B\right)p^3$$

$$+\left(4|A-B+C| + 2A - 4B\right)p$$

$$N''(p) = 3\left(|A-B+2C+4D| - 2|A-B+C| - 4A + 4B\right)p^2$$

$$+\left(4|A-B+C| + 2A - 4B\right)$$

Now $N''(p) < 0$ for $p = 0$, so maximum value occurs at $p = 0$, and hence

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{4(3-A_3)^2}.$$

This result is sharp for the functions

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = \sqrt{1+z}$$

or

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = \sqrt{1+z^2}$$

**Theorem 2.5.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_2a_3 - a_4\right| \le \frac{1}{2\left(4-A_4\right)}.$$

**Proof.** By similar arguments as used in last theorem, we get the required result.

**Lemma 2.6.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|a_2\right| \le \frac{1}{2\left(2-A_2\right)}, \quad \left|a_3\right| \le \frac{1}{2\left(3-A_3\right)},$$

$$\left|a_4\right| \le \frac{1}{2\left(4-A_4\right)}, \quad \left|a_5\right| \le \frac{1}{2\left(5-A_5\right)}.$$

This estimations are sharp.

**Proof.** From (2.1), we can write

$$\left(s-t\right)^2\left(zf'(z)\right)^2 = \left(1+w(z)\right)\left(f(z)\right)^2,$$

or, equivalently

$$\left(s-t\right)^2\left(z + \sum_{n=2}^{\infty} na_nz^n\right)^2 = \left(1+\sum_{n=1}^{\infty} d_nz^n\right)\left(z+\sum_{n=2}^{\infty} a_nz^n\right)^2$$

Comparing the coefficients of like powers and then simple computation gives the required result.

**Theorem 2.7.** Let $f(z) \in SL(s,t)$ be the form (1.1). Then

$$\left|H_3(1)\right| \le \frac{1}{4}\left[\frac{1}{2\left(3-A_3\right)^3} + \frac{1}{\left(4-A_4\right)^2} + \frac{1}{\left(3-A_3\right)\left(5-A_5\right)}\right].$$

**Proof.** Since

$$\left|H_3(1)\right| \le \left|a_3\right|\left|a_2a_4 - a_3^2\right| + \left|a_4\right|\left|a_2a_3 - a_4\right| + \left|a_5\right|\left|a_1a_3 - a_2^2\right|$$

Using Corollary 2.3, Theorem 2.5 and Lemma 2.6, we obtain

$$\left|H_3(1)\right| \le \frac{1}{8\left(3-A_3\right)^3} + \frac{1}{4\left(4-A_4\right)^2} + \frac{1}{4\left(3-A_3\right)\left(5-A_5\right)}.$$

Then by simple computation we obtain the required result. For $s = 1$ and $t = 0$, we obtain the result proved in [16].

**Corollary 2.8.** Let $f(z) \in SL$ be the form (1.1). Then

$$576\left|H_3(1)\right| \le 43.$$

For $s = 1$ and $t = -1$, we obtain the following result

**Corollary 2.9.** Let $f(z) \in SL(1,-1)$ be the form (1.1). Then

$$\left|H_3(1)\right| \le \frac{1}{16}.$$

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