SOME WEIGHTED INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GEOMATRICALLY-QUASI CONVEX FUNCTIONS ON THE ARGUMENTS

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ABSTRACT: *In the present paper, the concept of geometrical quasi convex functions on the argument is instigated. By using the concept of geometrically quasi convexity of a function on the argument, Hermite-Hadamard type inequalities for this category of functions are established.*

Key words: Convex function, argumented GA-convex function, Hölder's integral integral inequality, Geometrically quasi convex functions, Hermite-Hadamard type inequality

1. INTRODUCTION

An operation $f: L \subseteq \mathbb{R} \to \mathbb{R}$ forenamed as convex in a classical touch [1], if this lack of equality

$$f(\sigma e + (1-\sigma)k) \le \sigma f(e) + (1-\sigma) f(k)$$

exists for every $e,\ k \in L$ and $\sigma \in [0,1]$.

Indeed, a vast literature has been registered on inequalities utilizing traditional convexity, but one of the most celebrated is the hermit-Hadammard inequality. This double inequality is broadcast as succeeding :

Authorize $f: L \subseteq \mathbb{R} \to \mathbb{R}$ to be a function and $\mu,\ w \in L$ with $\mu < w$. Then f is convex on $[\mu, w]$ if

$$f\left(\frac{\mu+w}{2}\right) \le \frac{1}{w-\mu}\int_{\mu}^{w} f(e)de \le \frac{f(\mu)+f(w)}{2}. \qquad (1)$$

It is well-known that notion of quasi-convex functions as given in the definition below, which generalizes the notion of convex functions. Evidently, every convex function is a quasi-convex function.

Definition 1: *A function $f: L \subseteq \mathbb{R} \to \mathbb{R}$ aforesaid as quasi-convex if*

$$f(\sigma\mu + (1-\sigma)w) \le \max\{f(\mu), f(w)\} \qquad (2)$$

exists for all $\mu, w \in L$ and $\sigma \in [0,1]$.

In what follows, we will consider $I = [m, b]$ and $J = [c, d]$ with $m < b$ and $c < d$.

Definition 2 [2]**:** A function $f: \Delta \to \mathbb{R}$ is aforementioned to be convax over arguments on Δ in case that partial mappings $f_k: I \to \mathbb{R},\ f_k(u) = f(u, k)$ *and* $f_e: J \to \mathbb{R}$, $f_e(v) = f(e, v)$ *are convex where denominate for every* $e \in I,\ k \in J$.

Remark 1 [3]**:** *It is clear that if a function $f: \Delta \to \mathbb{R}$ is convex over* arguments *on Δ. Then*

$$f(e + (1-\sigma)z, \rho k + (1-\rho)w)$$
$$\le \sigma\rho f(e,k) + \sigma(1-\rho) f(e,w)$$
$$+\rho(1-\sigma) f(z,k) + (1-\sigma)(1-\rho) f(z,w),$$

holds for all $(\sigma, \rho) \in [0,1] \times [0,1]$ and $e, z \in$ I, $k, w \in$ J.

It is well-known that every convex averaging $f: \Delta \to \mathbb{R}$ is convex on the arguments, but counter pole may not permitted to be accurate [2].

The upcoming inequalities of Hermmite-Hadammard sort of augmented convex functions on the rectangle from the plane $\mathbb{R}^2$ were settled in [5].

Theorem 1: *Pretend $f: \Delta \to \mathbb{R}$ to be* argumented *convex on Δ,* [4] *then*

$$f\left(\frac{m+b}{2}, \frac{c+d}{2}\right)$$
$$\le \frac{1}{2}\left[\frac{1}{b-m}\int_m^b f\left(e, \frac{c+d}{2}\right) de + \frac{1}{d-c}\int_c^d f\left(\frac{m+b}{2}, k\right) dk\right]$$
$$\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(e,k)dkde$$
$$\le \frac{1}{4}\left[\frac{1}{b-m}\int_a^b [f(e,c) + f(e,d)]de + \frac{1}{d-c}\int_c^d [f(m,k) + f(b,k)]dk\right]$$
$$\le \frac{f(m,c) + f(m,d) + f(b,c) + f(b,d)}{4} \qquad (3)$$

The raised inequalities are sharp.

Definition 3 [5]**:** A action $f: \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is geometrically quasi-convex on Δ, with $m < b$ and $c < d$ if

$$f(\rho e + (1-\rho)z, \rho k + (1-\rho)w)$$
$$\le \max\{f(e,k), f(e,w), f(z,k), f(z,w)\}$$

holds for all $(e,k), (z,w) \in \Delta$ and $\rho \in [0,1]$.

2. WEIGHTED INEQUALITIES FOR ARGUMENTED GEOMETRICALLY QUASI CONVEX FUNCTIONS

Now we will introduce the definition of the geometrically quasi-convex functions.

Definition 4: A activity $f: \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is geometrically quasi-convex on the arguments on Δ, with $m < b$ and $c < d$ if

$$f(e^t z^{1-t}, k^\rho w^{1-\rho})$$
$$\le \max\{f(e,k), f(e,w), f(z,k), f(z,w)\}$$

holds for all $(e,k), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

and

$$f(\rho e + (1-\rho)z, \rho k + (1-\rho)w)$$
$$\le f(e^\rho z^{1-\rho}, k^\rho w^{1-\rho})$$
$$\le \max\{f(e,k), f(e,w), f(z,k), f(z,w)\} \quad (4)$$

We will use the following notations, in regard of amenity

$$\psi_1(t) = m^{\frac{1+t}{2}} b^{\frac{1-t}{2}}, \psi_2(s) = c^{\frac{1+s}{2}} d^{\frac{1-s}{2}},$$
$$\Omega_1(t) = m^{\frac{1-t}{2}} b^{\frac{1+t}{2}}, \Omega_2(s) = c^{\frac{1-s}{2}} d^{\frac{1+s}{2}}.$$

To obtain ou principal emanation, we first establish the following weighted identity.

Lemma 1: Suppose that $f: \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ has second order partial derivatives on Δ° and $I \times J \subseteq \Delta^\circ$. If $h: I \times J \to [0,\infty)$ is twice partially differentiable mapping and $f_{ts} \in L(I \times J)$, then the upcoming identity exists

$$\Phi(m, b, c, d; f, h) = h(m,c) f(m,c)$$
$$-h(m,d) f(m,d) - h(b,c) f(b,c) + h(b,d) f(b,d)$$
$$+\int_c^d h_k(m,k) f(m,k)dk - \int_c^d h_k(b,k) f(b,k)dk$$
$$-\int_a^b h_e(e,d) f(e,d)de + \int_a^b h_e(e,c) f(e,c)de$$

$$+\int_m^b\int_c^d h_{ek}(e,k)f(e,k)dkde$$

$$=\frac{(\ln b-\ln m)(\ln d-\ln c)}{4}$$
$$\times\Big[\int_0^1\int_0^1\psi_1(t)\psi_2(s)h(\psi_1(t),\psi_2(s))$$
$$\times f_{ts}(\psi_1(t),\psi_2(s))dsdt$$
$$+\int_0^1\int_0^1\Omega_1(t)\psi_2(s)h(\Omega_1(t),\psi_2(s))$$
$$\times f_{ts}(\Omega_1(t),\psi_2(s))dsdt$$
$$+\int_0^1\int_0^1\psi_1(t)\Omega_2(s)h(\psi_1(t),\Omega_2(s))$$
$$\times f_{ts}(\psi_1(t),\Omega_2(s))dsdt$$
$$+\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)h(\Omega_1(t),\Omega_2(s))$$
$$\times f_{ts}(\Omega_1(t),\Omega_2(s))dsdt\Big]. \qquad (5)$$

Proof. By switching of the variables $e=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}$, $k=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}}$ and by integration by parts with respect to k and then with respect to e, we have

$$\frac{(\ln b-\ln m)(\ln d-\ln c)}{4}\int_0^1\int_0^1\psi_1(t)\psi_2(s)h(\psi_1(t),\psi_2(s))$$
$$\times f_{ts}(\psi_1(t),\psi_2(s))dsdt$$
$$=\int_a^{\sqrt{mb}}\int_c^{\sqrt{cd}}h(e,k)f_{ek}(e,k)dkde$$
$$=h(\sqrt{mb},\sqrt{cd})f(\sqrt{mb},\sqrt{cd})$$
$$-h(m,\sqrt{cd})f(m,\sqrt{cd})-h(\sqrt{mb},c)f(\sqrt{mb},c)$$
$$+h(m,c)f(m,c)+\int_c^{\sqrt{cd}}h_y(m,y)f(m,y)dy$$
$$-\int_c^{\sqrt{cd}}h_k(\sqrt{mb},k)f(\sqrt{mb},k)dk$$
$$-\int_m^{\sqrt{mb}}h_e(e,\sqrt{cd})f(e,\sqrt{cd})de$$
$$+\int_m^{\sqrt{mb}}h_e(e,c)f(e,c)de$$
$$+\int_m^{\sqrt{mb}}\int_c^{\sqrt{cd}}h_{ek}(e,k)f(e,k)dkde. \qquad (6)$$

Similarly, we obtain

$$\frac{(\ln b-\ln m)(\ln d-\ln c)}{4}\int_0^1\int_0^1\Omega_1(t)\psi_2(s)h(\Omega_1(t),\psi_2(s))$$
$$\times f_{ts}(\Omega_1(t),\psi_2(s))dsdt$$
$$=h(b,\sqrt{cd})f(b,\sqrt{cd})-h(b,c)f(b,c)$$
$$-h(\sqrt{mb},\sqrt{cd})f(\sqrt{mb},\sqrt{cd})$$
$$+h(\sqrt{mb},c)f(\sqrt{mb},c)$$
$$-\int_c^{\sqrt{cd}}h_k(b,k)f(b,k)dk$$
$$+\int_c^{\sqrt{cd}}h_k(\sqrt{mb},k)f(\sqrt{mb},k)dk$$
$$-\int_{\sqrt{mb}}^b h_e(e,\sqrt{cd})f(e,\sqrt{cd})de$$
$$+\int_{\sqrt{mb}}^b h_e(e,c)f(e,c)de$$
$$+\int_{\sqrt{mb}}^b\int_c^{\sqrt{cd}}h_{ek}(e,k)f(e,k)dkde, \qquad (7)$$

$$\frac{(\ln b-\ln m)(\ln d-\ln c)}{4}\int_0^1\int_0^1\psi_1(t)\Omega_2(s)h(\psi_1(t),\Omega_2(s))$$
$$\times f_{ts}(\psi_1(t),\Omega_2(s))dsdt$$
$$=h(\sqrt{mb},d)f(\sqrt{mb},d)-h(m,d)f(m,d)$$
$$-h(\sqrt{mb},\sqrt{cd})f(\sqrt{mb},\sqrt{cd})+h(m,\sqrt{cd})f(m,\sqrt{cd})$$
$$-\int_c^{\sqrt{cd}}h_k(\sqrt{mb},k)f(\sqrt{mb},k)dk$$
$$+\int_c^{\sqrt{cd}}h_k(m,k)f(m,k)dk$$
$$-\int_m^{\sqrt{mb}}h_e(e,d)f(e,d)de$$
$$+\int_m^{\sqrt{mb}}h_x(e,\sqrt{cd})f(e,\sqrt{cd})de$$
$$+\int_m^{\sqrt{mb}}\int_{\sqrt{cd}}^d h_{ek}(e,k)f(e,k)dkde \qquad (8)$$

and

$$\frac{(\ln b-\ln m)(\ln d-\ln c)}{4}\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)h(\Omega_1(t),\Omega_2(s))$$
$$\times f_{ts}(\Omega_1(t),\Omega_2(s))dsdt$$
$$=h(b,d)f(b,d)-h(b,\sqrt{cd})f(b,\sqrt{cd})$$
$$-h(\sqrt{mb},d)f(\sqrt{mb},d)+h(\sqrt{mb},\sqrt{cd})f(\sqrt{mb},\sqrt{cd})$$
$$-\int_{\sqrt{cd}}^d h_k(b,k)f(b,k)dk+\int_{\sqrt{cd}}^d h_k(\sqrt{mb},k)f(\sqrt{mb},k)dk$$
$$-\int_{\sqrt{mb}}^b h_e(e,d)f(e,d)de$$
$$+\int_{\sqrt{mb}}^b h_e(e,\sqrt{cd})f(e,\sqrt{cd})de$$
$$+\int_{\sqrt{mb}}^b\int_{\sqrt{cd}}^d h_{ek}(e,k)f(e,k)dkde. \qquad (9)$$

Adding (6)-(9), we get the wanted identification. This accomplishes the proof of the Lemma.

Lemma 2: *Pretent $u,\ v>0,\ \eta,\epsilon\in\mathbb{R}$ and $\eta\neq 0$. Then*

$$\zeta(u,v;\epsilon,\eta)=\int_0^1(1-\epsilon t)u^{\frac{1}{2}+\eta t}v^{\frac{1}{2}-\eta t}dt$$
$$=\begin{cases}\frac{\epsilon v^{\frac{1}{2}-\eta}u^{\frac{1}{2}}[L(u^\eta,v^\eta)-u^\eta]}{\eta(\ln u-\ln v)}+v^{\frac{1}{2}-\eta}u^{\frac{1}{2}}L(u^\eta,v^\eta), & u\neq v,\\ \frac{u[1-(1-\epsilon)^2]}{2\epsilon}, & u=v,\end{cases}$$

where $L(u,v)$ is the logarithmic mean

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v.\end{cases}$$

Proof. The proof follows by integration by parts.

Theorem 2: *Permit $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $I\times J\subseteq\Delta^\circ$. If $h:I\times J\to[0,\infty)$ is a twice partially differentiable mapping such that $f_{ts}\in L(I\times J)$ and $|f_{ts}|^q$ is geometrically quasi-convex on the* arguments *for $q\geq 1$, then it authority the inequality:*

$$|\Phi(m,b,c,d;f,h)|\leq(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty$$
$$\times\Big\{\zeta\left(m,b;0,\tfrac{1}{2}\right)\zeta\left(c,d;0,\tfrac{1}{2}\right)M_1(f)$$

$$+\zeta\left(m,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)M_2(f)$$
$$+\zeta\left(m,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)M_3(f)$$
$$+\zeta\left(m,b;0,-\tfrac{1}{2}\right)\zeta\left(c,d;0,-\tfrac{1}{2}\right)M_4(f)\Big\},\quad (10)$$

where $\|h\|_\infty=\sup_{(e,k)\in I\times J} h(e,k)$ and $\zeta(u,v;\epsilon,\eta)$ is prescribed in Lemma 2, and

$$M_1(f)=\max\{|f_{ts}(\sqrt{mb},\sqrt{cd})|,|f_{ts}(\sqrt{mb},c)|,$$
$$|f_{ts}(m,\sqrt{cd})|,|f_{ts}(m,c)|\},$$
$$M_2(f)=\max\{|f_{ts}(\sqrt{mb},\sqrt{cd})|,|f_{ts}(\sqrt{mb},c)|,$$
$$|f_{ts}(b,\sqrt{cd})|,|f_{ts}(b,c)|\},$$

$$M_3(f)=\max\{|f_{ts}(\sqrt{mb},\sqrt{cd})|,|f_{ts}(\sqrt{mb},d)|,$$
$$|f_{ts}(m,\sqrt{cd})|,|f_{ts}(m,d)|\}$$

and

$$M_4(f)=\max\{|f_{ts}(\sqrt{mb},\sqrt{cd})|,|f_{ts}(\sqrt{mb},d)|,$$
$$|f_{ts}(b,\sqrt{cd})|,|f_{ts}(b,d)|\}.$$

Proof. From Lemma 1, we have

$$|\Phi(m,b,c,d;f,h)|\le\frac{(\ln b-\ln m)(\ln d-\ln c)\|h\|_\infty}{4}$$
$$\times\left[\int_0^1\int_0^1\psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt\right.$$
$$+\int_0^1\int_0^1\Omega_1(t)\psi_2(s)\left|f_{ts}(\Omega_1(t),\psi_2(s))\right|dsdt$$
$$+\int_0^1\int_0^1\psi_1(t)\Omega_2(s)\left|f_{ts}(\psi_1(t),\Omega_2(s))\right|dsdt$$
$$\left.+\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt\right].\quad (11)$$

Now by using Hölder's inequality for double integrals and by the geometrically quasi-convexity of $|f_{ts}|^q$ on the arguments on $I\times J$ for $q\ge 1$, we acquire

$$\int_0^1\int_0^1\psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt$$
$$\le\left(\int_0^1\int_0^1\psi_1(t)\psi_2(s)dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\le\left(\int_0^1\int_0^1\psi_1(t)\psi_2(s)dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(M^q(f)\int_0^1\int_0^1\psi_1(t)\psi_2(s)dsdt\right)^{\frac{1}{q}}$$
$$\le\zeta\left(m,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)M_1(f).$$

Correspondingly

$$\int_0^1\int_0^1\Omega_1(t)\psi_2(s)\left|f_{ts}(\Omega_1(t),\psi_2(s))\right|dsdt$$
$$\le\zeta\left(m,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)M_2(f),$$
$$\int_0^1\int_0^1\psi_1(t)\Omega_2(s)\left|f_{ts}(\psi_1(t),\Omega_2(s))\right|dsdt$$
$$\le\zeta\left(m,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)M_3(f)$$

and

$$\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt$$
$$\le\zeta\left(m,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)M_4(f).$$

Using the above four inequalities in (11) and by resolution, we get (10) and proof is completed.

Corollary 1: *If we consider* $h(e,k)=\frac{1}{(lnb-lna)(lnd-lnc)}$, $(e,k)\in I\times J$ *in Theorem 2, then*

$$\left|\Phi\left(m,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right|$$
$$\le\Big\{\zeta\left(m,b;0,\tfrac{1}{2}\right)\zeta\left(c,d;0,\tfrac{1}{2}\right)M_1(f)$$
$$+\zeta\left(m,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)M_2(f)$$
$$+\zeta\left(m,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)M_3(f)$$
$$+\zeta\left(m,b;0,-\tfrac{1}{2}\right)\zeta\left(c,d;0,-\tfrac{1}{2}\right)M_4(f)\Big\}.\quad (12)$$

Theorem 3: *Suppose* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $I\times J\subseteq\Delta^\circ$. *Further let* $h:I\times J\to[0,\infty)$ *be a twice partially differentiable mapping. If* $f_{ts}\in L(I\times J)$ *and* $|f_{ts}|^q$ *is geometrically quasi-convex on the* arguments *on* $I\times J$ *for* $q>1$, *then we have inequality of the form:*

$$|\Phi(m,b,c,d;f,h)|\le(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty$$
$$\times\left\{\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\tfrac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\tfrac{1}{2}\right)\right]^{1-\frac{1}{q}}M_1(f)\right.$$
$$+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\tfrac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\tfrac{1}{2}\right)\right]^{1-\frac{1}{q}}M_2(f)$$
$$+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\tfrac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\tfrac{1}{2}\right)\right]^{1-\frac{1}{q}}M_3(f)$$
$$\left.+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\tfrac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\tfrac{1}{2}\right)\right]^{1-\frac{1}{q}}M_4(f)\right\},$$
(13)

where $\|h\|_\infty=\sup_{(e,k)\in I\times J} h(e,k)$ and $\zeta(u,v;\epsilon,\eta)$ is characterized in Lemma 2.

Proof. From Lemma 1, we may write

$$|\Phi(m,b,c,d;f,h)|\frac{(\ln b-\ln m)(\ln d-\ln c)\|h\|_\infty}{4}$$
$$\times\left[\int_0^1\int_0^1\psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt\right.$$
$$+\int_0^1\int_0^1\Omega_1(t)\psi_2(s)\left|f_{ts}(\Omega_1(t),\psi_2(s))\right|dsdt$$
$$+\int_0^1\int_0^1\psi_1(t)\Omega_2(s)\left|f_{ts}(\psi_1(t),\Omega_2(s))\right|dsdt$$
$$\left.+\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt\right].\ (14)$$

Now by using Hölder's inequality for double integrals, Lemma 2 and by the geometrically quasi-convexity of $|f_{ts}|^q$ on the arguments on $I\times J$ for $q>1$, consequently we have

$$\int_0^1\int_0^1\psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt$$

$$\leq \left[\int_0^1\int_0^1 (\psi_1(t)\psi_2(s))^{\frac{q}{q-1}}dsdt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\int_0^1 \left|f_{ts}(\psi_1(t),\psi_2(s))\right|^q dsdt\right]^{\frac{1}{q}}$$
$$\leq \left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_1(f).$$

In addition

$$\int_0^1\int_0^1 \Omega_1(t)\psi_2(s)\left|f_{ts}(\Omega_1(t),\psi_2(s))\right|dsdt$$
$$\leq \left[\int_0^1\int_0^1 (\Omega_1(t)\psi_2(s))^{\frac{q}{q-1}}dsdt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\int_0^1 \left|f_{ts}(\Omega_1(t),\psi_2(s))\right|^q dsdt\right]^{\frac{1}{q}}$$
$$\leq \left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_2(f),$$
$$\int_0^1\int_0^1 \psi_1(t)\Omega_2(s)\left|f_{ts}(\psi_1(t),\Omega_2(s))\right|dsdt$$
$$\leq \left[\int_0^1\int_0^1 (\psi_1(t)\Omega_2(s))^{\frac{q}{q-1}}dsdt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\int_0^1 \left|f_{ts}(\psi_1(t),\Omega_2(s))\right|^q dsdt\right]^{\frac{1}{q}}$$
$$\leq \left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_3(f),$$

equivalently

$$\int_0^1\int_0^1 \Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt$$
$$\leq \left[\int_0^1\int_0^1 (\Omega_1(t)\Omega_2(s))^{\frac{q}{q-1}}dsdt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\int_0^1 \left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|^q dsdt\right]^{\frac{1}{q}}\leq$$
$$\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_4(f).$$

Using the above four inequalities in (14) and simplifying, we get the required inequality (13).

Corollary 2: *If we take* $h(e,k)=\frac{1}{(lnb-lnm)(lnd-lnc)}$, $(e,k)\in I\times J$ *in Theorem 3, then*

$$\left|\Phi\left(m,b,c,d;f,\frac{1}{(\ln b-\ln m)(\ln d-\ln c)}\right)\right|$$
$$\leq$$
$$\left\{\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_1(f)\right.$$
$$+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_2(f)$$
$$+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_3(f)$$
$$\left.+\left[\zeta\left(m^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_4(f)\right\}.(15)$$

Theorem 4: *Consider* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *to be a twice partially differentiable mapping on* Δ° *and* $I\times J\subseteq\Delta^\circ$. *Further let* $h: I\times J\to[0,\infty)$ *is a twice partially differentiable maapping. If* $f_{ts}\in L(I\times J)$ *and* $|f_{ts}|^q$ *is geometrically quasi-convex on the* arguments *on* $I\times J$ *for* $q>1$ *and* $q\geq r\geq 0$ *, then we attain the following inequality:*

$$|\Phi(m,b,c,d;f,h)|\leq(\ln b-\ln m)(\ln d-\ln c)\|h\|_\infty$$
$$\times\left\{\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\right.$$
$$\times\left[\zeta\left(m^r,b^r;0,\frac{1}{2}\right)\zeta\left(c^r,d^r;0,\frac{1}{2}\right)\right]^{\frac{1}{q}}M_1(f)$$
$$+\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\zeta\left(m^r,b^r;0,-\frac{1}{2}\right)\zeta\left(c^r,d^r;0,\frac{1}{2}\right)\right]^{\frac{1}{q}}M_2(f)$$
$$+\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\zeta\left(m^r,b^r;0,\frac{1}{2}\right)\zeta\left(c^r,d^r;0,-\frac{1}{2}\right)\right]^{\frac{1}{q}}M_3(f)$$
$$+\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$
$$\left.\times\left[\zeta\left(m^r,b^r;0,-\frac{1}{2}\right)\zeta\left(c^r,d^r;0,-\frac{1}{2}\right)\right]^{\frac{1}{q}}\right\}M_4(f),(16)$$

where $\|h\|_\infty=\sup_{(e,k)\in I\times J}h(e,k)$ and $\zeta(u,v;\epsilon,\eta)$ is defined in Lemma 2.

Proof. From Lemma 1, it follows that

$$|\Phi(m,b,c,d;f,h)|\leq\frac{(\ln b-\ln m)(\ln d-\ln c)\|h\|_\infty}{4}$$
$$\times\left[\int_0^1\int_0^1 \psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt\right.$$
$$+\int_0^1\int_0^1 \Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt$$
$$+\int_0^1\int_0^1 \psi_1(t)\Omega_2(s)\left|f_{ts}(\psi_1(t),\Omega_2(s))\right|dsdt$$
$$\left.+\int_0^1\int_0^1 \Omega_1(t)\Omega_2(s)\left|f_{ts}(\Omega_1(t),\Omega_2(s))\right|dsdt\right].\qquad(17)$$

Now by virtue of geometrically quasi-convexity of $|f_{ts}|^q$ on the arguments on $I\times J$ for $q>1$, Lemma 2 and by the Hölder's inequality for double integrals, so got in hand

$$\int_0^1\int_0^1 \psi_1(t)\psi_2(s)\left|f_{ts}(\psi_1(t),\psi_2(s))\right|dsdt$$
$$\leq\left(\int_0^1\int_0^1 (\psi_1(t)\psi_2(s))^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 (\psi_1(t)\psi_2(s))^r\left|f_{ts}(\psi_1(t),\psi_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\zeta\left(m^r,b^r;0,\frac{1}{2}\right)\zeta\left(c^r,d^r;0,\frac{1}{2}\right)\right]^{\frac{1}{q}}M_1(f).$$

Similarly

$$\int_0^1\int_0^1\Omega_1(t)\psi_2(s)\left|f_{ts}\left(\Omega_1(t),\psi_2(s)\right)\right|dsdt$$

$$\leq\left(\int_0^1\int_0^1\left(\Omega_1(t)\psi_2(s)\right)^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1\left(\Omega_1(t)\psi_2(s)\right)^r\left|f_{ts}\left(\Omega_1(t),\psi_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\zeta\left(m^r,b^r;0,-\frac{1}{2}\right)\zeta\left(c^r,d^r;0,\frac{1}{2}\right)\right]^{\frac{1}{q}}M_2(f)$$

$$\int_0^1\int_0^1\psi_1(t)\Omega_2(s)\left|f_{ts}\left(\psi_1(t),\Omega_2(s)\right)\right|dsdt$$

$$\leq\left(\int_0^1\int_0^1\left(\psi_1(t)\Omega_2(s)\right)^{\frac{q-r}{q-q}}dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1\left(\psi_1(t)\Omega_2(s)\right)^r\left|f_{ts}\left(\psi_1(t),\Omega_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\zeta\left(m^r,b^r;0,\frac{1}{2}\right)\zeta\left(c^r,d^r;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_3(f)$$

and

$$\int_0^1\int_0^1\Omega_1(t)\Omega_2(s)\left|f_{ts}\left(\Omega_1(t),\Omega_2(s)\right)\right|dsdt$$

$$\leq\left(\int_0^1\int_0^1\left(\Omega_1(t)\Omega_2(s)\right)^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1\left(\Omega_1(t)\Omega_2(s)\right)^r\left|f_{ts}\left(\Omega_1(t),\Omega_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq\left[\zeta\left(m^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\zeta\left(m^r,b^r;0,-\frac{1}{2}\right)\zeta\left(c^r,d^r;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}M_4(f)$$

Using the above four inequalities in (17) and simplifying, we obtained the required inequality (16).

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