E^h-b-PREINVEX FUNCTIONS AND LOCAL E^h-b-PREINVEX PROGRAMMINGS

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ABSTRACT *In this paper, the classes of B-preinvex and E-B-preinvex functions are extended to the classes of b^h-preinvex and E^h-b-preinvex functions, respectively. In this extension the effect of the functions $h: [0,1] \to \mathbb{R}$ and $b: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^+$ are taken into consideration. Some basic properties for the new functions are discussed and some optimality properties for local E^h-b-preinvex nonlinear optimization problems involving E^h-b-preinvex functions are established. The new results can be considered as an extension to several results that are introduced in the literature.*

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1. INTRODUCTION

Convex analysis is studied and employed widely in applied mathematics, especially, in optimization and optimal control (see e.g., [1-6]). Several attempts were made in the literature to generalize and extend convex sets and convex functions [7-26]. The class of convex sets (functions) were generalized to the class of invex sets [7,8], the class of preinvex functions [9,10], the class of B-vex functions [11], and to the class of B-preinvex [12]. Recently, Youness [13] introduced the concepts of E-convex sets (functions) and E-convex optimization problems. Youness inspired many researchers to extend many concepts from convex analysis into E-convexity and applying E-convexity in optimization problems see (e.g., [14-17]. Fulga and Perda [18] introduced the class of E-preinvex functions by combining the classes of preinvex and E-convex functions and also introduced E-prequasiinvex functions. Fulga and Perda applied the new classes to non-linear optimization problems. On the other hand, Syau *et. al.* [19] defined E-B-preinvex as a generalizations of E-convex and B-preinvex functions. More recently, h-strongly E-convex functions [20] was defined as a combination of strongly E-convex functions [14] and h-convex functions [21]. In this paper, we introduce the class of E^h-b-preinvex and local E^h-b-preinvex functions by combining the classes of h-preinvex [22] and E-B-preinvex functions. The class of b^h-preinvex functions is also defined by extending the classes of h-preinvex and B-preinvex functions. In section 2, some preliminary definitions studied in the literature are recalled and the new generalized convex functions are introduced. In section 3, some properties of E^h-b-preinvex functions are discussed and two characterizations of this class are provided using b^h-preinvex and E-prequasiinvex functions (see Propositions 3.4-3.5). We give a new characterization of E-prequasiinvex functions using the invexity of the level set $D_{\gamma,E}$ (see Proposition 3.9). In section 4, we provide some optimality properties of non-linear optimization problems for which the functions are local E^h-b-preinvex functions and the constraint set is local E-invex set.

2. Preliminaries

In this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^+$ be a set of non-negative real numbers. For brevity in writing the statements, the following assumption is needed.

Assumption Let $\emptyset \neq D \subseteq \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}$, $h: [0,1] \to \mathbb{R}$ be two real valued functions such that. Assume that $E: \mathbb{R}^n \to \mathbb{R}^n$ $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $b: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^+$ are given mappings where $\lambda b(x, y, \lambda) \in [0,1]$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Next, some preliminaries and related concepts that to develop the new functions are given. Throughout the paper, D, f, E, ψ and b are defined as in above Assumption unless otherwise stated.

Definition 2.1 Let $D, E,$ and ψ are defined as in the Assumption then, $\forall d_1, d_2 \in D$ and $\forall \lambda \in [0,1]$, D is said to be

1. E-convex if $\lambda E(d_1) + (1-\lambda)E(d_2) \in D$. [13]
2. An invex set with respect to ψ (for short, D is an invex w.r.t. ψ) if $d_2 + \lambda\psi(d_1, d_2) \in D$. [8]
3. An E-invex set w.r.t. ψ if $E(d_2) + \lambda\psi(E(d_1), E(d_2)) \in D$. [18]
4. A local E-invex set w.r.t. ψ if for each $d_1, d_2 \in D$ there exists $\zeta_{d_1,d_2} \in (0,1]$ such that $E(d_2) + \lambda\psi(E(d_1), E(d_2)) \in D \quad \forall\lambda \in [0, \zeta_{d_1,d_2}]$. [18]

Definition 2.2 [18] Let H and K be two subsets of $\mathbb{R}^n$ and ψ is defined as in the Assumption. Then

1. H is said to be slack-invex w.r.t. K if, for every $h, k \in H \cap K$ and every $0 \leq \lambda \leq 1$ such that $k + \lambda\psi(h, k) \in K$ we get $k + \lambda\psi(h, k) \in H$.
2. H is said to be local slack-invex w.r.t. K if, for every $h, k \in H \cap K$ there exists $\zeta_{h,k} \in (0,1]$ such that $\forall\lambda \in [0, \zeta_{h,k}]$ if $k + \lambda\psi(h, k) \in K$ we get $k + \lambda\psi(h, k) \in H$.

Proposition 2.3 [18] Let D and E are defined as in the Assumption such that D is an E-invex (respectively, local E-invex) set. Then, $E(D) \subseteq D$.

Definition 2.4 Let D, f, E, b and ψ are defined as in the Assumption then, $\forall d_1, d_2 \in D$ and for every $\lambda \in [0,1]$, f is called

1. preinvex function w.r.t. ψ on the invex set D if, $f(d_2 + \lambda\psi(d_1, d_2)) \leq \lambda f(d_1) + (1-\lambda)f(d_2)$. [9, 10]
2. -convex function on the E-convex set D if, $f(\lambda E(d_1) + (1-\lambda)E(d_2)) \leq \lambda f(E(d_1)) + (1-\lambda)f(E(d_2))$. [13]
3. E-preinvex function w.r.t. ψ on the E-invex set D if, $f(E(d_2) + \lambda\psi(E(d_1), E(d_2))) \leq \lambda f(E(d_1)) + (1-\lambda)f(E(d_2))$. [18]
4. B-preinvex function w.r.t. ψ on the invex set D if, $f(d_2 + \lambda\psi(d_1, d_2)) \leq \lambda b(d_1, d_2, \lambda)\ f(d_1) + (1-\lambda b(d_1, d_2, \lambda)\ )f(d_2)$, for $\lambda b(d_1, d_2, \lambda) \in [0,1]$ [12]

5. E-B-preinvex function w.r.t. ψ on the E-invex set D w.r.t. ψ if,

$f(E(d_2) + \lambda\psi(E(d_1), E(d_2))) \leq$
$\lambda b(d_1, d_2, \lambda)\ f(E(d_1)) + (1 - \lambda b(d_1, d_2, \lambda)\ )f(E(d_2))$,
for $\lambda b(d_1, d_2, \lambda) \in [0,1]$. [19]

6. E-prequasiinvex w.r.t. ψ on the E-invex set D if,
$f(E(d_2) + \lambda\psi(E(d_1), E(d_2))) \leq$
$\max\{f(E(d_1)), f(E(d_2))\}$. [18]

Remark 2.5 For simplicity in appearance

1. We omit in the proofs and calculations the parentheses from $E(x)$, and writing it instead as Ex whenever it seems convenient.
2. E-invex set w.r.t. ψ and (preinvex, E-preinvex, B-preinvex, E-B-preinvex, E-prequasiinvex) functions w.r.t. ψ will be called E-invex set, (preinvex, E-preinvex, B-preinvex, E-B-preinvex, E-prequasiinvex) functions.
3. We discard the argument of the mapping b and express $b(x, y, \lambda)$ as b wherever it appears in the paper.

Definition 2.6 A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is called

1. sublinear if $f(\alpha_1 x_1 + \alpha_1 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ $\forall x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. [1]
2. idempotent if $f^2(x) = f(x)$ $\forall x \in \mathbb{R}^n$. [23]
3. non-decreasing if whenever $x, y \in \mathbb{R}^n$ such that $x \preccurlyeq y$ (i.e., $x_i \leq y_i$, $\forall i = 1, \ldots, n$) we get $f(x) \leq f(y)$. [24, Definition 5.2.1]

n the literature, different types of γ-level sets associated with f and E are defined. Some of these sets are listed below.

Definition 2.7 Let $\gamma \in \mathbb{R}$. Then,

1. $D_\gamma = \{d \in D: f(d) \leq \gamma\}$. [1]
2. E-$D_\gamma = \{d \in D: f(Ed) \leq \gamma\}$. [16]
3. $D_{\gamma,E} = \{E(d) \in E(D): f(Ed) \leq \gamma\}$. [16]

Next, the definition of h-convex introduced in [21] is recalled. Noting that, in [25,26] other versions of h-convex functions are defined.

Definition 2.8 [21] Let $h: [0,1] \to \mathbb{R}$ be a function. Then $f: D \to \mathbb{R}$ is said to be h-convex function if for each $d_1, d_2 \in D$, and each $0 \leq \lambda \leq 1$ we have $f(\lambda d_1 + (1 - \lambda)d_2) \leq h(\lambda)f(d_1) + h(1 - \lambda)f(d_2)$.

By making use of a non-negative h-convex function and a preinvex function, Matloka in 2014 introduced the class of h-preinvex function.

Definition 2.9 [22] Let $h: [0,1] \to \mathbb{R}$ be a positive function. Then a positive function $f: D \to \mathbb{R}$ is said to be h-preinvex on the invex set D if for each $d_1, d_2 \in D$, and each $0 \leq \lambda \leq 1$ we have $f(d_2 + \lambda\psi(d_1, d_2)) \leq h(\lambda)f(d_1) + h(1 - \lambda)f(d_2)$.

By extending the definitions of B-preinvex and h-preinvex functions, we introduce the b^h-preinvex function as follows.

Definition 2.10 Let D, f, b, ψ and h are defined as in the Assumption such that D is an invex set. Then f is said to be b^h-preinvex function on D if for each $d_1, d_2 \in D$, and each $0 \leq \lambda \leq 1$

$$f(d_2 + \lambda\psi(d_1, d_2)) \leq h(\lambda b)f(d_1) + h(1 - \lambda b)f(d_2).$$

Using the definitions of h-preinvex function and E-B-preinvex function, we define the E^h-b-preinvex and local E^h-b-preinvex functions.

Definition 2.11 Let D, f, E, b, ψ and h are defined as in the Assumption such that D is an E-invex set. Then

1. f is said to be E^h-b-preinvex function on D if for each $d_1, d_2 \in D$, and each $0 \leq \lambda \leq 1$
$$f(Ed_2 + \lambda\psi(Ed_1, Ed_2)) \leq h(\lambda b)f(Ed_1) + h(1 - \lambda b)f(Ed_2).$$
2. f is said to be local E^h-b-preinvex function on D if for each $d_1, d_2 \in D$ there exists $\lambda_{d_1, d_2} \in (0, \zeta_{d_1, d_2}]$ such that $\forall \lambda \in [0, \lambda_{d_1, d_2}]$
$$f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \leq h(\lambda b)f(Ed_1) + h(1 - \lambda b)f(Ed_2),$$
where $\zeta_{d_1, d_2} \in (0,1]$.

Note that b^h-preinvex and E^h-b-preinvex functions are considered as extentions of B-preinvex and E-B-preinvex functions, respectively in the following sense.

Remark 2.12 If $h = I$ the identity function. Then

1. Every B-preinvex function is b^h-preinvex function.
2. Every E-B-preinvex function is E^h-b-preinvex.

Next, we show an example of E^h-b-preinvex function that is not E-B-preinvex .

Example 2.13 Let $f, E: \mathbb{R} \to \mathbb{R}$ and $\psi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as follows.

$$f(x) = \begin{cases} 1 & x \in [-2,0] \\ \frac{1}{2} & otherwise \end{cases},$$

$$\psi(x, y) = \begin{cases} x - y & x, y \in [-2,0] \ or \ x, y \in (0,2] \\ y - x & otherwise \end{cases}$$

and $E(x) = e^x$ $\forall x \in \mathbb{R}$. Assume that $h: [0,1] \to \mathbb{R}$ is defined as $h(\lambda) = 2\lambda$ $\forall \lambda \in [0,1]$ and $D = [-2,2] \subseteq \mathbb{R}$. We observe that f is E^h-b-preinvex on D but not E-B-preinvex function on D. Indeed, consider $x, y \in (0,2]$ such that $x = 2$, $y = 0.7$ and and $\lambda = 1$. Then $\psi(e^x, e^y) = e^y - e^x$. Hence,

$$f\left(Ey + \lambda\ \psi(Ex, Ey)\right) = f\left(e^y + \lambda\psi(e^x, e^y)\right)$$
$$= f(e^y + (e^y - e^x))$$
$$= f(-e^x + 2e^y) = 1.$$

On the other hand,

$$\lambda bf(Ex) + (1 - \lambda b)f(Ey) = \lambda bf(e^x) + (1 - \lambda b)f(e^y)$$
$$= \lambda b\left(\tfrac{1}{2}\right) + (1 - \lambda b)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$

This shows, $f\left(Ey + \lambda\ \psi(Ex, Ey)\right) > \lambda bf(Ex) + (1 - \lambda b)f(Ey)$ which means f is not E-B-preinvex on D.

Next, we show that f is E^h-b-preinvex on D. Direct calculations yields $D = [-2,2]$ is an E-invex set. Now, considering $x, y \in [-2,2]$ and $\lambda \in [0,1]$, we have four possible cases:

Case (1): If $x, y > 0$, i.e., $x,y \in (0,2]$ and $e^x, e^y \in (1, e^2]$;

then $f\left(Ey + \lambda\psi(Ex, Ey)\right) = f\left(e^y + \lambda\psi(e^x, e^y)\right)$ such that

$$\psi(e^x, e^y) = \begin{cases} e^x - e^y & e^x, e^y \in (1,,2] \\ e^y - e^x & o.w \end{cases},$$

Now,

$$f\left(e^y + \lambda\psi(e^x, e^y)\right) = \begin{cases} f\left(e^y + \lambda(e^x - e^y)\right) & e^x, e^y \in (1,,2] \\ f\left(e^y + \lambda(e^y - e^x)\right) & o.w. \end{cases}$$

$$= \begin{cases} f(\lambda e^x + (1 - \lambda)e^y) & e^x, e^y \in (1,,2] \\ f(-\lambda e^x + (1 + \lambda)e^y) & o.w. \end{cases},$$

$$=\begin{cases}1 & e^x, e^y \in (1,,2] \\ 1 \ or \ \frac{1}{2} & o.w.\end{cases},$$

and

$h(\lambda b)f(Ex) + h(1-\lambda b)f(Ey) = 2\lambda b f(e^x) + 2(1-\lambda b)f(e^y) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)\left(\frac{1}{2}\right) = 1.$

Case (2): If $x, y \leq 0$, i.e., $x, y \in [-2,0]$ and $e^x, e^y \in [0.1,1]$. This means $\psi(e^x, e^y) = e^x - e^y$, then

$$f\left(Ey + \lambda\psi(Ex, Ey)\right) = f(e^y + \lambda(e^x - e^y)) = f(\lambda e^x + (1-\lambda)e^y) = \frac{1}{2},$$

and

$h(\lambda b)f(Ex) + h(1-\lambda b)f(Ey) = 2\lambda b f(e^x) + 2(1-\lambda b)f(e^y) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)\left(\frac{1}{2}\right) = 1.$

Case (3): If $x > 0$, $y \leq 0$, i.e., $x \in (0,2]$ and $y \in [-2,0]$. Then $e^x \in (1, e^2]$ and $e^y \in [e^{-2}, 1]$ this means $\psi(Ex, Ey) = e^y - e^x$. Thus,

$$f\left(Ey + \lambda\psi(Ex, Ey)\right) = f\left(e^y + \lambda(e^y - e^x)\right) = f(-\lambda e^x + (1+\lambda)e^y) = \frac{1}{2} \text{ or } 1,$$

and $h(\lambda b)f(Ex) + h(1-\lambda b)f(Ey) = 2\lambda b f(e^x) + 2(1-\lambda b) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)\left(\frac{1}{2}\right) = 1.$

Case (4): If ≤ 0 , $y > 0$; i.e., $x \in [-2,0]$ and $y \in [0,2]$. Then $e^x \in [0.1,1]$ and $e^y \in (1, e^2]$. This means $\psi(Ex, Ey) = e^y - e^x$,

then $f\left(Ey + \lambda\psi(Ex, Ey)\right) = f\left(e^y + \lambda\psi(e^x, e^y)\right) = f(-\lambda e^x + (1+\lambda)e^y) = \frac{1}{2}$,

and $h(\lambda b)f(Ex) + h(1-\lambda b)f(Ey) = 2\lambda b f(e^x) + 2(1-\lambda b)f(e^y) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)\left(\frac{1}{2}\right) = 1,$

In all cases, we have $f\left(Ey + \lambda\psi(Ex, Ey)\right) \leq h(\lambda b)f(Ex) + h(1-\lambda b)f(Ey)$ as it is required to show. ■

3. Some properties of E^h-b-preinvex functions

In this section, we discuss some basic properties of E^h-b-preinvex functions. We start first by showing that the class of E^h-b-preinvex functions is closed under non-negative scalar multiplication and addition. Same property holds for classical convex functions.

Proposition 3.1 Let D,f,E ,b ,ψ and h are defined as in the Assumption. Assume that $g: \mathbb{R}^n \to \mathbb{R}$ be a function such that D is an E-invex and f, g are E^h-b-preinvex functions, then $\alpha f + \beta g$ is an E^h-b-preinvex function. $\forall \alpha, \beta \geq 0$.

Proof. Let $d_1, d_2 \in D$, and $\lambda \in [0,1]$. Set $u = Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D$. Using the fact that f and g are E^h-b-preinvex functions, we obtain

$(\alpha f + \beta g)(u) = \alpha f(u) + \beta g(u) \leq \alpha h(\lambda b)f(Ed_1) + \alpha h(1-\lambda b)f(Ed_2) + \beta h(\lambda b)g(Ed_1) + \beta h(1-\lambda b)g(Ed_2),$

$= h(\lambda b)(\alpha f + \beta g)(Ed_1) + h(1-\lambda b)(\alpha f + \beta g)(Ed_2).$

Hence, $\alpha f + \beta g$ is an E^h-b-preinvex on D. ■

Proposition 3.2 Let Let D, E ,ψ, b and h are defined as in the Assumption such that D is an E-invex set and h is positive, $f_i: \mathbb{R}^n \longrightarrow \mathbb{R}$ is bounded from above for each $i \in \Lambda$. Define, $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f = \sup_{i\in\Lambda} f_i$. If f_i is E^h-b-preinvex function on D w.r.t. the same ψ, b, h for each $i \in \Lambda$, then f is E^h-b-preinvex on D .

Proof. Since f_i is a E^h-b-preinvex on $\forall i \in \Lambda$, then, for each $d_1, d_2 \in D$ and $0 \leq \lambda \leq 1$ we have

$$f_i\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \leq h(\lambda b)f_i(Ed_1) + h(1-\lambda b)f_i(Ed_2) \quad \forall i \in \Lambda.$$

Taking the supremum to the right-hand side of the inequality above, we get

$$f_i\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \leq \sup_{i\in\Lambda}[h(\lambda b)f_i(Ed_1) + h(1-\lambda b)f_i(Ed_2)] \quad \forall i \in \Lambda.$$

Then,

$$\sup_{i\in\Lambda} f_i\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \leq \sup_{i\in\Lambda}[h(\lambda b)f_i(Ed_1) + h(1-\lambda b)f_i(Ed_2)].$$

From the fact that h is positive and sup$M$ and $s$up$N$ are finite, we get sup$(M+N)$ =sup$M$ +sup$N$. Thus, the last inequality yields, $f(Ed_2 + \lambda\psi(Ed_1, Ed_2)) \leq h(\lambda b)\sup_{i\in\Lambda} f_i(Ed_1) + h(1-\lambda b)\sup_{i\in\Lambda} f_i(Ed_2)$

$$= h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2).$$

Then, we get f is an E^h-b-preinvex. ■

The composite property is also held if f is an E^h-b-preinvex function as we show next.

Proposition 3.3 Let D,f,E,ψ,b and h are defined as in the Assumption such that D is an E-invex and f is an E^h-b-preinvex on D. Assume that $G: \mathbb{R} \to \mathbb{R}$ is a non-decreasing sublinear function. Then $G \circ f$ is an E^h-b-preinvex on D.

Proof. Let $d_1, d_2 \in D$, and $\lambda \in [0,1]$. Since, f is E^h-b-preinvex on the E-invex set D, then $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D$ and

$f(Ed_2 + \lambda\psi(Ed_1, Ed_2)) \leq h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2),$

$\mathrm{G}\left(f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right)\right) \leq \mathrm{G}(h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2)).$

The last inequality holds because G is a non-decreasing function. Using the sublinearity assumption of G, the right-hand side of the last inequality yields,

$$\mathrm{G}(f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right)) \leq h(\lambda b)\mathrm{G}(f(Ed_1)) + h(1-\lambda b)\mathrm{G}(f(Ed_2)),$$

i.e., $(\mathrm{G}o f)(Ed_2 + \lambda\psi(Ed_1, Ed_2)) \leq h(\lambda b)(\mathrm{G}o f)(Ed_1) + h(1-\lambda b)(\mathrm{G}o f)(Ed_2).$

Thus, Gof is a E^h-b-preinvex on D. ■

The next propositions provide necessary and sufficient conditions for f to be an E^h-b-preinvex function.

Proposition 3.4 Let D, f, E, ψ,b and h are defined as in the Assumption such that D is an E-invex set and $E(D)$ is an invex. Then, a function f is E^h-b-preinvex on D if and only if its restriction $\tilde{f}: E(D) \to \mathbb{R}$ is b^h-preinvex on $E(D)$.

Proof. Let $\widetilde{d_1}, \widetilde{d_2} \in E(D)$ then there exist d_1,$d_2 \in D$ such that $\widetilde{d_1} = Ed_1$ and $\widetilde{d_2} = Ed_2$. Since $E(D)$ is an invex set then $\widetilde{d_2} + \lambda\,\psi(\widetilde{d_1}, \widetilde{d_2}) \in E(D)$. First, we prove $\tilde{f}$ is b^h-preinvex on $E(D)$ where $\tilde{f}(\tilde{d}) = f(\tilde{d})$ $\forall \tilde{d} \in E(D)$. From the definition of $\tilde{f}$ and the assumption on f, we get

$$\tilde{f}(\widetilde{d_2} + \lambda\psi(\widetilde{d_1}, \widetilde{d_2})) = f(\widetilde{d_2} + \lambda\psi(\widetilde{d_1}, \widetilde{d_2}))$$
$$= f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right)$$
$$\leq h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2)$$
$$= h(\lambda b)f(\widetilde{d_1}) + h(1-\lambda b)f(\widetilde{d_2})$$

$= h(\lambda b)\tilde{f}(\widetilde{d_1}) + h(1-\lambda b)\tilde{f}(\widetilde{d_2}).$

Thus, $\tilde{f}$ is b^h-preinvex on $E(D)$. For obtaining the reverse implication, we use the definition of f and the assumption on $\tilde{f}$, $\forall\lambda \in [0,1]$ and $\lambda b \in [0,1]$ we have

$$f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) = \tilde{f}\left(\widetilde{d_2} + \lambda\psi(\widetilde{d_1}, \widetilde{d_2})\right) \le h(\lambda b)\tilde{f}(\widetilde{d_1}) + h(1-\lambda b)\tilde{f}(\widetilde{d_2})$$
$$= h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2),$$

which implies that f is E^h-b-preinvex on D. ■

Proposition 3.5 Let D, f, E, ψ, b and h are defined as in the Assumption such that D is an E-invex set. Assume that $h(\alpha) \le \alpha\ \forall\alpha \in [0,1]$, h is a non-negative function, and $h(1) = 1$.Then, f is E^h-b-preinvex on D for some b and h if and only if f is an E-prequasiinvex on D.

Proof. Let f be an E^h-b-preinvex on D w.r.t. some b and h. Note that for all $d_1, d_2 \in D$, $\lambda \in [0,1]$ and $\lambda b \in [0,1]$ we have

$$f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \le h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2)$$

From the assumptions on h, the right-hand side of the above inequality becomes

$$f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \le \lambda b f(Ed_1) + (1-\lambda b)f(Ed_2) \le \max\{f(Ed_1), f(Ed_2)\},$$

where the right-hand side of the last inequality obtained by considering $\lambda b = 1$ or 0. This yields that f is an E-prequasiinvex on D. Assume now that f is an E-prequasiinvex on D and define $b(d_1, d_2, \lambda)$ by

$$b(d_1, d_2, \lambda) = \begin{cases} \frac{1}{\lambda} & if\ \lambda \in (0,1]\ and\ \ f(Ed_1) \ge f(Ed_2) \\ 0 & if\ \ \lambda = 0\ \ \ or\ \ f(Ed_1) < f(Ed_2) \end{cases}$$

$$\lambda b(d_1, d_2, \lambda) = \begin{cases} 1 & if\ \lambda \in (0,1]\ and\ \ f(Ed_1) \ge f(Ed_2) \\ 0 & if\ \ \lambda = 0\ \ \ or\ \ f(Ed_1) < f(Ed_2) \end{cases} \quad (1)$$

From the assumption on f, $f\left(Ed_2 + \lambda\psi(Ed_1, Ed_2)\right) \le \max\{f(Ed_1), f(Ed_2)\}$. To show f is E^h-b-preinvex on D for some b, it is enough to prove that

$$\max\{f(Ed_1), f(Ed_2)\} = h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2) \quad (2)$$

If $\max\{f(Ed_1), f(Ed_2)\} = f(Ed_1)$, hence from (1), $\lambda b = 1$.Then $h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2)$

$$= h(1)f(Ed_1) + h(0)f(Ed_2) = f(Ed_1) = \max\{f(Ed_1), f(Ed_2)\} \quad (3)$$

where we used, in (3), the fact that $h(1) = 1$ and $h(0) \le 0$ (which forces $h(0) = 0$).On the other hand, if $\max\{f(Ed_1), f(Ed_2)\} = f(Ed_2)$, i.e., $\lambda b = 0$. Then

$$h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2) = h(0)f(Ed_1) + h(1)f(Ed_2) = f(Ed_2) = \max\{f(Ed_1), f(Ed_2)\} \quad (4)$$

From (3) and (4), we obtain (2) which implies that f is E^h-b-preinvex on D w.r.t. some b. ■

Some properties related to the γ-level sets are given next. First, two necessary conditions for f to be E^h-b-preinvex using the invexity of the level set $D_{\gamma,E}$ and the slack invexity of the level set D_γ, respectively, are stated as follows.

Proposition 3.6 Let D, f, E, ψ, b and h are defined as in the Assumption such that D is an E-invex set and $E(D)$ is an invex set. Assume that $h(\alpha) \le \alpha\ \forall\alpha \in [0,1]$, h is a non-negative function. If f is E^h-b-preinvex on D. Then $D_{\gamma,E}$ is an invex set for all $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and let $Ed_1, Ed_2 \in D_{\gamma,E}$ such that $f(Ed_1) \le \gamma$ and $f(Ed_2) \le \gamma$. Since $E(D)$ is an invex set then $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in E(D)$ (5)

From the assumption property on f and h, we have

$$f(Ed_2 + \lambda\eta(Ed_1, Ed_2)) \le h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2)$$
$$\le \lambda b\gamma + (1-\lambda b)\gamma = \gamma \quad (6)$$

From (5) and (6), we get $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D_{\gamma,E}$. Therefore, $D_{\gamma,E}$ is an invex set as required. ■

Proposition 3.7 Let D, f, E, ψ, b and h are defined as in the Assumption such that D is an E-invex set. Assume also that h is sublinear function and $h(1) = 1$. If f is E^h-b-preinvex on D. Then D_γ is a slack invex set w.r.t. $E(D)$ for all $\gamma \in \mathbb{R}$.

Proof. Let $u, \bar{u} \in D_\gamma \cap E(D)$ and $\lambda \in [0,1]$ such that $u, \bar{u} \in E(D)$, $f(u) \le \gamma$, $f(\bar{u}) \le \gamma$, and

$$\bar{u} + \lambda\psi(u, \bar{u}) \in E(D) \subseteq D \quad (7)$$

Since f is E^h-b-preinvex on D, then

$$f\left(\bar{u} + \lambda\psi(u, \bar{u})\right) \le h(\lambda b)f(u) + h(1-\lambda b)f(\bar{u})$$

Since h is sublinear and $h(1) = 1$, the right-hand side of the above inequality yields

$$\le h(\lambda b)f(u) + h(1)f(\bar{u}) - h(\lambda b)f(\bar{u}) \le \gamma. \quad (8)$$

From (7) and (8) we get $\bar{u} + \lambda\psi(u, \bar{u}) \in D_\gamma$ as required to prove.

Another necessary condition for f to be E^h-b-preinvex using the E-invexity of the level set E-D_γ is given next.

Proposition 3.8 Let D, f, E, ψ, b and h are defined as in the Assumption such that D is an E-invex set w.r.t. $E \circ \psi$ and f is an E^h-b-preinvex on D. Assume that E is linear and idempotent, h is sublinear, and $h(1) = 1$. Then, E-D_γ is an E-invex set w.r.t. $E \circ \psi$ for each $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $d_1, d_2 \in E$-D_γ. Then $f(Ed_1) \le \gamma$ and $f(Ed_2) \le \gamma$. Since D is an E-invex set w.r.t. $E \circ \psi$ then $Ed_2 + \lambda(E \circ \psi)(Ed_1, Ed_2) \in D$ (9)

and $f(E\left(Ed_2 + \lambda(E \circ \psi)(Ed_1, Ed_2)\right) = f\left(E^2 d_2 + \lambda(E^2 \circ \psi)(Ed_1, Ed_2)\right) = f\left(Ed_2 + \lambda(E \circ \psi)(Ed_1, Ed_2)\right)$

where we used in the last statements the assumptions on E. Applying now the assumptions on f and h, the last equality yields

$$\le h(\lambda b)f(Ed_1) + h(1-\lambda b)f(Ed_2) \le \gamma. \quad (10)$$

From (9) and (10), we have E-D_γ is an E-invex set w.r.t. $E \circ \psi$. ■

Next proposition introduces a characterization of E-prequasiinvex function.

Proposition 3.9 Let D, f, E, ψ, b and h are defined as in the Assumption such that D is an E-invex set and $E(D)$ is an invex set. Then f is an E-prequasiinvex if and only if $D_{\gamma,E}$ is an invex set.

Proof. First, we prove that $D_{\gamma,E}$ is an invex set. Let $\gamma \in \mathbb{R}$ and let $Ed_1, Ed_2 \in D_{\gamma,E}$ such that $f(Ed_1) \le \gamma$ and $f(Ed_2) \le \gamma$. Since $E(D)$ is an invex set and D is an E-invex set, then for each $\lambda \in [0,1]$ we have

$Ed_2 + \lambda\psi(Ed_1, Ed_2) \in E(D) \subseteq D$ (11)

From the hypothesis, f is an E-prequasiinvex on D. Then $f\big(Ed_2 + \lambda\psi(Ed_1, Ed_2)\big) \le \max\{f(Ed_1), f(Ed_2)\} \le \gamma$ (12)

Using (11) and (12), we obtain $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D_{\gamma,E}$. Let us now show that f is an E-prequasiinvex on D. Let $d_1, d_2 \in D$, $\lambda \in [0,1]$ and $\gamma \in \mathbb{R}$. By setting $\gamma = \max\{f(Ed_1), f(Ed_2)\}$, we conclude $Ed_1, Ed_2 \in D_{\gamma,E}$. Since $D_{\gamma,E}$ is an invex set . Then, $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D_{\gamma,E}$ and

$f\big(Ed_2 + \lambda\psi(Ed_1, Ed_2)\big) \le \gamma = \max\{f(Ex), f(Ey)\}$,

where, from Proposition 2.3, $Ed_2 + \lambda\psi(Ed_1, Ed_2) \in D$. This shows f is E-prequasiinvex. ■

4. Applications to local E^h-b-preinvex non-linear optimization problems

In this section, we consider the following non-linear optimization problem which will be denoted as (NLP)

$$\min f(d)$$
$$s.t. \quad g_i(d) \le b_i, \quad i = 1,..,r$$
$$d \in D,$$

where D, f and E are defined as in the Assumption, $b_i \in \mathbb{R}$ $\forall i = 1,\dots,r$, and g_i: $D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued functions $\forall i = 1,\dots,r$ such that D is a local E-invex set and f and g_i are local E^h-b-preinvex functions on D $\forall i = 1,\dots,r$. The problem (NLP) is referred to as local E^h-b-preinvex optimization problem.

Remark 4.1 In the Problem (NLP), if D is an E-invex set and f, g_i $\forall i = 1,\dots,r$ are E^h-b-preinvex functions on D then the Problem (NLP) is called E^h-b-preinvex optimization problem.

Definition 4.2 In the Problem (NLP)

1.

he set of feasible solutions is denoted by

$$F = \{d \in D: g_i(d) \le b_i, i = 1,\dots,r\}$$

2. The set of all optimal solutions (or global minimum) is denoted by $argmin_D f$ and is defined as $argmin_D f = \{d^* \in D: f(d^*) \le f(d) \quad \forall d \in D\}$.

3. A point $d^* \in \mathbb{R}^n$ is said to be local minimum if there exists $\varepsilon > 0$ such that $f(d^*) \le f(d)$ $\forall d \in B(d^*, \varepsilon) \cap D$, where $B(d^*, \varepsilon) = \{d \in \mathbb{R}^n : ||d - d^*|| < \varepsilon\}$ is the neighborhood of d^* with radius ε.

Under certain assumptions, the feasible set and the set of the optimal values of the Problem (NLP) are local slack E-invex w.r.t. $E(D)$ as we show in the next propositions.

Proposition 4.3 Let D, f, E, ψ, b and h are defined as in the Assumption such that $E(D)$ is a local invex and $F \cap E(D) \neq \phi$. assume that h is a sublinear and $h(1) = 1$. Then the set of feasible solutions F of the problem (NLP) is local slack E-invex w.r.t. $E(D)$.

Proof. Let $\widetilde{d_1}, \widetilde{d_2} \in F \cap E(D)$, i.e. $\widetilde{d_1}, \widetilde{d_2}$, then there exists $d_1, d_2 \in D$ such that $\widetilde{d_1} = Ed_1, \widetilde{d_2} = Ed_2$. From the local invexity of $E(D)$ there exists $\zeta_{d_1,d_2} \in (0,1]$ such that $\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in E(D)$ for each $\lambda \in \big[0, \zeta_{d_1,d_2}\big]$. From Proposition 2.3, $E(D) \subseteq D$, hence, $\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in D$. We need to show that $\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in F$. Fix $i \in \{1,2,\dots,r\}$. Since g_i is local E^h-b-preinvex, then there exists $\lambda^i_{d_1,d_2} \in [0, \zeta_{d_1,d_2}]$ such that $\forall\lambda \in [0, \lambda^i_{d_1,d_2}]$

$g_i(\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big)) \le h(\lambda b) g_i\big(\widetilde{d_1}\big) + h(1 - \lambda b) g_i\big(\widetilde{d_2}\big)$

Then, using the assumptions on h, the last inequality yields

$\le h(\lambda b) g_i(Ex) + h(1 - \lambda b) g_i(Ey) \le b_i$.

Take $\bar{\lambda} = \min_{1 \le i \le r} \{\lambda^i_{d_1,d_2}\}$. Then, from the definition of the feasible set we obtain

$\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in F$ $\forall\lambda \in \big[0, \bar{\lambda}\big]$ as required. ■

Proposition 4.4 Let D, f, E, ψ, b and h are defined as in the Assumption such that $E(D)$ is a local invex. If h is a sublinear and $h(1) = 1$. Then the set of $argmin_D f$ of the problem (NLP) is a local slack E-invex w.r.t. $E(D)$.

Proof. Let $\widetilde{d_1}, \widetilde{d_2} \in argmin_D f \cap E(D)$ then $f\big(\widetilde{d_1}\big) = f\big(\widetilde{d_2}\big) = f^*$. Using Proposition 4.3, there exists $\bar{\lambda} \in (0,1]$ such that $\forall\lambda \in [0, \bar{\lambda}]$ we have $\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in E(D)$. Using the local-$E^h$-$b$-preinvexity of f yields there exists $\lambda_\circ \in [0,1]$ such that $\forall\lambda \in [0, \lambda_\circ]$, $f\Big(\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big)\Big) \le h(\lambda b) f\big(\widetilde{d_1}\big) + h(1 - \lambda b) f\big(\widetilde{d_2}\big)$

$\le h(\lambda b) f^* + h(1) f^* - h(\lambda b) f^* = f^*$.

By taking $\tilde{\lambda} = \min\{\bar{\lambda}, \lambda_\circ\}$, we get $f(\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big)) = f^*$. Thus, $\widetilde{d_1} + \lambda\psi\big(\widetilde{d_1}, \widetilde{d_2}\big) \in argmin_D f$ for any $\lambda \in \big[0, \tilde{\lambda}\big]$. ■ ■

In the Problem (NLP), a sufficient condition for a local minimum to be global is given in the following result.

Proposition 4.5 Let D, f, E, ψ, b and h are defined as in the Assumption. Assume h is a sublinear and $h(1) = 1$. If $d^* \in intF$ be a local minimum point for f and $F \subset E(D)$. Then d^* is a global minimum of the problem (NLP).

Proof. Let $d^* \subset intF \subset F \subset E(D)$ is a local minmum point then there exists $\varepsilon > 0$ such that $B(d^*, \varepsilon) \subset E(D)$ and $f(d^*) \le f(d)$ $\forall d \in U = B(d^*, \varepsilon) \cap F$. (13)

To complete the proof, it is enough to show that $f(d^*) \le f(d)$ $\forall d \in F \backslash U$. On contrary, assume that there is $\bar{d} \in F$, $\bar{d} \neq d^*$ such that $f(\bar{d}) < f(d^*)$. (14)

From (13), $\bar{d} \notin B(d^*, \varepsilon)$ and $\|\bar{d} - d^*\| \ge \varepsilon$. Let $\widetilde{d_1}, \widetilde{d_2} \in D$ such that $\bar{d} = E\,\widetilde{d_1}$, $d^* = E\widetilde{d_2}$. Since D is a local E-invex, there exists $\zeta_{\widetilde{d_1},\widetilde{d_2}} \in (0,1]$ such that $\forall\lambda \in [0, \zeta_{\widetilde{d_1},\widetilde{d_2}}]$

$$E\widetilde{d_2} + \lambda\psi\big(E\widetilde{d_1}, E\widetilde{d_2}\big) \in D$$

From the assumption of Problem (NLP), f is a local E^h-b-preinvex on D, hence there exists $\lambda_{\widetilde{d_1},\widetilde{d_2}} \in (0, \zeta_{\widetilde{d_1},\widetilde{d_2}}]$such that $\forall\lambda \in [0, \lambda_{\widetilde{d_1},\widetilde{d_2}}]$,

$f(E\widetilde{d_2} + \lambda\psi\big(E\widetilde{d_1}, E\widetilde{d_2}\big)) \le h(\lambda b) f\big(E\widetilde{d_1}\big) + h(1 - \lambda b) f\big(E\widetilde{d_2}\big)$.

Applying (14) and the assumptions on h, the last inequality gives

$f(d^* + \lambda\psi\big(\bar{d}, d^*\big)) \le h(\lambda b) f\big(\bar{d}\big) + h(1) f(d^*) - h(\lambda b) f(d^*)$

$< h(\lambda b) f(d^*) + f(d^*) - h(\lambda b) f(d^*) = f(d^*)$. (15)

If $\psi\big(\bar{d}, d^*\big) = 0$. Then for any $\lambda \in [0, \lambda_{\widetilde{d_1},\widetilde{d_2}}]$, it yields $f\Big(d^* + \lambda\psi\big(\bar{d}, d^*\big)\Big) = f(d^*)$ which contradicts (15). If $\psi\big(\bar{d}, d^*\big) \neq 0$. Choose $\varepsilon > 0$ sufficiently small such that $\frac{\varepsilon}{\|\psi(\bar{d}, d^*)\|} \le 1$. Set $\bar{\lambda} = \min\{\lambda_{\widetilde{d_1},\widetilde{d_2}}, \frac{\varepsilon}{\|\psi(\bar{d}, d^*)\|}\}$. Then for any $\lambda \in (0, \bar{\lambda}]$ we get $\big\|d^* - [d^* + \lambda\psi\big(\bar{d}, d^*\big)]\big\| = \big\|\lambda\psi\big(\bar{d}, d^*\big)\big\| \le \bar{\lambda}\big\|\psi\big(\bar{d}, d^*\big)\big\| \le \varepsilon$,

i.e $d^* + \lambda\psi\big(\bar{d}, d^*\big) \in B\ (d^*, \varepsilon) \subset E(D)$. Using the last asseration and the fact that $F \subset E(D)$, then we are in condition of applying Proposition 4.3, i.e., $d^* +$

$\lambda\psi(\bar{d}, d^*) \in F$. Again (15) contradicts the fact that d^* is a local minimum on F. ■

Remark 4.6 Propositions 4.3-4.5 are held in case (NLP) is E^h-b-preinvex optimization problem as follows.

Proposition 4.7 Consider E^h-b-preinvex optimization problem (NLP). Then

1.

et D, f, E, ψ, b and h are defined as in the Assumption such that $E(D)$ is an invex and $F \cap E(D) \neq \phi$. assume that h is a sublinear and $h(1) = 1$. Then the set of feasible solutions F of problem (NLP) is slack E-invex w.r.t. $E(D)$.

2.

et D, f, E, ψ, b and h are defined as in the Assumption such that $E(D)$ is an invex set. If h is a sublinear and $h(1) = 1$. Then the set of $argmin_D f$ of the problem (NLP) is slack E-invex w.r.t. $E(D)$.

3.

et D, f, E, ψ, b and h are defined as in the Assumption. Assume h is a sublinear and $h(1) = 1$. If $d^* \in intF$ be a local minimum point for f and $F \subset E(D)$. Then d^* is a global minimum of the problem (NLP).

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