ON WEAKER FORMS OF WALC-Spaces

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ABSTRACT: *The purpose of this paper, we present a new definition of four weaker forms of* $WALC-spaces$ $(AL_s - spaces \quad s=1,2,3,4)$ *and Study and find their relationships with some other classes of topological spaces as well as among themselves.*

KEYWORDS: $LC-space$, $ALC-space$, $WALC-space$, Lindelöf, almost Lindelöf, locally Lindelöf spaces

1. INTRODUCTION

In 2002, Sarsak $[16]$ introduced the concept of $ALC-spaces$ as spaces M in which $\forall \mathrm{K} \subseteq \mathrm{M}$ which is almost Lindelöf in M is closed.

In 2008, Hdeib and Sarsak $[7]$ introduced the concept of weakly $ALC-spaces$ as spaces M in which $\forall$ almost Lindelöf $\mathrm{K} \subseteq \mathrm{M}$ is closed. Note that $ALC-spaces \Rightarrow$ Weakly $ALC-spaces \Rightarrow LC-spaces$.

In this research we give we present a new definition of four weaker forms of $WALC-spaces$ and we continue the investigation of more relationships.

2.Preliminaries

Definition2.1:

A space M is an LC if $\forall$ Lindelöf $\mathrm{K} \subseteq \mathrm{M}$ is close is closed $[4]$, $[11]$.Also $LC-space$ As is called $L-closed$ $[5]$, $[6]$, $[8]$ and $[9]$.

Definition 2.2[18]:

A space M is a KC if $\forall$ compact $\mathrm{K} \subseteq \mathrm{M}$ is closed,for example the usual topology (R, Γ_U) is a KC.

Definition 2.3[7] :

A space M is $cid-space$ if $\forall$ countable $\mathrm{K} \subseteq \mathrm{M}$ is closed and discrete.

Definition2.4 [10]:

A space M is P if $\forall G_\sigma - open$ set in M is open.

Definition 2.5[1]:

Al space M is called a $Q-set$ if $\forall C \subseteq \mathrm{M}$ is an $F_\sigma - closed$ sets.

Definition2.6 [2]: A space M is called a weak P if any countable union of regular closed sets is closed. M is a weak P $\Leftrightarrow$ $\forall \{O_k : k \in \Lambda\}$ of open sets, $cl\left(\bigcup_{k\in\Lambda} O_k\right) = \bigcup_{k\in\Lambda} clO_k$.

Definition2.7: A space M is almost Lindelöf if $\forall$ open cover Ω of M $\exists$ a countable subfamily $\psi \subset \Omega$ such that $\mathrm{M} = \bigcup_{O\in\psi} \overline{O}$. From the definition that Lindelöf space $\Rightarrow$ almost Lindelöf $[3]$, $[17]$. If N is an almost Lindelöf subspace of a space M, then N is almost Lindelöf in M but not conversely $[16]$.

Definition2.8[2]: A topological space M is locally Lindelöf (resp. weakly locally Lindelöf) if $\forall$ point of M has a closed Lindelöf (resp. Lindelöf) neighborhood. From the definition that every locally Lindelöf space $\Rightarrow$ a weakly locally Lindelöf and the converse is not true.

Definition2.9 [2] :

A space M is called

(1) an L_1 if $\forall$ Lindelöf $F_\sigma - closed$ is closed.

(2) an L_2 if $\mathrm{K} \subseteq \mathrm{M}$ is Lindelöf, then $cl\mathrm{K}$ is Lindelöf.

(3) an L_3 if $\forall$ Lindelöf $\mathrm{K} \subseteq \mathrm{M}$ is an $F_\sigma - closed$.

(4) an L_4 if $\mathrm{K} \subseteq \mathrm{M}$ is Lindelöf, then $\exists$ Lindelöf $F_\sigma - closed$ B with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl\mathrm{K}$.

Theorem2.10[2]:

(i) $LC-space \Rightarrow L_s$ s=1,2,3,4.

(ii) $Q-set$ space $\Rightarrow L_3$.

(iii) hereditarily Lindelöf $L_3 - space \Rightarrow Q-set$.

(iv) Every $T_1 L_1$ – space is cid.

Corollary2.11[14]:

(i) $L_1 Q-set$ space $\Rightarrow cid$. (ii) $L_1 L_3 -$ space $\Rightarrow cid$.

(iii) Hausdorff L_1 space $\Rightarrow cid$.(iv) L_1 $KC-$ space $\Rightarrow cid$.

Corollary2.12[14]: Every $Q-set$ $L_1 - space$ is an LC .

Theorem2.13[14]:

For a Lindelöf $L_3 -$ space M , M is an LC . $\Leftrightarrow$ M is a P .

Corollary2.14[13]: $P-space \Rightarrow$ weak P .

Theorem 2.15[12]: A regular almost Lindelöf space is Lindelöf.

Theorem2.16 [15]: For a Hausdorff Lindelöf space M , M is a P . $\Leftrightarrow$ M is an L_1 and an L_2 . $\Leftrightarrow$ M is an LC $\Leftrightarrow$ M is an $WALC$.

Corollary2.17 [13]: For a Lindelöf space M , M is locally Lindelöf $\Leftrightarrow$ M is a weakly locally Lindelöf

Theorem2.18 [2]: For locally Lindelöf space M , M is an L_1 $\Leftrightarrow$ M is a P .

Corollary2.19 [7]: a Hausdorff weak $P-space$. $\Rightarrow$ $WALC$.

Theorem2.20 [19]: If $B \subseteq N \subseteq M$, where $N \subset M$ is a clopen, then B is almost Lindelöf in $N \Leftrightarrow B$ is almost Lindelöf in M.

Theorem2.21[13]:
For a locally Lindelöf $Q-set$ space M, M is an LC. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an L_1.

Corollary2.22[13]:
For a weakly locally Lindelöf $L_2-space$ M, M is an $L_1-space$. $\Leftrightarrow$ M is a P.

Theorem2.23 [15]: For a regular space M, M is an LC. $\Leftrightarrow$ M is an $WALC$.

Theorem 2.24[14]: For a Lindelöf $space$ M, M is an L_1. $\Leftrightarrow$ M is a P.

Theorem2.25 [12]: Every clopen subset of almost Lindelöf M is almost Lindelöf.

3. ON WEAKER FORMS OF WALC-Spaces

Definition3.1: A space M is called

(1) an AL_1 if $\forall$ almost Lindelöf $F_\sigma-closed$ is closed.

(2) an AL_2 if $K \subseteq M$ is almost Lindelöf, then clK is almost Lindelöf.

(3) an AL_3 if $\forall$ almost Lindelöf $K \subseteq M$ is an $F_\sigma-closed$.

(4) an AL_4 if $K \subseteq M$ is almost Lindelöf, then $\exists$ almost Lindelöf $F_\sigma-closed$ B with

$K \subseteq B \subseteq clK$. Some results are directly from the Definition3.1.

Theorem3.2:

(i) $WALC-space \Rightarrow AL_s$ s=1,2,3,4.

(ii) AL_1 $AL_3-space$ $\Rightarrow WALC$.

(iii) AL_1 $AL_4-space$ $\Rightarrow AL_2$.

(iv) $AL_2-space$ $\Rightarrow AL_4$ and $AL_3-space$ $\Rightarrow$ AL_4.

Proof. (i)Where if $i=1$, let K be an almost Lindelöf $F_\sigma-closed$ in M, which is an $WALC-space$, then K is a closed so M is $AL_1-space$.

If $i=2$, let $K \subseteq M$ be an almost Lindelöf, which is an $WALC$, so K is a closed i.e. $K=clK$, then clK is an almost Lindelöf, hence M is an $AL_2-space$.

If $i=3$, let $K \subseteq M$ be an almost Lindelöf, which is an $WALC-space$, so K is a closed so K is an $F_\sigma-closed$.

If $i=4$, let K be an almost Lindelöf subset in M, which is an $WALC-space$, then K is a closed so K is an $F_\sigma-closed$ set, take $B=K$ then B is an almost Lindelöf $F_\sigma-closed$ subset and so $K \subseteq K = B \subseteq clK$, hence M is an $AL_4-space$.

(ii) If K is an almost Lindelöf subset in M, which is an $AL_3-space$,

then K is an $F_\sigma-closed$, and M is an AL_1, so K a closed, hence

M is an $WALC-spaces$.

(iii) If $K \subseteq M$ is an almost Lindelöf, which is an AL_4, then $\exists$ a set B which is an almost Lindelöf $F_\sigma-closed$ $\ni K \subseteq B \subseteq clK$, B is closed set, (M is an AL_1), then $B=clK$, so clK is an almost Lindelöf, hence M is an $AL_2-space$. (iv) Obvious.

Corollary3.3:

(i) If (M,Γ) is an AL_1 $AL_3-space$, then (M,Γ) is an LC.

(ii) If (M,Γ) is an AL_1 $AL_3-space$, then (M,Γ) is a KC.

(iii) If (M,Γ) is an AL_1 $AL_3-space$, then (M,Γ) is Cid.

(iv) If (M,Γ) is an AL_1 $AL_3-space$, then (M,Γ) is a locally LC.

Example3.4:

let (R,Γ_U) be usual topology,then (R,Γ_U) is an $AL_2-space$ but neither $AL_3-space$ nor $WALC-space$. R is an $AL_2-space$ since if $K \subseteq R$ is a almost Lindelöf, R is a Lindelöf space, then clK is a Lindelöf subspace, so clK is a almost Lindelöf, hence R is an $AL_2-space$. R is not an $AL_3-space$, since Q^C is a Lindelöf, so Q^C is a almost Lindelöf to prove Q^C is not $F_\sigma-closed$, suppose that Q^C is $F_\sigma-closed$ so Q is $G_\sigma-open$ i.e. $Q=\bigcap_{i=1}^{\infty} O_i$, where O_i is an open set in R, $Q \subseteq O_i$, for each i, but the only open set containing Q is R, i.e. $O_i = R$, for each i $Q=\bigcap_{i=1}^{\infty} O_i = \bigcap R = R$ which is a contradiction,so Q is not $G_\sigma-open$, therefore Q^C is not $F_\sigma-closed$ Now to prove R is not an

$WALC-space$, suppose it is an $WALC-space$, since Q^C is a almost Lindelöf in R, then Q^C is closed, so Q is open which is a contradiction, so R is not an $WALC-space$.

Theorem3.5:
For a hereditarily compact Hausdorff space M, M is an $WALC \Leftrightarrow \mathrm{M}$ is an AL_1 AL_2.

Proof. (a) $\Longrightarrow$(b) : This is clear by Theorem 3.2(i).

(b) $\Longrightarrow$(a) : Let $\mathrm{K}\subseteq\mathrm{M}$ be an almost Lindelöf,and $y\notin\mathrm{K}$.

Since M is Hausdorff, for each $z\in\mathrm{K}$ there exists a clopen set

O_z containing z with $y\notin clO_z$ .Clearly $\{O_z : z\in\mathrm{K}\}$ is cover of K ,and therefore $\exists$ a countable set $\mathrm{E}\subseteq\mathrm{K}$ such that $\mathrm{K}\subseteq\bigcup_{z\in\mathrm{E}} clO_z = \bigcup_{z\in\mathrm{E}} O_z$.

For each $z\in\mathrm{E}$, $\mathrm{K}\bigcap clO_z$ is almost Lindelöf and so $cl(\mathrm{K}\bigcap clO_z)$ is a lmost Lindelöf since M is an $AL_2-space$.

Furthermore, if $W=\bigcup_{z\in\mathrm{E}} cl(\mathrm{K}\bigcap clO_z)$,then W is almostLindelöf $F_\sigma-closed$, since M is an $AL_1-space$, W is a closed almost Lindelöf set not containing y .So $y\notin cl\mathrm{K}$. Hence K is closed in M .

Theorem3.6:
For a $T_3-space$ M, M is an LC . $\Leftrightarrow \mathrm{M}$ is an AL_1 AL_2.

Proof. (a) $\Longrightarrow$(b): This is obvious by Theorem2.23 and Theorem 3.2(i).

(b) $\Longrightarrow$ (a): Let K be a Lindelöf subset of M ,and $y\notin\mathrm{K}$ .Since M is Hausdorff, for each $z\in\mathrm{K}$ there exists an open set O_z containing z with $y\notin clO_z$. Clearly $\{O_z : z\in\mathrm{K}\}$ is cover of K ,and therefore $\exists$ a countable set $\mathrm{E}\subseteq\mathrm{K}$ such that $\mathrm{K}\subseteq\bigcup_{z\in\mathrm{E}} clO_z$. For each $z\in\mathrm{E}$, $\mathrm{K}\bigcap clO_z$ is Lindelöf and $\mathrm{K}\bigcap clO_z$ is almost Lindelöf and so $cl(\mathrm{K}\bigcap clO_z)$is almost Lindelöf since M is an $AL_2-space$.

Furthermore, if $W=\bigcup_{z\in\mathrm{E}} cl(\mathrm{K}\bigcap clO_z)$,then W is almost Lindelöf $F_\sigma-closed$,

thus W is a closed almost Lindelöf not containing y (M is an AL_1).

So $y\notin cl\mathrm{K}$. Hence K is closed in M .

Corollary3.7: For a $T_3-space$ M, M is an $WALC-space$. $\Leftrightarrow \mathrm{M}$ is an AL_1 AL_2.

Proof. This is clear by Theorem 3.2 (i) and Theorem 3.6.

Corollary3.8:
For a hereditarily compact Hausdorff Lindelöf space M, M is a P . $\Leftrightarrow \mathrm{M}$ is an L_1 L_2. $\Leftrightarrow \mathrm{M}$ is an LC . $\Leftrightarrow \mathrm{M}$ is an $WALC$. $\Leftrightarrow$ M is an AL_1 AL_2.

Proof. This is clear by Theorem 2.16and Theorem 3.5.

4. $AL_1-Spaces$

Theorem4.1: Every $AL_1-space$ is an $L_1-space$.

Proof. Let $\mathrm{K}\subseteq\mathrm{M}$ be a Lindelöf $F_\sigma-closed$, therefore K is an almost Lindelöf $F_\sigma-closed$, which is an $AL_1-space$, then K is a closed,hence M is $L_1-space$.

Theorem4.2: For a regular space M, M is an $AL_1-space.$ $\Leftrightarrow \mathrm{M}$ is an $L_1-space$.

Proof. (a) $\Longrightarrow$(b): This is clear by Theorem 4.1.

(b) $\Longrightarrow$(a): Let $\mathrm{K}\subseteq\mathrm{M}$ be an almost Lindelöf $F_\sigma-closed$, therefore K is a Lindelöf $F_\sigma-closed$ subset of M (M is a regular),

so K is a closed (M is an L_1), so M is an AL_1 .

Corollary4.3:
(i) Every AL_1Q-set space is cid .(ii) Every AL_1 Hausdorff space is cid .
(iii) Every AL_1 KC $-space$ is cid .(iv) Every AL_1AL_3 space is cid .
(v) Every T_1 AL_1 space is cid .

Proof. This is obvious Theorem 4.1, Corollary 2.11 and Theorem 2.10(iv).

Corollary4.4: $P-space \Rightarrow AL_1$.

Proof. This is clear by Definition.3.1.

Corollary4.5: $Q-set$ $AL_1-space \Rightarrow LC$.

Proof. This is clear Theorem 4.1 and Corollary 2.12.

Theorem4.6: For a $Q-set$ space M, M is an $WALC$ $\Leftrightarrow \mathrm{M}$ is an AL_1.

Proof. (a) $\Longrightarrow$(b): This is clear by Theorem 3.2(i).

(b) $\Rightarrow$(a): Let $K \subseteq M$ be an almost Lindelöf,therefore K is an almost Lindelöf $F_\sigma - closed$ set (M is a $Q-set$), thus K is a closed (M is an AL_1),so M is an $WALC$.

Theorem4.7:
For a locally Lindelöf space M, M is an AL_1. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an L_1.

Proof. (a) $\Rightarrow$(b): Let $B \subseteq M$ be an $F_\sigma - closed$. If $y \notin B$ choose a closed Lindelöf neighborhood O of y. Then $O \cap B$ is an almost Lindelöf $F_\sigma - closed$ in M and thus closed, (M is an $AL_1 - space$). Hence

$O-(O \cap B)$ is a neighborhood of y disjoint from B .therefore

B is closed and hence is a $P-space$.The remainder of theory proves by

Corollary 4.4 and Theorem 2.18

Theorem4.8: Every Huasdorff $P-space$ is an $WALC-space$.

Proof. Let K be an almost Lindelöf subset in M, and let $y \notin K$. Then for each $z \in K$ $\exists$ two disjoint open sets U_z, V_z $\ni y \in U_z$ and $z \in V_z$ with $y \notin clV_z$ (as M is Hausdorff).Clearly $\{V_z : z \in K\}$ is cover of K and since K is almost Lindelöf in M, there exist $z_1, z_2, \ldots \in K$ such that $K \subset \bigcup_{i\in\omega} clV_{z_i}$.

Let $U = \bigcap_{i\in\omega} U_{z_i}$ so U is an open neighborhood of y which is disjoint from K (as M is a P). Hence K is closed in M.

Corollary 4.9:
For a Hausdorff locally Lindelöf space M the following are equivalent:
.(a) M is an $LC-space$.(b) M is an $L_1 - space$.
(c) M is a $P-space$.
(d) M is a weak $P-space$.(e) M is an $AL_1 - space$.(f) M is an $WALC-space$.

Proof. This is clear Theorem 2.10(i), Theorem 2.18, Corollary 2.14 Theorem 4.7, Theorem 4.8 Theorem 3.2(i) and Corollary 2.19.

Theorem4.10: The property AL_1 is hereditary on clopen sets.

Proof. If N is clopen $F_\sigma - closed$ subspace of M,suppose that K is an almost Lindelöf and $F_\sigma - closed$ subset of N,then K is an almost Lindelöf subset of M by Theorem 2.20and $K = \bigcup_{i\in\omega} B_i^*$ such that $\{B_i^*\}$ a family of closed subsets in N.Let $B_i^* = N \cap B_i$ where B_i is closed subsets in M for each i,thus

$$K = \bigcup_{i\in\Omega}(N \cap B_i) = N \cap \left(\bigcup_{i\in\Omega} B_i\right) = \left(\bigcup_{j\in\Omega} E_j\right) \cap \left(\bigcup_{i\in\Omega} B_i\right) = \bigcup_{\substack{i\in\Omega \\ j\in\Omega}} (E_j \cap B_i)$$

where E_j is closed subsets in M, so K is almost Lindelöf and $F_\sigma - closed$ in M,which is an $AL_1 - space$,then K is closed in M,so K is closed in N,therefore N is an $AL_1 - space$.

Theorem4.11: P $Q-s$ space $\Rightarrow$ $WALC-space$.

Proof. Let M a space and $K \subseteq M$ be an almost Lindelöf, then K is an $F_\sigma - closed$ set(M is a $Q-set$ space), so K is a closed set(M is a P), hence M is an $WALC$.

Corollary4.12:
locally Lindelöf $WALC-space \Rightarrow$ a $P-space$.

Corollary 4.13:
For a locally Lindelöf $Q-set$ space M,
M is an LC. $\Leftrightarrow$ M is an L_1. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an $WALC$. $\Leftrightarrow$ M is an AL_1.

Proof. This is clear Theorem 2.10(i), Theorem 2.21, Theorem 2.18,Theorem 4.11 and Theorem 4.7.

Corollary4.14:
For a weakly locally Lindelöf $L_2 - space$ M,
M is an L_1. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an AL_1.

Proof. This is clear Corollary 2.22 and Corollary 4.4.

Theorem4.15: Lindelof $AL_1 - space \Rightarrow$ a $P-space$.

Proof. For each $n \in \Omega$, let D_n be closed in Lindelöf $AL_1 - space$ M and $D = \bigcup_{n\in\Omega} D_n$,then $D_n \subseteq M$ is a Lindelöf and thus $D \subseteq M$ is a Lindelöf and is an almost Lindelöf by ("Countable union of Lindelöf subset is Lindelöf".) and by Definition 2.7. Therefore D is closed in M (M is an AL_1),so M is a P.

Corollary4.16: For a Lindelöf space M, M is an L_1. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an AL_1.

Proof. This is clear by Theorem 2.24, Corollary 4.4and Theorem 4.12.

Theorem4.17:
Every continuous function h from Lindelof space M into AL_1 $Q-set$ $space$ N is a closed function.

Proof. Let $B \subseteq M$ be a closed then B is a Lindelöf in M (M which is a Lindelöf), so $h(B)$ is a Lindelöf in N ("continuous image of a Lindelöf is Lindelöf"),which is a $Q-set$,then $h(B)$ is an almost Lindelöf $F_\sigma - closed$ subset in space N ,hence $h(B)$ is a closed subset in a space N (since N is an $AL_1 - space$), therefore h a closed function.

Theorem4.18:

If $h: M \longrightarrow N$ is a continuous injective function from a space M into AL_1 $Q-set$ space N then M is an $WALC$.

Proof. Let $K \subseteq M$ be any almost Lindelöf,then $h(K)$ is an almost Lindelöf in N ("A continuous image of an almost Lindelöf is almost Lindelöf"), which is a $Q-set$ $space$, then $h(K) \subseteq N$ is an almost Lindelöf $F_\sigma - closed$ since N is an $AL_1 - space$, then $h(K) \subseteq N$ is a closed,therefore $h^{-1}(h(K)) = K$ is a closed subset of M (because h is a continuous injective function),thus M is an $WALC - space$.

Theorem4.19:

If function $h: M \longrightarrow N$ is a continuous closed injective from a space M into an $AL_1 - space$ N ,then M is an $AL_1 - space$.

Proof. $\forall n \in \Omega$,let K_n be closed in M and $K = \bigcup_{n \in \omega} K_n \subseteq M$ be almost Lindelöf,,then $h(K)$ is a almost Lindelöf subset of N ("A continuous image of an almost Lindelöf is almost Lindelöf") and $h(K) = \bigcup_{n \in \omega} h(K_n)$ such that $h(K_n)$ is closed subsets of N (since h is a closed function),since N is an $AL_1 - space$, then $h(K) \subseteq N$ is a closed, therefore $h^{-1}(h(K)) = K \subseteq M$ is a closed (because h is a continuous injective function),thus M is an $AL_1 - space$.

Corollary4.20: If function $h: M \longrightarrow N$ is a continuous closed injective from a space M into an $AL_1 - space$ N ,then M is an $L_1 - space$.

5. $AL_2 - Spaces$

Theorem5.1: Every Lindelöf space is an AL_2.

Proof. Let $K \subseteq M$ be an almost Lindelöf, since clK is a closed in M , then clK is a Lindelöf (closed in Lindelöf is Lindelöf), so clK is an almost Lindelöf, hence M is an $AL_2 - space$.

Corollary5.2: Every 2^{nd} countable (C_{11}) space is an $AL_2 - space$.

Proof. Let M be a 2^{nd} countable space, then M is a Lindelof, hence M is n $AL_2 - space$ by Theorem 5.1.

Theorem5.3:

Let (M,Γ) be $AL_2 - space$ and $K \subseteq M$ be a Lindelöf dense, then (M,Γ) is almost Lindelöf.

Proof. Since $K \subseteq M$ is a Lindelöf, so K is an almost Lindelöf and $clK = M$,but M is an $AL_2 - space$, then clK is an almost Lindelöf.

Corollary5.4:

Let (M,Γ) be regular $AL_2 - space$ and $K \subseteq M$ be a Lindelöf dense, then $(M,\Gamma)$is Lindelöf.

Corollary5.5: regular almost Lindelöf $\Rightarrow AL_2 - space$.

Proof. This is clear by Theorem 2.15 and Theorem 5.1.

Theorem5.6: Let M be an almost Lindelöf space and every closure set is open, then M is an AL_2.

Proof. Let $K \subseteq M$ be an almost Lindelöf, so clK is clopen in M , then clK is an almost Lindelöf("If M is almost Lindelöf, then any clopen subset of M is almost Lindelöf"), hence M is an $AL_2 - space$.

Theorem5.7: Let M be an almost Lindelöf space and every closure set is regular closed, then M is an $AL_2 - space$.

Proof. Let $K \subseteq M$ be an almost Lindelöf M , so clK is regular closed in M , then clK is an almost Lindelöf("A regular closed subset of an almost Lindelöf space M is almost Lindelöf"), hence M is an $AL_2 - space$.

Theorem5.8: Every regular $AL_2 - space$ is an L_2.

Proof. Let $K \subseteq M$ be a Lindelöf, then K is an almost Lindelöf subset of M, so clK is an almost Lindelöf (since M is an $AL_2 - space$),then K is a Lindelöf (since M is a regular), hence M is an $L_2 - space$.

Theorem5.9: For a regular M , M is an AL_2. $\Leftrightarrow$ M is an L_2.

Proof. (a) $\Longrightarrow$(b): This is clear by Theorem 5.8. (b) $\Longrightarrow$ (a): Let $K \subseteq M$ be an almost Lindelöf, which is a regular space, then K is an Lindelöf subset of M, so clK is a Lindelöf (since M is an $L_2 - space$),then clK is an almost Lindelöf,

Hence M is an $AL_2 - space$.

Theorem5.10: Every $Q-set$ $AL_1 - space$ is an AL_2.

Proof. Let $K \subseteq M$ be an almost Lindelöf, then K is an $F_\sigma - closed$ (M is a $Q - set$), thus K is closed (M is an AL_1), so $K = clK$ and clK is an almost Lindelöf. Therefore M is an AL_2.

Theorem5.11: Every P $AL_3 - space$ is an $AL_2 - space$.

Proof. Let $K \subseteq M$ be an almost Lindelöf, then K is an $F_\sigma - closed$ (M is an AL_3), thus K is closed (M is a P), so $K = clK$ and clK is an almost Lindelöf. Therefore M is an AL_2.

Theorem5.12: Let M be a space and $N \subseteq M$, $N = \bigcup_{s=1}^{n} N_s$, where N_s, $s = 1,2,...,n$ are clopen $AL_2 - subspace$s in M, then N is an $AL_2 - subspace$.

Proof. Let K be an almost Lindelöf subset of N, then $K \cap N_s$, $s = 1,2,...,n$ are clopen in K, which is almost Lindelöf, so $K \cap N_s$, $s = 1,2,...,n$ are almost Lindelöf subset of N_s, $s = 1,2,...,n$. Since $K \cap N_s$ is subset of N_s, $s = 1,2,...,n$ which is $AL_2 - space$, then $cl(K \cap N_s)$ is a almost Lindelöf in N_s, $s = 1,2,...,n$, so $cl(K \cap N_s)$ is a almost Lindelöf in N, $s = 1,2,...,n$. But $clK = cl\left(\bigcup_{s=1}^{n}(K \cap N_i)\right) = \bigcup_{s=1}^{n} cl(K \cap N_i)$, so clK is almost Lindelöf in N, hence N is an $AL_2 - subspace$.

Theorem5.13: The property AL_2 is hereditary on clopen sets.

Proof. If N is clopen subspace of M, suppose that $K \subseteq N$ is an almost Lindelöf, then $K \subset M$ is an almost Lindelöf by Theorem 6.5.9, which is AL_2, so $cl_M K$ is an almost Lindelöf and $cl_N K = cl_M K \cap N$. Then $cl_M K \cap N$ is an almost Lindelöf subset of N by Theorem 2.20, so $cl_N K$ is an almost Lindelöf subset of N. Hence N is an $AL_2 - space$.

Theorem5.14: For a regular $AL_2 - space$ M,
M is locally Lindelöf. $\Leftrightarrow$ M is a weakly locally Lindelöf.

Proof. (a) $\Rightarrow$(b): This is clear by Definition 2.8. b) $\Rightarrow$(a): If $y \in M$ then has Lindelöf neighborhood K, so K is an almost Lindelöf, but M is an $AL_2 - space$ then clK is almost Lindelöf, so clK is Lindelöf(since M is a regular space), but clK is closed, so y has a closed Lindelöf neighborhood, therefore M is locally Lindelöf.

Corollary5.15: For a regular space M, and M having a dense Lindelöf subset,
M is a Lindelöf. $\Leftrightarrow$ M is an almost Lindelöf. $\Leftrightarrow$ M is an AL_2. $\Leftrightarrow$ M is an L_2.

Proof. This is clear by Definition.2.8., Theorem 2.15,Theorem 5.1,Theorem5.4 and Theorem 5.9.

Theorem5.16:
For a Lindelöf $Q - set$ space M, M is an AL_1. $\Leftrightarrow$ M is a P AL_2.

Proof. This is clear by Theorem 5.9,Theorem4.12, Theorem4.11and Theorem 3.2.(i)

Theorem5.17:
For a regular weak $P - space$ M,
M is locally Lindelöf $\Leftrightarrow$ M is a weakly locally Lindelöf AL_2.

Proof.(a) $\Rightarrow$(b): Let $K \subseteq M$ be an almost Lindelöf, so K is Lindelöf (M is a regular space). $\forall$ point of K has an open neighborhood O_z $\ni clO_z$ is Lindelöf. Select a countable subset E of K $\ni K \subseteq \bigcup_{z \in E} O_z$.

Since M is a weak P and $clK \subseteq \bigcup_{z \in E} clO_z = W$.
Since W is Lindelöf we infer that clK is Lindelöf and, so clK is almost Lindelöf, hence M is an $AL_2 - space$.

Theorem5.18: If function $h : M \longrightarrow N$ is a continuous open injective from a space M into an $AL_2 - space$ N then M is an $AL_2 - space$.

Proof. Let $K \subseteq M$ be almost Lindelöf, then $h(K)$ is an almost Lindelöf in N ("A continuous image of an almost Lindelöf is almost Lindelöf"), since N is an $AL_2 - space$, then $cl_N(h(K))$ is an almost Lindelöf subset of N, hence $h^{-1}(cl_N(h(K)))$ is an almost Lindelöf subset of a space M (since h is an open function), but
$h^{-1}(cl_N(h(K))) = cl_M(h^{-1}h(K)) = cl_M(K)$, so
$cl_M(K)$ is an almost Lindelöf subset of M, thus M is an $AL_2 - space$.

6. $AL_3 - Spaces$

Theorem6.1: Every $AL_3 - space$ is an $L_3 - space$.

Proof. Let $K \subseteq M$ be a Lindelöf,then K is an almost Lindelöf, which is an AL_3, then K is an $F_\sigma - closed$, hence M is an $L_3 - space$.

Theorem6.2: For a regular space M, M is an AL_3. $\Leftrightarrow$ M is an L_3.

Proof.

(b) $\Rightarrow$ (a): Let $K \subseteq M$ be an almost Lindelöf, then $K \subseteq M$ is a Lindelöf (M is a regular), so K is an $F_\sigma - closed$ (M is an L_3),therefore M is an AL_3.

Corollary6.3: Every $Q - set$ space is an $AL_3 - space$.

Corollary6.4: Every hereditarily Lindelöf $AL_3 - space$ is a $Q - set$ space.

Corollary6.5:

For a hereditarily Lindelöf space M, M is an AL_3. $\Leftrightarrow$ M is a $Q - set$.

Corollary6.6: For a 2^{nd} countable space M, M is an AL_3. $\Leftrightarrow$ M is a $Q - set$.

Proof. (a) $\Rightarrow$(b): Let M be a $AL_3 - space$.Since M is a 2^{nd} countable space, then M is a hereditarily Lindelöf, hence M is a $Q - set$ space by Corollary 6.4. (b) $\Rightarrow$(a): This is clesr by Corollary 6.3.

Corollary6.7: For a countable space M, M is an AL_3. $\Leftrightarrow$ M is a $Q - set$.

Proof. (a) $\Rightarrow$(b): Let M be an $AL_3 - space$. Since M is a countable space, then M is a hereditarily Lindelöf, hence M is a $Q - set$ space by Corollary 6.4.

Theorem6.8: For a $P - space$ M, M is an $WALC$. $\Leftrightarrow$ M is an AL_3.

Proof. (a) $\Rightarrow$(b): This is obvious by Theorem 3.2(i).

(b) $\Rightarrow$(a): Let $K \subseteq M$ be an almost Lindelöf,then K is an $F_\sigma - closed$ (M is an AL_3),thus K is closed set (M is a P), therefore M is an $WALC$.

Theorem6.9: The property AL_3 is hereditary on clopen sets.

Proof. If N is a subspace of M,suppose that $K \subseteq N$ is an almost Lindelöf,then $K \subseteq M$ is an almost Lindelöf by Theorem 2.20, which is an AL_3 then K is an $F_\sigma - closed$ i.e. $K = \bigcup_{i \in \Omega} B_i \ni \{B_i\}$ a family of closed subsets in M. Put $B_i^* = N \bigcap B_i$,then B_i^* is closed subsets in N for each i and

$$K = K \cap N = \left(\bigcup_{i \in \Omega} B_i\right) \cap N = \bigcup_{i \in \Omega} (B_i \cap N) = \bigcup_{i \in \Omega} B_i^*$$

, so K is an $F_\sigma - closed$ in N, therefore N is an $AL_3 - space$.

Corollary6.10:

For a hereditarily Lindelöf space M, M is an L_3. $\Leftrightarrow$ M is a $Q - set$. $\Leftrightarrow$ M is an AL_3.

Proof. This is clear by Corollary 6.5., Theorem 2.10(ii) and Theorem 2.10(iii)

Corollary6.11:

For a Lindelof $AL_3 - space$ M, M is an LC. $\Leftrightarrow$ M is a P. $\Leftrightarrow$ M is an $WALC$..

Proof. This is clear by Theorem 2.13., Theorem 6.1 and Theorem 6.8.

Theorem6.12:

Every continuous function h from Lindelöf space M into AL_3 $P - space$ N is a closed function.

Proof. Let $B \subseteq M$ be a closed then B is a Lindelöf in M (M is a Lindelöf),so $h(B)$ is a Lindelöf in N ("continuous image of a Lindelöf is Lindelöf"), then $h(B) \subseteq N$ is an almost Lindelöf, so $h(B)$ is an $F_\sigma -$ closed subset in a space N (N is an $AL_3 - space$),hence $h(B)$ is a closed subset in a space N (since N is a $P - space$), therefore h a closed function.

Theorem6.13: If function $h : M \longrightarrow N$ is a continuous injective from a space M into an $AL_3 - space$ N then M is an $AL_3 - space$.

Proof. Let $K \subseteq M$ be almost Lindelöf, then $h(K)$ is an almost Lindelöf in N ("continuous image of an almost Lindelöf is almost Lindelöf"),since N is an $AL_3 - space$, then $h(K)$ is an $F_\sigma -$ closed subset of N, $\Omega n \in \omega$,let K_n be closed in N and $h(K) = \bigcup_{n \in \Omega} K_n$, so $K = \bigcup_{n \in \Omega} h^{-1}(K_n)$ such that $h^{-1}(K_n)$ is closed subsets of M, therefore K is an $F_\sigma - closed$ subset of M ,thus M is an $AL_3 - space$.

Corollary6.14: If function $h: \mathrm{M} \longrightarrow \mathrm{N}$ is a continuous injective from a space M into an $AL_3 - space$ N then M is an $L_3 - space$.

7. $AL_4 - Spaces$

Theorem7.1: Every regular $AL_4 - space$ is an $L_4 - space$.

Proof. Let $\mathrm{K} \subseteq \mathrm{M}$ be a Lindelöf then $\mathrm{K} \subseteq \mathrm{M}$ is an almost Lindelöf, since M is an AL_4, then there is an almost Lindelöf $F_\sigma - closed$ B with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl\mathrm{K}$, since M is a regular, then B is a Lindelöf $F_\sigma - closed$ with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl\mathrm{K}$, hence M is an $L_4 - space$.

Theorem7.2: For a regular M, M is an $AL_4 - space$. $\Leftrightarrow$ M is an $L_4 - space$.

Proof.. (b) $\Longrightarrow$(a): Let $\mathrm{K} \subseteq \mathrm{M}$ be an almost Lindelöf, then $\mathrm{K} \subseteq \mathrm{M}$ is an Lindelöf (M is a regular), since M is an L_4, then there is a Lindelöf $F_\sigma - closed$ B with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl\mathrm{K}$, so B is an almost Lindelöf $F_\sigma - closed$ with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl\mathrm{K}$, hence M is an $AL_4 - space$.

Theorem7.3: The property AL_4 is hereditary on clopen sets.

Proof. If N is clopen subspace of M, suppose that $\mathrm{K} \subseteq \mathrm{N}$ is an almost Lindelöf,then $\mathrm{K} \subseteq \mathrm{M}$ is an almost Lindelöf by theorem 2.20,which is AL_4, then there is almost Lindelöf $F_\sigma - closed$ B with $\mathrm{K} \subseteq \mathrm{B} \subseteq cl_{\mathrm{M}}\mathrm{K}$,let $\mathrm{B} = \bigcup_{j \in \Omega} \mathrm{B}_j$ such that $\{\mathrm{B}_j\}$ a family of closed subsets in M .Put $\mathrm{B}_j^* = Y \bigcap \mathrm{B}_j$,then B_j^* is closed subsets in N for each j, B_j^* is clopen subsets in B for each i and

$$\mathrm{B}^* = \mathrm{B} \bigcap \mathrm{N} = \left(\bigcup_{j \in \Omega} \mathrm{B}_j \right) \bigcap \mathrm{N} = \bigcup_{j \in \Omega} \left(\mathrm{B}_j \bigcap \mathrm{N} \right) = \bigcup_{j \in \Omega} \mathrm{B}_j^* .$$

Then B^* is an almost Lindelöf subset of N by Theorem 2.25and is an $F_\sigma - closed$ in N with $\mathrm{K} \subseteq \mathrm{B}^* \subseteq cl_{\mathrm{N}}\mathrm{K}$, Hence N is an $AL_4 - space$.

Corollary7.4: For a hereditarily compact Hausdorff $AL_1 - space$ M,
M is an AL_4. $\Leftrightarrow$ M is an AL_3. $\Leftrightarrow$ M is an AL_2.

Theorem7.5: If function $h: \mathrm{M} \longrightarrow \mathrm{N}$ is a continuous open injective from a space M into an $AL_4 - space$ N then M is an $AL_4 - space$.

Proof. Let $\mathrm{K} \subseteq \mathrm{M}$ be any almost Lindelöf, then $h(\mathrm{K}) \subseteq \mathrm{N}$ is an almost Lindelöf ("continuous image of an almost Lindelöf is almost Lindelöf"),since N is an $AL_4 - space$, then $\exists$ an almost Lindelöf $F_\sigma - closed$ B (let $\mathrm{B} = \bigcup_{n \in \Omega} \mathrm{B}_n$ such that $\{\mathrm{B}_n\}$ a family of closed subsets in N)with

$$h(\mathrm{K}) \subseteq \mathrm{B} = \bigcup_{n \in \Omega} \mathrm{B}_n \subseteq cl_{\mathrm{N}}(h(\mathrm{K})),$$ so

$$\mathrm{K} \subseteq h^{-1}(\mathrm{B}) = \bigcup_{n \in \Omega} h^{-1}(\mathrm{B}_n) \subseteq cl_{\mathrm{M}}(\mathrm{K}),$$

therefore $h^{-1}(\mathrm{B})$ is an almost Lindelöf $F_\sigma - closed$ subset of M ,thus M is an $AL_4 - space$.

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