CERTAIN TYPES OF TOPOLOGICAL VECTOR SPACes

Jalal Hatem Hussein Bayati.
The University of Baghdad, College of Science for Woman, Department of Mathematics, Iraq.
jalalintuch@yahoo.com

ABSTRACT. *In this study, the concepts of gsp-topological vector space, δ-topological vector space and δ-homogeneous space was introduced and studied, and some properties of them are given.*

Keywords: topological vector space, gsp-closed set, gsp-continuous mapping.

1. INTRODUCTION

Functional analysis in its traditional sense deals primarily with Banach spaces and, in particular, Hilbert spaces. However, many classical vector spaces have natural topologies that are not given by the norm. Such, for example, are many spaces of smooth, holomorphic and generalized functions. The theory of topological vector spaces is the science of spaces of precisely this kind. Recently, the concept of generalization of topological vector spaces was studied by many authors. In 2001, AL-Nayef and AL-Hawary defined the spaces of Irresolute Topological Vector Spaces[2] also N.Rajesh and Thanjarur[3] defined the notion of strongly pre irresolute topological vector spaces. in 2015 Moiz and Azabm[4] defined and investigated s-topological vector spaces which are another generalization of topological vector spaces. Radhi I. M. Ali, J. H. Hussein. and S. K. Hameed defined the concept of On Semi -Pre irresolute Topological Vector Space[8]. Recall that a topology τ with a vector space χ is said to be irresolute topological vector space (resp, strongly pre irresolute topological vector spaces [3], s-topological vector spaces[4], Semi -Pre irresolute Topological Vector Space[8]). whenever The vector addition map $S:X\times X\to X$ and The scalar multiplication map $M:F\times X\to X$ Are both irresolute (resp. semi-continuous, p-continuous, Semi -Pre irresolute) mapping. For more details on irresolute (resp. semi-continuous, p-continuous, Semi -Pre irresolute) mapping, we refer to [9,10,4,11]. By a space X we mean a topological space. Recall that The vector space X over the field of complex numbers C, on which the topology τ is given, is called topological vector space [1], if the addition map $m: X \times X \to X$ define by $m((x, y)) = x + y$, and the scalar multiplication $M: F \times X \to X$ define by $M((\lambda, x)) = \lambda.x$ is continuous $for\ each\ \lambda\ in\ F\ and\ y, x\ in\ X$, such that the domain and codomain of these mappings are (X, τ) . An equivalent definition of topological vector space state as follows: (X, τ) is a topological vector space if the following conditions are satisfied:

1- If for any elements x, y belong to a space X and any open neighborhood $W\ of\ x + y$, there exist open neighborhoods U of x and V of y in X , such that U+V⊂W.
2- $For\ each\ \lambda\ in\ F, x\ in\ X$ and each open neighborhood $W\ in\ X$ contains λ.x there exist open neighborhood U of λ in F and open neighborhood V of x in X such that $U.V\subset W$.

For any x, y in a vector space X , denote ${}_xT:X\to X$, defined by $y\to x + y$, and denote $T_x:X\to X$, defined by $y \to x + y$. Then, ${}_xT$ and T_x denote the left and right translation by x, respectively [4]. In order to define the concept of gsp-open set, we need to define semi-preopen sets. Recall that a subset A of a space X is called semi-preopen [5] if $A \subseteq cl(int(cl(A)))$, the intersection of all semi-preopen sets containing A is called $spcl(A)$. Now we are ready to define gsp-open set. A subset A of a space X is called generalized semi-preclosed (briefly gsp-closed) set [6] if $spcl(A) \subseteq U$ whenever $\subseteq U$ and U is open in (X, τ). The complement of generalized semi-preclosed set is called generalized semi-preopen denoted by gsp-open. Also in [6] the concept of gsp-continuous function was defined. Recall that mapping is gsp-continuous if the inverse image of every open set is gsp-open. In the same way, the concept of gsp-homeomorphism was defined, a bijective mapping f is gsp-homeomorphism if f and f^{-1} are gsp-continuous. A subset A of a point x in a space X is called gsp-neighborhood [6] if there exists gsp-open set G such that $x\in G\subset A$. A point in a space X is called gsp -interior point of a subset A if there exist a gsp-open set B in X such that $x\in B\subset A$. A subset A of a space X is δ-set [7] if and only if $int(cl(A))$ subset of $cl(int(A))$. Recall that mapping f is called δ-continuous if the inverse image of every δ-set is open equivalently a mapping f from a space X into a space Y is called δ-continuous at a point $x\in X$ if for all $\delta-set\ D$ containing f(x) there is an open set U containing x such that $f(U)\subset D$, in the same way but via gsp-open set, we can define an equivalent definition of the concept of gsp-continuous mapping, a subset A of a space X is δ-neighborhood of a point $x\in X$ if there exists a δ-set G such that $x\in G\subset A$.

1. gsp -Topological vector space.

Definition 1.1: A vector space X over a field F with a topology τ is called a gsp-topological vector spaces (briefly $gsp - TVS$) if the vector addition mapping $S:X\times X\to X$ and the scalar multiplication mapping $M:F\times X\to X$ are gsp-continuous equivalent definition of gsp-TVS can be satisfied if two condition must satisfied, the first condition is : if x,y belong to X such that $x+y\in W$ where W is open neighborhood of x+y then there exist gsp-neighborhoods U and V of x and y respectively such that U+V⊂W, and the second condition is : for any $\lambda\in F$ and x belong to X and for each open neighborhood W of $\lambda.x$ there exist an gsp-neighborhood U of λ in F and $gsp -$ neighborhood V of x in X such that $U.V \subseteq W$.

Lemma 1.2: For any gsp-TVS the translation mapping and the multiplication mapping are gsp -continuous.

Proof: Let $T_x:X\to X$ and $M_\lambda: X\to X$ be the translation and the multiplication mappings on a gsp-TVS respectively. Let y,x be any points belong to X and W be an open-neighborhood of x+y such that $T_x(y)=y+x$, then there exist gsp-neighborhoods U and V of x and y respectively such that U+V⊂W then $U + x \subseteq W$ then $T_x(U) = U + x$ that implies $T_x(U) \subseteq W$ this result show that T_x is $gsp - continuous$.,let W be an open neighborhood of $\lambda.x = M_\lambda(x)$for each λ in F ,x in X, then there is an gsp-open neighborhood U of λ in F and an gsp $-$ $neighborhood\ V$ containing $x\ in\ X\ such\ that\ \lambda.V \subseteq W$, now $M_\lambda(V) = \lambda.V \subseteq W$ implies that $M_\lambda(V) \subseteq W$ this show that the multiplication mapping is gsp-continuous.

Theorem 1.3 Let A be any open set in the gsp-TVS X , then for any point x in X , A+x is gsp-open.

Proof: Let a be any point in A+x and $a=a_0+a_1$ such that a_0 belong to A and a_1 belong to X by gsp-continuity of the translation mapping $T_{-a}: X \to X$ we have $T_{-a}(a)= a_0 + a_1- a_1= a_0$. For the open set A of a_0 , there exists gsp- neighborhood U_a of a such that $T_{-a}(U_a)= U_a$ + ($-a_1$)$\subseteq A$ thus $U_a \subseteq A + x$ then a is gsp-interior point of $A + x$. Hence $A + x$ is gsp -open set.

In the same way, we can prove the following result.

Theorem 1.4 let λ be any non-zero element in F then for each x belong to X, λA is gsp-open such that A is open in a $gsp - TVS$ X.

Theorem 1.5. The translation and multiplication mappings in a gsp-TVS are gsp-homeomorphism.

Proof: By Lemma 1.2 the translation and multiplication mappings are gsp-continuous, and they bijective, so all we have to do is to prove that the inverse is also gsp-continuous. Let A be an open-neighborhood of x, then $T_x(A)=A+x$ then , by Theorem 1.3, A+x is gsp-open in X then $(T_x)^{-1}$ is gsp-continuous in the same way we can prove that the multiplication mapping is gsp-homeomorphism.

2. δ-Topological Vector Spaces

Definition 2.1: A vector space X over a field F with a topology τ is called δ-topological vector spaces(briefly δ-TVS) if the following conditions satisfied: The vector addition and scalar multiplication mappings are δ-continuous, equivalently the following two conditions are satisfied: (1) for any two points x,y belong to X and any open neighborhood W of $x + y$ in X there exist a δ-neighborhoods U and V of x and y respectively such that $U+V\subset W$ (2) for any $\lambda\in F$ and any $x\in X$ and for each open neighborhood W of λ.x there exist a **δ**-neighborhoods U in F and V in X of λ and x respectively such that $U.V\subset W$.

Theorem 2.2. The translation and the multiplication mappings on any δ-TVS is δ-continuous.

Proof : Let y be an element in a δ-TVS X and let W be an open neighborhood of x+y such that $Tx(y) = y + x$ then there exist a δ-neighborhoods U and V of y and x in X respectively such that $U + V \subset W$ then we can write $U + V$ as $U + x$ then $U + x\subset W$, and since $T_x(y)=y+x$ then $Tx(U) = U + x$, therefore, $Tx(U)\subset W$ then T_x is δ-continuous.

To prove the multiplication mapping is δ-continuous, let W be an open neighborhood of $M_\lambda(x)=\lambda.x$ such that λ in F and x in X, but X is δ-continuous then there exist δ-neighborhood U of λ in F and δ-neighborhood V of x in X such that $\lambda.V\subseteq W$, then $M\lambda(V) = \lambda.V\subset W$ then M_λ is **δ**-continuous.

We can prove the following theorem in the same way of the proving of Theorems 1.3.

Theorem 2.3: For any open set A in a $\delta - TVS\ X$ and foe each $x\in X$, $A + x$ is δ-set.

Definition 2.4[7]: A bijective mapping f from a topological space to itself is called δ- homeomorphism if $f$and its inverse are δ-continuous.

Theorem 2.5: Let (X, τ) be $\delta - TVS$ then the translation mapping $T_x: X \to X$ and the multiplication mapping $M_\lambda: X \to X$ are δ- homeomorphism.

Proof: the bijective of the mappings is clear and the δ-continuity of the translation and multiplication mappings proved in Theorem 2.2, so we have to show now the inverse of two maps are δ-continuous. Suppose that U be any open neighborhood of y .Then $T_x^{-1}(U)=U+x$ by the Theorem 2.3 U+x is δ-set then T_x is δ-continuous. Therefore T_x is δ-homeomorphism mapping, in the same way, we can show that the multiplication mapping is δ-homeomorphism.

We conclude our study by the notion of δ- homogenous, which is defined in the definition below.

Definition 2.6: A space X is called δ- homogenous if for any points x and y in X there exist a δ- homeomorphism mapping $f: X\to X$ such that f(x)=y.

Theorem 2.7: Let X be a $\delta - TVS$ and S a subspace of X, if S contains a non-empty open set, then S is δ-set set in X.

Proof: Let $V \neq \emptyset$ *be an open subset in X, such that* $V \subseteq S$, for any $x \in S$, the set $V + x = T_x(V)$ is a δ-set in X and V subset of S, then the subspace S equal to $S = \bigcup\{(V+ x): x\in S\}$ is δ-set , as the union of δ-set is δ-set .

Proposition 2.8: Let X be $\delta - TVS$ for any neighborhood V of 0 there exists a $\delta-$ neighborhood O of 0 such that $O + O \subseteq V$.

Proof: Let V be an open neighborhood of 0 + 0= 0, by the definition of $\delta - TVS$, there exists a δ-neighborhood U of 0 such that $O + O \subseteq V$.

Theorem 2.9: Every δ-TVS (X,τ) is δ- homogenous space.

Proof. Let $z = (-x) + y$, then by the Theorem 2.5 Tz is δ-homeomorphism and $Tz(x) = x - x + y = y$, then X is δ-homogenous space .

REFERENCES

[1] A. N. Kolmogorov, Zur Normierbierbarkeit eines topologischen linearen Raumes, *Studia Mathematica*, Vol 1. No 5., PP 29-33, 1934

[2] T. Al-Hawary, A. Al-Nayef, , Irresolute-Topological Vector Spaces, *Math. Sci. Res. Hot-Line* Vol 1,No 5., pp 49-53, 2001

[3] N. Rajesh, V. Tanjavur, , On Strongly Preirresolute Topological Vector Space, *Mathematica Bohemica*, Vol. 138, No.1, pp 37-42, 2013

[4] D. Moiz, S. Azam , M. S. Bosan , s-Topological vector ctor Spaces, Journal of Linear and Topological Algebra, Vol.04, No.02, ,153-158, 2015

[5] **M,** Abd El-Monsef**,** β-open sets and β-continuous **m**appings - *Bull. Fac. Sci. Assiut Univ.* , Vol.12, No.2, ,77-90, 1983

[6] J. Dontchev, On generalized semi-pre-open sets, *Mem. Fac. Sci. Kochi. Univ., Ser. A. Math*., Vol 16 No3. pp 35-48 ,1995.

[7] C. Chattopadhyay, U.K. Roy, δ - sets, irresolvable and resolvable spaces, *Math. Slovaca*, Vol. 42, No 3,pp 371 – 378, 1992.

[8] I. M. Radhi, J. H. Hussein, S. K. Hameed ,On Semi - Pre irresolute Topological Vector Space, *Diy. of pur. Sci* V.14 No.3 pp183-191, 2018.

[9] S. G. Crossely, S. K. Hildebrand, Semi-topological properties, *Fund. Math.,* Vol. 74 No 1, pp 233–254, 1972.

[10] N. Levine, Semi-open sets and semi-continuity in topological space, *Amer. Math. Monthly*, 70(1963), 36-41.

[11] G.B. Navalagi, Semi-precontinuous functions and properties of generalized semipreclosed sets in topological spaces, *IJMMS* Vol. 29 No. 2 85-98, 2002.