NUMERICAL SOLUTION FOR CLASSICAL OPTIMAL CONTROL PROBLEM GOVERNING BY HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION VIA GALERKIN FINITE ELEMENT WITH GRADIENT AND FRANK WOLFE METHODS

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ABSTRACT: *This paper deals with the study of the discrete classical optimal control problem (DCOCP) for systems of linear hyperbolic partial differential equations (LHPDEs) with initial (ICs) and boundary (BC) conditions. At first, the existence theorem of a unique discrete solution for the discrete state equation when the discrete control is fixed is proved using the Galerkin finite element method (GFEM) in space variable and the implicit finite difference scheme (IFDS) in time variable, which will be denoted by (GFEIM). Second, the existence theorem of a discrete classical optimal control (piecewise constants (PCs)) is stated and proved. The necessary condition for the optimality of the DCOCP problem is proved. Finally an algorithm is given and a computer program is coded to find the numerical solution of the DCOCP problem, the discrete state, and discrete adjoint are found using the GFEIM, while the Gradient method (GM) with Armijo step search (GARM) and the Frank Wolfe method (FWM) with Armijo step search (FWARM) are used to obtain the discrete classical optimal control, some illustrative examples are given to show the efficiency of these methods.*

Keywords: Numerical classical optimal control, hyperbolic boundary value problem, finite element method, Gradient method, Frank Wolfe method.

1. INTRODUCTION

Optimal control problems for partial differential equations PDEs have wide applications in many real-life problems for example in Electromagnetic waves, robotics, Dynamical elasticity, air traffic optimization and in many others fields.

Due to the importance of the optimal control problems, many researchers were interested to study the numerical solution of optimal control problems governing by a nonlinear ordinary differential equation as in [1] or governing by PDEs of the semilinear parabolic type as in [2,3], or by nonlinear elliptic PDEs as in [4,5].

These studies and many others encourage us to study the numerical solution for the CCOCP for systems of (LHPDEs) with (ICs) and (BC). The problem is discretized into the DCOCO using the GFEM in space variable and IFDS in time variable, these mixed two methods will be denote by (GFEIM), at first the existence theorem of a unique discrete solution for the discrete state equation is proved using the Galerkin finite element method (GFEM) in space variable and the implicit finite difference scheme (IFDS) in time variable, which we denote them by (GFEIM). Second, the existence theorem of a discrete classical optimal control (piecewise constants (PCs)) is stated and proved. The necessary condition for the optimality of the DCOCP problem is proved. Finally an algorithm is given and a computer program is coded in Matlab software to find the numerical solution of the DCOCP problem, both the discrete state and adjoint are obtained using the GFEIM, while the Gradient method with Armijo step search (GARM) and the Frank Wolfe method with Armijo step search (FWARM) are used both to obtain the discrete classical optimal control, some illustrative examples are given to show the comparison and the efficiency of these methods.

2. Description of the CCOCP [6]

Let $\Lambda \subset \mathbb{R}^2$ be a bounded and open region with Lipschitz boundary $\partial\Lambda$, and let $\tau = (0,T)$, $0<T<\infty$, $P = \Lambda \times \tau$ and $\partial P = \partial\Lambda \times [0,T]$. The CCOCP of LHPDEs consists of the following Hyperbolic PDE:

$$y_{tt} + B(t)y = h(\vec{x},t) + y + v - v_d, \text{ in } P, \vec{x} = (x_1,x_2) \quad (1)$$

with the BC

$$y(\vec{x},t) = 0, \text{ in } \partial P \quad (2)$$

and the ICs

$$y(\vec{x},0) = y^0(\vec{x}), \text{ in } \Lambda \quad (3)$$

$$y_t(\vec{x},0) = y^1(\vec{x}), \text{ in } \Lambda \quad (4)$$

Where $y = y_v(\vec{x},t) \in C^2(\bar{P})$. is the state which corresponds to the CCC $v = v(\vec{x},t) \in L^2(P)$, $v_d = v_d(\vec{x},t) \in L^2(P)$ is the desired control, $h = h(\vec{x},t) \in L^2(P)$ is a given function and B(t) is the second order elliptic operator.

$$B(t)y = -\sum_{i,j=1}^{2} \frac{\partial^2 y}{\partial x_i \partial x_j}$$

The set of the CCCs is $v \in M$, $M \subset L^2(P)$

where $M = \{ v \in L^2(P) | v(\vec{x},t) \in U, \text{ a.e. in } P \}$, $U \subset \mathbb{R}^d$ is a convex and compact (usually $d = 1$ or $d = 2$).

The cost functional is

$$G_0(v) = \int_P [\tfrac{1}{2}(y - y_d)^2 + \tfrac{1}{2}(v - v_d)^2] d\vec{x}dt \quad (5)$$

Where $y_d = y_d(\vec{x},t)$ and $v_d = v_d(\vec{x},t)$ are the desired state and control respectively.

The CCOCP is to minimize the cost functional (5) subject to $v \in M$.

In this work, the inner product and the norm in $L^2(\Lambda)$ are denoted by $(.,.)_\Lambda$ and $\|\cdot\|_\Lambda$ respectively, the norm in Sobolev space $\Psi = H^1(\Lambda)$ by $\|\cdot\|_1$, and the norm in $L^2(P)$ by $\|\cdot\|_P$.

Now, the weak form (WF) of the problem (1-4) for $y \in H_0^1(\Lambda)$ is

$$(y_{tt},\psi)_\Lambda + b(t,y,\psi) = (h(t),\psi)_\Lambda + (y,\psi)_\Lambda + (v,\psi)_\Lambda - (v_d,\psi)_\Lambda, \forall\psi \in \Psi \quad (6)$$

$$y(0) = y^0, \text{ in } \Lambda \quad (7)$$

$$y_t(0) = y^1, \text{ in } \Lambda \quad (8)$$

where $b(t,y,\psi) = (\nabla y, \nabla\psi)_\Lambda$ is a symmetric bilinear form, and satisfies the following assumptions, $\forall y,\psi \in \Psi, t \in \bar{\tau}$ and for some c_1 and c_2.

(i) $|b(t,y,\psi)| \le c_2 \parallel y \parallel_1 \parallel \psi \parallel_1$
(ii) $|b(t,\psi,\psi)| \ge c_1 \parallel \psi \parallel_1^2$
Suppose $y_t = w$, then equations (6-8) can be rewritten as

$$(w_t,\psi)_\Lambda + b(t,y,\psi) = (h(t),\psi)_\Lambda + (y,\psi)_\Lambda + (v,\psi)_\Lambda - (v_d,\psi)_\Lambda,\ \forall \psi \in \Psi \quad (6a)$$

$$y(0) = y^0, \text{ in } \Lambda \quad (7a)$$

$$w(0) = y^1, \text{ in } \Lambda \quad (8a)$$

3. Description of the DCOCP [7]

In this part the CCOC is discretized by using the GFEIM as follows: Assume that the domain Λ is a polyhedron. Let for every integer (m), $\{K_i^m\}_{i=1}^{N(m)}$ be an admissible regular triangulation of $\overline{\Lambda}$ into closed d-simplices [10], $\{\tau_j^m\}_{j=0}^{S(m)-1}$ be a subdivision of the interval $\overline{\tau}$ into S(m) intervals, where $\tau_j^m = [t_j^m, t_{j+1}^m]$ of equal lengths($\Delta t = \frac{T}{S}$). The subspace $\Psi_m \subset \Psi = H_0^1(\Lambda)$ be the space of continuous piecewise affine mapping (CPAM) in Λ. Let the set of discrete classical controls M^m with $P_{ij} := K_i^m \times I_j^m$ is

$$M^m = \{\bar{v} = \bar{v}^m \in M | \bar{v}(\vec{x},t) = \bar{v}_{ij} \in U^m \text{ in } P_{ij}\}$$

For each $\psi \in \Psi_m$, and for $j = 0,1,\dots,S-1$, the discrete state equations (DSEs) of (1-4) is given by

$$\left(w_{j+1}^m - w_j^m, \psi\right)_\Lambda + \Delta t\, b\left(y_{j+1}^m, \psi\right) = \Delta t\left(h(t_j^m),\psi\right)_\Lambda + \Delta t\left(y_{j+1}^m,\psi\right)_\Lambda + \Delta t\left(v_j^m,\psi\right)_\Lambda - \Delta t\left(v_d(t_j^m),\psi\right)_\Lambda \quad (9)$$

$$y_{j+1}^m - y_j^m = \Delta t w_{j+1}^m \quad (10)$$

$$(y_0^m,\psi)_\Lambda = (y^0,\psi)_\Lambda \quad (11)$$

$$(w_0^m,\psi)_\Lambda = (y^1,\psi)_\Lambda \quad (12)$$

where $y_j^m = y(t_j^m)$, $w_j^m = w(t_j^m) \in \Psi_m$ for $j = 0,1,\dots,S$, and $y^0 \in \Psi$ and $y^1 \in L^2(\Lambda)$ are given.

The discrete cost functional (DCF) $G_0^m(v^m)$ is defined by

$$G_0^m(v^m) = \Delta t \sum_{j=0}^{S-1} \int_\Lambda \frac{1}{2}[(y_{j+1}^m - y_d)^2 + (v_j^m - v_d)^2] d\vec{x} \quad (13)$$

The DCOCP is to find $v^m \in M^m$, such that

$$G_0^m(v^m) = \min_{\bar{v}^m \in M^m} G_0^m(\bar{v}^m)$$

4. Applying the GFEIM for the problem

4.1 Theorem: For any fixed j ($0 \le j \le S-1$), and $\forall\, v^m \in M^m$, the DSEs (9-12) has a unique solution $y_{v^m}^m = y^m = (y_0^m, y_1^m, \dots, y_S^m)$ for sufficiently small Δt.

Proof: To find the solution $y^m = (y_0^m, y_1^m, \dots, y_S^m)$ for any fixed j ($0 \le j \le S-1$), let $(\psi_i(\vec{x})$, $i = 1,\dots,N$ are CPAM in Λ with $\psi_i(\vec{x}) = 0$ on $\partial\Lambda$) be a finite basis of Ψ_m , then (9-12) can be written in the following form for any $i = 1,\dots,N$ and y_j^m, w_j^m , y_{j+1}^m ,$w_{j+1}^m \in \Psi_m$,

$$\left(w_{j+1}^m - w_j^m, \psi_i\right)_\Lambda + \Delta t\, b\left(y_{j+1}^m, \psi_i\right) = \Delta t\left(h(t_j^m),\psi_i\right)_\Lambda + \Delta t\left(y_{j+1}^m,\psi_i\right)_\Lambda + \Delta t\left(v_j^m,\psi_i\right)_\Lambda - \Delta t\left(v_d(t_j^m),\psi_i\right)_\Lambda \quad (14)$$

$$y_{j+1}^m - y_j^m = \Delta t w_{j+1}^m \quad (15)$$

$$(y_0^m,\psi_i)_\Lambda = (y^0,\psi_i)_\Lambda \quad (16)$$

$$(w_0^m,\psi_i)_\Lambda = (y^1,\psi_i)_\Lambda \quad (17)$$

Using (15), then (14) can be written as:

$$\left(y_{j+1}^m,\psi_i\right)_\Lambda + (\Delta t)^2\, b\left(y_{j+1}^m,\psi_i\right) - (\Delta t)^2\left(y_{j+1}^m,\psi_i\right)_\Lambda = \left(y_j^m,\psi_i\right)_\Lambda + \Delta t\left(w_j^m,\psi_i\right)_\Lambda + (\Delta t)^2\left(h(t_j^m),\psi_i\right)_\Lambda + (\Delta t)^2\left(v_j^m,\psi_i\right)_\Lambda - (\Delta t)^2\left(v_d,\psi_i\right)_\Lambda \quad (18)$$

Now, using the GFEIM, we write

$$y_0^m = \sum_{k=1}^N p_k^0 \psi_k,\quad y_j^m = \sum_{k=1}^N p_k^j \psi_k,\quad y_{j+1}^m = \sum_{k=1}^N p_k^{j+1} \psi_k,$$

$$w_0^m = \sum_{k=1}^N q_k^0 \psi_k,\ w_j^m = \sum_{k=1}^N q_k^j \psi_k \text{ and } w_{j+1}^m = \sum_{k=1}^N q_k^{j+1} \psi_k$$

where $p_k^j = p_k(t_k^j)$ and $q_k^j = q_k(t_k^j)$ are unknown constants, for each $j = 1,2,\dots,S$.

Substituting y_0^m , y_j^m , y_{j+1}^m , w_0^m , w_j^m and w_{j+1}^m in equations (18,15 ,16 and 17), the following system of 1st ODEs is obtained (for $j = 1,2,\dots,S-1$)

$$(E + (\Delta t)^2 F - (\Delta t)^2 E)p^{j+1} = Ep^j + \Delta t\, E\, q^j + (\Delta t)^2 \vec{b}_1(t_j) + (\Delta t)^2 \vec{b}_2(t_j) \quad (19)$$

$$g^{j+1} = \frac{p^{j+1} - p^j}{\Delta t} \quad (20)$$

$$Ep^0 = e^0 \quad (21)$$

$$Eq^0 = e^1 \quad (22)$$

where $E = (a_{ik})_{N\times N}$, $a_{ik} = (\psi_i,\psi_k)$, $F = (b_{ik})_{N\times N}$, $b_{ik} = b(t,\psi_i,\psi_k)$, $p_{N\times 1}^{j+r} = (p_1^{j+r}, p_2^{j+r},\dots,p_N^{j+r})^T$, $q_{N\times 1}^{j+r} = (q_1^{j+r}, q_2^{j+r},\dots,q_N^{j+r})^T$, (for $r = 0,1$), $e^0 = (e_i^0)_{N\times 1}$, $e^1 = (e_i^1)_{N\times 1}$, $e_i^0 = (y^0,\psi_i), e_i^1 = (y^1,\psi_i)$, $b_{1i} = (h(t_j),\psi_i)$, $\vec{b}_2 = (b_{2i})_{N\times 1}$, $b_{2i} = (v_j^m - v_d, \psi_i)$, $\forall\, i,k = 1,\dots,N$.

From the assumption on the operator $b(.,.)$ we have the matrices E and F is positive definite (PD), then $E + (\Delta t)^2 F - (\Delta t)^2 E$ is PD (has positive eigenvalues), therefore it is regular, then (19-22) has a unique solution. □

5. The existence of the DCOCP

5.1 Theorem:-The operator $v^m \mapsto y^m = y_{v^m}^m$ is continuous.

Proof: Let

$v^m = (v_0^m, v_1^m,\dots,v_{S-1}^m)$, $v^{mn} = (v_0^{mn}, v_1^{mn},\dots,v_{S-1}^{mn})$
$y^m = (y_0^m, y_1^m,\dots,y_{S-1}^m)$, $y^{mn} = (y_0^{mn}, y_1^{mn},\dots,y_{S-1}^{mn})$
$w^m = (w_0^m, w_1^m,\dots,w_{S-1}^m)$, and
$w^{mn} = (w_0^{mn}, w_1^{mn},\dots,w_{S-1}^{mn})$

We want to prove that if $v^{mn} \longrightarrow v^m$ as $n \longrightarrow \infty$ then $y_{v^{mn}}^m = y^{mn} \longrightarrow y^m = y_{v^m}^m$, i.e. if $v_j^{mn} \longrightarrow v_j^m$, ∀ j as $n \longrightarrow \infty$ then $y_j^{mn} \longrightarrow y_j^m$, ∀ j, as $n \longrightarrow \infty$. this will prove it by mathematical induction.

First from the ICs (16 and 17) and the projection theory, we have

$y_0^{mn} \longrightarrow y_0^m$, and $w_0^{mn} \longrightarrow w_0^m$, as $n \longrightarrow \infty$.

Second, suppose for any fixed j, that $y_j^{mn} \longrightarrow y_j^m$ and $w_j^{mn} \longrightarrow w_j^m$ as $n \longrightarrow \infty$, and we prove that $y_{j+1}^{mn} \longrightarrow y_{j+1}^m$ as $n \longrightarrow \infty$.

Let $y_{j+1}^m = L(y_j^m, w_j^m, v_j^m)$, and $y_{j+1}^{mn} = L(y_j^{mn}, w_j^{mn}, v_j^{mn})$, then

$$\parallel y_{j+1}^{mn} - y_{j+1}^m \parallel_\Lambda = \parallel L\left(y_j^{mn}, w_j^{mn}, v_j^{mn}\right) - L\left(y_j^m, w_j^m, v_j^m\right) \parallel_\Lambda$$

$$= \parallel y_j^m - y_j^m \parallel_\Lambda = 0$$

$\Longrightarrow y_{j+1}^{mn} \longrightarrow y_{j+1}^m$, ∀ j $\Longrightarrow y_j^{mn} \longrightarrow y_j^m$, ∀ j, i.e.

the operator $v^m \mapsto y^m = y_{v^m}^m$ is continuous. □

5.1 Lemma [9]: The norm $\parallel\cdot\parallel_\Lambda$ is weakly lower semicontinuous (WLSc).

5.2 Lemma [9]: The DCF that is given by (13) is WLSc.

5.3 Lemma: If the DCCs. v^m ,$\tilde{v}^m$ are bounded in $L^2(P)$, and y_j^m , $y_{\varepsilon j}^m = y_j^m + \Delta_\varepsilon y_j^m$ (ϵ is a small positive number) are corresponding discrete states solutions to the DCCs v_j^m and $v_{\varepsilon j}^m = v_j^m + \varepsilon\Delta\, v_j^m$ respectively, then ($\forall j = 1,2,\dots,S$):

$\| \Delta_\varepsilon y_\hbar^m \|_1^2 \le \eta\varepsilon^2 \| \Delta\, v^m \|_P^2$ and $\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 \le \eta\varepsilon^2 \| \Delta\, v^m \|_P^2$

Or $\| \Delta_\varepsilon y_\hbar^m \|_1^2 \le \eta$, and $\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 \le \eta$ (23)

Proof: From the DSEs (9-12), and for j=1,2,…,S-1, we have

$$\left(\Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m, \psi\right)_\Lambda + \Delta t\, b\left(\Delta_\varepsilon y_{j+1}^m, \psi\right)$$
$$= \Delta t\left(\Delta_\varepsilon y_{j+1}^m, \psi\right)_\Lambda + \Delta t\left(\varepsilon\Delta\, v_j^m, \psi\right)_\Lambda \quad (24)$$
$$\Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m = \Delta t \Delta_\varepsilon w_{j+1}^m \quad (25)$$
$$\Delta_\varepsilon y_0^m = \Delta_\varepsilon w_0^m = 0 \quad (26)$$

Substituting $\psi = \Delta_\varepsilon w_{j+1}^m$ in (24), and then rewriting it in another way to get

$$\| \Delta_\varepsilon w_{j+1}^m \|_\Lambda^2 - \| \Delta_\varepsilon w_j^m \|_\Lambda^2 + \| \Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m \|_\Lambda^2 + 2\Delta t\, b\left(\Delta_\varepsilon y_{j+1}^m, \Delta_\varepsilon w_{j+1}^m\right)$$
$$\le c\Delta t \| \Delta_\varepsilon y_{j+1}^m \|_1^2 + 2\Delta t \| \Delta_\varepsilon w_{j+1}^m \|_\Lambda^2 + \Delta t\varepsilon^2 \| \Delta\, v_j^m \|_\Lambda^2 \quad (27)$$

Since

$$b\left(\Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m, \Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m\right) = (\Delta t)^2 b\left(\Delta_\varepsilon w_{j+1}^m, \Delta_\varepsilon w_{j+1}^m\right)$$

and

$$b\left(\Delta_\varepsilon y_{j+1}^m, \Delta_\varepsilon y_{j+1}^m\right) - b\left(\Delta_\varepsilon y_j^m, \Delta_\varepsilon y_j^m\right) = -(\Delta t)^2 b\left(\Delta_\varepsilon w_{j+1}^m, \Delta_\varepsilon w_{j+1}^m\right) + 2\Delta t\, b\left(\Delta_\varepsilon y_{j+1}^m, \Delta_\varepsilon w_{j+1}^m\right)$$

Then

$$2\Delta t\, b\left(\Delta_\varepsilon y_{j+1}^m, \Delta_\varepsilon w_{j+1}^m\right) = \left[b\left(\Delta_\varepsilon y_{j+1}^m, \Delta_\varepsilon y_{j+1}^m\right) - b\left(\Delta_\varepsilon y_j^m, \Delta_\varepsilon y_j^m\right) + b\left(\Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m, \Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m\right)\right] \quad (28)$$

By substituting (28) in the LHS (left-hand side) of (27), summing both sides of the obtained equation from j=0 to j= $\hbar - 1$, using (26), and applying assumption (i) on $b(.,.)$, we have

$$\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m \|_\Lambda^2 + c_2 \| \Delta_\varepsilon y_\hbar^m \|_1^2$$
$$+ c_2 \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m \|_1^2 \le \Delta t\varepsilon^2 \sum_{j=0}^{\hbar-1} \| \Delta\, v_j^m \|_\Lambda^2$$
$$+ \tilde{C}\Delta t \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon w_{j+1}^m \|_\Lambda^2 + c\Delta t \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon y_{j+1}^m \|_1^2 \quad (29)$$

But

$\| \Delta_\varepsilon y_{j+1}^m \|_1^2 \le 2 \| \Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m \|_1^2 + 2 \| \Delta_\varepsilon y_j^m \|_1^2$ and $\| \Delta_\varepsilon w_{j+1}^m \|_\Lambda^2 \le 2 \| \Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m \|_\Lambda^2 + 2 \| \Delta_\varepsilon w_j^m \|_\Lambda^2$

Substituting these inequalities in equation (29), to get

$$\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + \left(\bar{\bar{c}} - \bar{\bar{L}}\Delta t\right) \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m \|_\Lambda^2$$
$$+ c_2 \| \Delta_\varepsilon y_\hbar^m \|_1^2 + \left(\bar{\bar{c}} - \bar{\bar{L}}\,\Delta t\right) \sum_{j=0}^{\hbar-1} \| \Delta_\varepsilon y_{j+1}^m - \Delta_\varepsilon y_j^m \|_1^2$$
$$\le \Delta t\varepsilon^2 \sum_{j=0}^{\hbar-1} \| \Delta\, v_j^m \|_\Lambda^2 + 2\Delta t\ddot{C} \sum_{j=0}^{\hbar-1} \left(\| \Delta_\varepsilon w_j^m \|_\Lambda^2 + \| \Delta_\varepsilon y_j^m \|_1^2\right)$$
$$\le \varepsilon^2 \| \Delta\, v^m \|_P^2 + 2\Delta t\ddot{C} \sum_{j=0}^{\hbar-1} \left(\| \Delta_\varepsilon w_j^m \|_\Lambda^2 + \| \Delta_\varepsilon y_j^m \|_1^2\right) \quad (30)$$

where $\bar{\bar{c}} = \min\{1, c_2\}$, $\bar{\bar{L}} = \min\{2c, 2\tilde{C}\}$, $\ddot{C} = max\{c, \tilde{C}\}$. By choosing $\Delta t < \frac{\bar{\bar{c}}}{\bar{\bar{L}}}$, then the second and the fourth terms in the L.H.S. of (30) become positive, hence it gives

$$\bar{h}\left(\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + \| \Delta_\varepsilon y_\hbar^m \|_1^2\right) \le \| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + c_2 \| \Delta_\varepsilon y_\hbar^m \|_1^2$$
$$\le \varepsilon^2 \| \Delta\, v_j^m \|_P^2 + 2\Delta t\ddot{C} \sum_{j=0}^{\hbar-1} \left(\| \Delta_\varepsilon w_j^m \|_\Lambda^2 + \| \Delta_\varepsilon y_j^m \|_1^2\right) \quad (31)$$

Then

$$\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + \| \Delta_\varepsilon y_\hbar^m \|_1^2 \le \check{h}\varepsilon^2 \| \Delta\, v^m \|_P^2 + \acute{h}\Delta t \sum_{j=0}^{\hbar-1} \left(\| \Delta_\varepsilon w_j^m \|_\Lambda^2 + \| \Delta_\varepsilon y_j^m \|_1^2\right) \quad (32)$$

Applying the discrete Grownwall's inequality (DGI) [8] on(32) to give

$\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 + \| \Delta_\varepsilon y_\hbar^m \|_1^2 \le \dot{c}\check{h}\,\varepsilon^2 \| \Delta\, v^m \|_P^2$, which gives

$\| \Delta_\varepsilon w_\hbar^m \|_\Lambda^2 \le \dot{c}\check{h}\varepsilon^2 \| \Delta\, v^m \|_P^2$ and

$\| \Delta_\varepsilon y_\hbar^m \|_1^2 \le \dot{c}\check{h}\varepsilon^2 \| \Delta\, v^m \|_P^2$

since v^m and $\tilde{v}^m$ are bounded in $L^2(P)$, then (23) is satisfy. □

5.2 Theorem: Consider the DCF (13), assume U^m is convex. If $G_0^m(v^m)$ is coercive, then there exists a classical discrete optimal control.

Proof: Since U^m is convex, then W^m is convex. Since $G_0^m(v^m) \ge 0$ and $G_0^m(v^m)$ is coercive then there exists a minimizing sequence $\{v_j^{mk}\} \in W^m$, $\forall k, j$ such that

$$\lim_{k\to\infty} G_0^m(v^{mk}) = \inf_{k\to\infty} G_0^m(v^{mk})$$

and there exists a constant $\bar{\bar{C}}$ such that $\| v_j^{mk} \|_0 \le \bar{\bar{C}}$, $\forall k, j$, then by Alaoglue theorem[11], there exists a subsequence of $\{v_j^{mk}\}$ (for simplicity say again$\{v_j^{mk}\}$) such that $\{v_j^{mk}\} \longrightarrow \{v_j^m\}$ weakly in $L^2(\Lambda)$. But theorem (2.1) tell us for each control v_j^{mk}, the discrete state equations has a unique solution $y^{mk} = y_{v^{mk}}^m$

Now, to prove $\{y_j^{mk}\}$, $\{y_{j+1}^{mk}\}$, $\{w_j^{mk}\}$ and $\{w_{j+1}^{mk}\}$ are bounded for in Ψ_m, $\forall k$ (for j=0,1,…,S-1).

Set $\psi = w_{j+1}^m$ in (9), and then the first term in the LHS of the obtained equation can be rewritten as

$$\| w_{j+1}^m \|_\Lambda^2 - \| w_j^m \|_\Lambda^2 + \| w_{j+1}^m - \Delta_\varepsilon w_j^m \|_\Lambda^2$$
$$+ 2\Delta t b\left(y_{j+1}^m, w_{j+1}^m\right) \le c\Delta t \| y_{j+1}^m \|_1^2 + \Delta t \| h\left(t_j^m\right) \|_\Lambda^2$$
$$+ \bar{\bar{C}}\Delta t \| w_{j+1}^m \|_\Lambda^2 + \Delta t \| v_j^m \|_\Lambda^2$$
$$+ \Delta t \| v_d\left(t_j^m\right) \|_\Lambda^2 \quad (33)$$

Since

$$b\left(y_{j+1}^m - y_j^m, y_{j+1}^m - y_j^m\right) = (\Delta t)^2 b\left(w_{j+1}^m, w_{j+1}^m\right)$$

and

$$b\left(y_{j+1}^m, y_{j+1}^m\right) - b\left(y_j^m, y_j^m\right) = -(\Delta t)^2 b\left(w_{j+1}^m, w_{j+1}^m\right) + 2\Delta t\, b\left(y_{j+1}^m, w_{j+1}^m\right)$$

Then

$$2\Delta t\, b\left(y_{j+1}^m, w_{j+1}^m\right) = \left[b\left(y_{j+1}^m, y_{j+1}^m\right) - b\left(y_j^m, y_j^m\right) + b\left(y_{j+1}^m - y_j^m, y_{j+1}^m - y_j^m\right)\right] \quad (34)$$

By substituting (34) in the LHS of (33), taking the summing for their both sides from j=0 to j= $\hbar - 1$, and applying the assumption (i) on $b(.,.)$, we get

$$\| w_\hbar^m \|_\Lambda^2 + \sum_{j=0}^{\hbar-1} \| w_{j+1}^m - w_j^m \|_\Lambda^2 + c_2 \| y_\hbar^m \|_1^2$$

$+c_2\sum_{j=0}^{\hbar-1} \| y_{j+1}^m - y_j^m \|_1^2$

$\leq \Delta t \sum_{j=0}^{\hbar-1} \| v_j^m \|_\Lambda^2 + \Delta t \sum_{j=0}^{\hbar-1} \| v_d(t_j^m) \|_\Lambda^2 + \Delta t \sum_{j=0}^{\hbar-1} \| h(t_j^m) \|_\Lambda^2$

$+\bar{\bar{C}}\Delta t \sum_{j=0}^{\hbar-1} \| w_{j+1}^m \|_\Lambda^2 + c\Delta t \sum_{j=0}^{\hbar-1} \| y_{j+1}^m \|_1^2$ (35)

But

$\| y_{j+1}^m \|_1^2 \leq 2 \| y_{j+1}^m - y_j^m \|_1^2 + 2 \| y_j^m \|_1^2$ and

$\| w_{j+1}^m \|_\Lambda^2 \leq 2 \| w_{j+1}^m - w_j^m \|_\Lambda^2 + 2 \| w_j^m \|_\Lambda^2$

Substituting the inequalities in (35), we get

$\| w_\hbar^m \|_\Lambda^2 + (\bar{\bar{c}} - \bar{\bar{K}}\Delta t)\sum_{j=0}^{\hbar-1} \| w_{j+1}^m - w_j^m \|_\Lambda^2 + c_2 \| y_\hbar^m \|_1^2$

$+(\bar{\bar{c}} - \bar{\bar{K}}\,\Delta t)\sum_{j=0}^{\hbar-1} \| y_{j+1}^m - y_j^m \|_1^2$

$\leq \Delta t \sum_{j=0}^{\hbar-1} \| v_j^m \|_\Lambda^2 + \Delta t \sum_{j=0}^{\hbar-1} \| v_d(t_j^m) \|_\Lambda^2$

$+\Delta t \sum_{j=0}^{\hbar-1} \| h(t_j^m) \|_\Lambda^2 + \Delta t \ddot{C} \sum_{j=0}^{\hbar-1} (\| w_j^m \|_\Lambda^2 + \| y_j^m \|_1^2)$ (36)

where $\bar{\bar{c}} = \min\{1, c_2\}$, $\bar{\bar{K}} = \min\{2c, 2\bar{\bar{C}}\}$ and $\ddot{C} = \max\{2c, 2\bar{\bar{C}}\}$.

By choosing $\Delta t < \frac{\bar{\bar{c}}}{\bar{\bar{K}}}$, then the second and the fourth terms in the LHS of (36) becomes positive, hence

$\bar{h}(\| w_\hbar^m \|_\Lambda^2 + \| y_\hbar^m \|_1^2) \leq \| w_\hbar^m \|_\Lambda^2 + c_2 \| y_\hbar^m \|_1^2$

$\leq \| h(t^m) \|_P^2 + \| v^m \|_P^2 + \| v_d(t^m) \|_P^2$

$+\Delta t \ddot{C} \sum_{j=0}^{\hbar-1} (\| w_j^m \|_\Lambda^2 + \| y_j^m \|_1^2)$

By applying the DGI on the above inequality, to get

$\| w_\hbar^m \|_\Lambda^2 + \| y_\hbar^m \|_1^2 \leq c$, which gives

$\| y_\hbar^m \|_1^2 \leq c$, and $\| w_\hbar^m \|_\Lambda^2 \leq c$, for any arbitrary index $\hbar$.

Then by Alaoglue theorem, there exists a subsequences of $\{y_{j+1}^{mk}\}$, $\{y_j^{mk}\}$, $\{w_{j+1}^{mk}\}$ and $\{w_j^{mk}\}$ (same notation for simplicity) such that $\{y_{j+1}^{mk}\} \longrightarrow \{y_{j+1}^m\}$ weakly in Ψ_m , $\{y_j^{mk}\} \longrightarrow \{y_j^m\}$ weakly in Ψ_m , $\{w_{j+1}^{mk}\} \longrightarrow \{w_{j+1}^m\}$ weakly in Ψ_m, and $\{w_j^{mk}\} \longrightarrow \{w_j^m\}$ weakly in Ψ_m, which are given $\{y_{j+1}^{mk}\} \longrightarrow \{y_{j+1}^m\}$ weakly in $L^2(\Lambda)$, $\{y_j^{mk}\} \longrightarrow \{y_j^m\}$ weakly in $L^2(\Lambda)$, $\{w_{j+1}^{mk}\} \longrightarrow \{w_{j+1}^m\}$ weakly in $L^2(\Lambda)$ and $\{w_j^{mk}\} \longrightarrow \{w_j^m\}$ weakly in $L^2(\Lambda)$.

For each k, $\{y_{j+1}^{mk}\}$ and $\{y_j^{mk}\}$ satisfy (18), then

$(y_{j+1}^{mk}, \psi_i)_\Lambda + (\Delta t)^2\, b(y_{j+1}^{mk}, \psi_i) - (\Delta t)^2\, (y_{j+1}^{mk}, \psi_i)_\Lambda$

$= (y_j^{mk}, \psi_i)_\Lambda + \Delta t(w_j^{mk}, \psi_i)_\Lambda + (\Delta t)^2(h(t_j^m), \psi_i)_\Lambda +$

$(\Delta t)^2(v_j^{mk}, \psi_i)_\Lambda - (\Delta t)^2\, (v_d(t_j^m), \psi_i)_\Lambda$

(37)

Now, to show that (37) converges to

$(y_{j+1}^m, \psi_i)_\Lambda + (\Delta t)^2\, b(y_{j+1}^m, \psi_i) - (\Delta t)^2\, (y_{j+1}^m, \psi_i)_\Lambda$

$= (y_j^m, \psi_i)_\Lambda + \Delta t(w_j^m, \psi_i)_\Lambda + (\Delta t)^2(h(t_j^m), \psi_i)_\Lambda$

$+(\Delta t)^2(v_j^m, \psi_i)_\Lambda - (\Delta t)^2(v_d, \psi_i)_\Lambda$ (38)

First, from the LHS of (37) and (38), we have

$|(y_{j+1}^{mk}, \psi_i)_\Lambda + (\Delta t)^2(\nabla y_{j+1}^{mk}, \nabla\varphi_i)_\Lambda - (\Delta t)^2\, (y_{j+1}^{mk}, \psi_i)_\Lambda$

$-(y_{j+1}^m, \psi_i)_\Lambda - (\Delta t)^2(\nabla y_{j+1}^m, \nabla\psi_i)_\Lambda + (\Delta t)^2\, (y_{j+1}^m, \psi_i)_\Lambda|$

$\leq \| y_{j+1}^{mk} - y_{j+1}^m \|_\Lambda \| \psi_i \|_\Lambda$

$+(\Delta t)^2 \| \nabla y_{j+1}^{mk} - \nabla y_{j+1}^m \|_\Lambda \| \nabla\psi_i \|_\Lambda$

$+(\Delta t)^2 \| y_{j+1}^{mk} - y_{j+1}^m \|_\Lambda \| \psi_i \|_\Lambda \longrightarrow 0$

Thus, the LHS of (37) converges to the LHS of (38)

Second, since

$\{y_j^{mk}\} \longrightarrow \{y_j^m\}$ weakly in $L^2(\Lambda)$, $\{w_j^{mk}\} \longrightarrow \{w_j^m\}$ weakly in $L^2(\Lambda)$, and $\{v_j^{mk}\} \longrightarrow \{v_j^m\}$ weakly in $L^2(\Lambda)$

then, the RHS (right hand side) of (37) converges to the R.H.S. of (38).

On the other hand, since $G_0^m(v^m)$ is WLSc from lemma (5.2)

$G_0^m(v^m) \leq \lim_{k\to\infty} \inf_{v^{mk}\in W^m} G_0^m(v^{mk}) = \lim_{k\to\infty} G_0^m(v^{mk})$

$= \inf_{v^{mk}\in W^m} G_0^m(v^{mk})$

$G_0^m(v^m) = \inf_{v^{mk}\in W^m} G_0^m(v^{mk})$, thus

v^m is a classical optimal control

□

6. The Necessary conditions for DCOC problem

6.1 Theorem: The discrete classical adjoint state $z_{v^m}^m = z^m = (z_0^m, z_1^m, \ldots, z_{S-1}^m)$ is given by (for j=S-1,S-2,…0)

$(\varphi_{j+1}^m - \varphi_j^m, \psi) + \Delta t\, b(z_j^m, \psi) = \Delta t(z_j^m, \psi)$

$+\Delta t\, (y_{j+1}^m - y_d, \psi)$ (39)

$z_{j+1}^m - z_j^m = \Delta t\, \varphi_j^m$ (40)

$z_S^m = \varphi_S^m = 0$ (41)

where $z_j^m, \varphi_j^m \in \Psi_m$ ($\forall j = 0,1, \ldots S$). The directional derivative of G is given by

$DG_0^m(v^m, v'^m - v^m) = \lim_{\varepsilon\to 0} \frac{G(v^m+\epsilon\Delta v^m) - G(v^m)}{\varepsilon}$

$= \Delta t \sum_{j=0}^{S-1} (H_v^m(t_j^m, y_{j+1}^m, z_j^m, v_j^m), \Delta v_j^m)_\Lambda$

$= \Delta t \sum_{j=0}^{S-1} (z_j^m + v_j^m - v_d, \Delta v_j^m)_\Lambda$

(42)

where $v'^m, v^m \in M^m$, $\Delta v_j^m = v'^m - v^m$ for (j=0,1,…,S), and H^m is called the Hamiltonian functional.

Proof: By using equation (24), with $\psi = z_j^m$, and summing over j (for j=0 to j=S-1), to get

$\Delta t \sum_{j=0}^{S-1} \frac{(\Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m, z_j^m)_\Lambda}{\Delta t} + \Delta t \sum_{j=0}^{S-1} b(\Delta_\varepsilon y_{j+1}^m, z_j^m)$

$= \Delta t \sum_{j=0}^{S-1} (\Delta_\varepsilon y_{j+1}^m, z_j^m)_\Lambda + \Delta t \sum_{j=0}^{S-1} (\varepsilon\Delta\, v_j^m, z_j^m)_\Lambda$

(43)

Set $\psi = \Delta_\varepsilon y_{j+1}^m$ in (35), and summing over j (for j=0 to j=S-1), to get

$\Delta t \sum_{j=0}^{S-1} \frac{(\varphi_{j+1}^m - \varphi_j^m, \Delta_\varepsilon y_{j+1}^m)_\Lambda}{\Delta t} + \Delta t \sum_{j=0}^{S-1} b(z_j^m, \Delta_\varepsilon y_{j+1}^m)$

$= \Delta t \sum_{j=0}^{S-1} (z_j^m, \Delta_\varepsilon y_{j+1}^m)_\Lambda + \Delta t \sum_{j=0}^{S-1} (y_{j+1}^m - y_d, \Delta_\varepsilon y_{j+1}^m)_\Lambda$ (44)

Then, subtracting (43) from (44), gives

$\Delta t \sum_{j=0}^{S-1} \frac{(\Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m, z_j^m)_\Lambda}{\Delta t} - \Delta t \sum_{j=0}^{S-1} \frac{(\varphi_{j+1}^m - \varphi_j^m, \Delta_\varepsilon y_{j+1}^m)_\Lambda}{\Delta t}$

$= \Delta t \sum_{j=0}^{S-1} (\varepsilon\Delta\, v_j^m, z_j^m)_\Lambda - \Delta t \sum_{j=0}^{S-1} (y_{j+1}^m - y_d, \Delta_\varepsilon y_{j+1}^m)_\Lambda$ (45)

Now, for any given values y_j^m, (j=0,1,...,S) in a vector space, the following functions are defined a.e. on $\bar{\tau}$ as:

$y_-^m(t) := y_j^m,\ t \in \tau_j^m$, for each $j = 0, \ldots, S$

$y_+^m(t) := y_{j+1}^m,\ t \in \tau_j^m$, for each $j = 0, \ldots, S-1$

$y_\wedge^m(t_j^m) :=$ The functions which is affine on each τ_j^m, such that

$y_\wedge^m(t_j^m) := y_j^m,\ \forall j = 0,1, \ldots, S$

These notations are used for y, w, z and φ in the LHS of (45), to get

$$\Delta t \sum_{j=0}^{S-1} \frac{\left(\Delta_\varepsilon w_{j+1}^m - \Delta_\varepsilon w_j^m, z_j^m\right)_\Lambda}{\Delta t} = \int_0^T ((\Delta_\varepsilon w_\wedge^m)', z_-^m)_\Lambda dt \qquad (46a)$$

and

$$\Delta t \sum_{j=0}^{S-1} \frac{\left(\varphi_{j+1}^m - \varphi_j^m, \Delta_\varepsilon y_{j+1}^m\right)_\Lambda}{\Delta t} = \int_0^T ((\varphi_\wedge^m)', \Delta_\varepsilon y_+^m)_\Lambda dt$$

(46b)

By using the discrete integral by parts twice to the integral in (46a), i.e.

$$\int_0^T ((\Delta_\varepsilon w_\wedge^m)', z_-^m)_\Lambda dt = -\int_0^T (\Delta_\varepsilon w_+^m, (z_\wedge^m)')_\Lambda dt + (\Delta_\varepsilon w_N^m, z_N^m)_\Lambda - (\Delta_\varepsilon w_0^m, z_0^m)_\Lambda$$

$$= -\int_0^T (\Delta_\varepsilon w_+^m, (z_\wedge^m)')_\Lambda dt \quad , \quad \text{(by (26)\& (37))}$$

$$= -\int_0^T ((\Delta_\varepsilon y_\wedge^m)', \varphi_-^m)_\Lambda dt \quad , \quad \text{(by (25)\& (40))}$$

$$= -\int_0^T (\Delta_\varepsilon y_+^m, (\varphi_\wedge^m)')_\Lambda dt + (\Delta_\varepsilon y_N^m, \varphi_N^m)_\Lambda - (\Delta_\varepsilon y_0^m, \varphi_0^m)_\Lambda$$

$$= -\int_0^T (\Delta_\varepsilon y_+^m, (\varphi_\wedge^m)')_\Lambda dt, \text{ by (26)\& (41))}$$

$$= \int_0^T ((\varphi_\wedge^m)', \Delta_\varepsilon y_+^m)_\Lambda dt$$

(47)

Using (47) in (45) , gives

$$\Delta t \sum_{j=0}^{S-1} \left(y_{j+1}^m - y_d, \Delta_\varepsilon y_{j+1}^m\right)_\Lambda = \Delta t \sum_{j=0}^{S-1} \left(\varepsilon \Delta\, v_j^m, z_j^m\right)_\Lambda$$

(48)

On the other hand, since the Frechét derivative of the cost function G exists, and then substituting (48) in the obtain equation.

$$G(v^m + \epsilon \Delta v^m) - G(v^m)$$
$$= \Delta t \sum_{j=0}^{S-1} \varepsilon \left(\Delta\, v_j^m, z_j^m\right)_\Lambda + \Delta t \sum_{j=0}^{S-1} \left(v_j^m - v_d, \varepsilon \Delta\, v_j^m\right)_\Lambda + O_1(\varepsilon) \parallel \Delta\ v^m \parallel_P^2 + O_2(\varepsilon) \parallel \Delta\ v^m \parallel_P^2$$
$$= \Delta t \sum_{j=0}^{S-1} \varepsilon \left(\Delta\, v_j^m, z_j^m\right)_\Lambda + \Delta t \sum_{j=0}^{S-1} \left(v_j^m - v_d, \varepsilon \Delta\, v_j^m\right)_\Lambda + O\,(\varepsilon) \parallel \Delta\ v^m \parallel_P^2 \qquad (49)$$

where $O\,(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$,

Dividing (49) by ε, and taking the limit when $\varepsilon \longrightarrow 0$, we get

$$DG_0^m(v^m, v'^m - v^m) = \Delta t \sum_{j=0}^{S-1} (z_j^m + v_j^m - v_d, \Delta v_j^m)_\Lambda$$

7. Numerical Examples

The following algorithm describes the GMARM and FWARM; we will use the norm $\|\cdot\|$ with respect to vector space V.

7.1 ALGORITHM : Let V be a vector space, U is a convex subset of an open set $\Lambda \subset \mathbb{R}^2$, $G\colon \Lambda \subset V \longrightarrow \mathbb{R}$, b,c $\in (0,1)$, $\{s_n\}$ be a sequence with $s_n \in (0,\infty)$, or $s_n \in (0,1]$, for each n. $\rho > 0$. and let $v_0 \in U$ be an initial control.

Step1: Set $n := 0$, solving WF(15-18) (adjoint WF (39-41)) by GFEM to get y_n (z_n), and Calculate $G'(v_n)$ in equation (42) and $G(v_n)$ in (13).

Step 2: Find a direction point $\omega_n \in U$, (i.e. a direction $\omega_n - v_n$) by using the following method:

GM: Find the unique $\omega_n \in U$, such that

$$\omega_n = v_n - \frac{1}{\rho} G'(v_n)$$

FWM: Find $\omega_n \in U$, such that

$$(G'(v_n), \omega_n - v_n) = \min_{v \in U} (G'(v_n), \omega - v_n)$$

Step 3: Solve the solution of WF (15-18) to find the state y_n corresponding to the new control ω_n

Step 4: Calculate $\zeta_n = -\frac{1}{\rho} \parallel G'(v_n) \parallel^2$

If $\zeta_n = 0$, stop.

Step 5: Choose α_n using the following method:

ARM: Assume an initial value $\alpha = s_n \in [0, +\infty)$ (or $\alpha = s_n \in [0,1]$). If α satisfies the inequality

$$\Phi_n(\alpha) = G\big(v_n + \alpha(\omega_n - v_n)\big) - G(v_n) \le \alpha b \zeta_n$$

We set $\alpha := \alpha/c$, and choose for α_n, the last $\alpha \in (0,\infty)$ that satisfies the above inequality. If not, we set $\alpha := \alpha c$, and chooser for α_n the first $\alpha \in (0,\infty)$ that satisfies this inequality.

Step 6: Set $v_{n+1} = v_n + \alpha(w_n - v_n)$, $n := n+1$ and we go to step 2.

The COCP in the following examples are solved using Algorithm (7.1), a computer program in Mat lab software version 8.1.0.604 is written to achieve the discrete solution. y_n (z_n) in step (1) is found using GFEM with $N = 9$, $S = 20$, ($\Delta t = \frac{1}{20}$), the parameters in Armijo method are taken the value $b = c = 0.5$, and the parameter $\rho = 0.5$ in the GM and FWM.

7.1 Example: Consider the following classical optimal control problem (COCP) governing by the linear hyperbolic equation

$y_{tt} - \Delta y = h(\vec{x}, t) + y + v - v_d$, in $\mathrm{P} = \Lambda \times$, $\vec{x} = (x_1, x_2)$

$y(\vec{x}, t) = 0$, in $\partial\mathrm{P} = \partial\Lambda \times [0, T]$.

With the ICs

$y(\vec{x}, 0) = 0.5\, x_1 x_2 (1 - x_1)(1 - x_2)$, in Λ

$y_t(\vec{x}, 0) = -0.5(x_1 x_2 (x_1 - 1)(x_2 - 1))$, in Λ

where τ =[0,1], $\Lambda = [0,1] \times [0,1]$, and

$h(\vec{x}, t) = -e^{-t}[x_1^2 - x_1 + x_2^2 - x_2]$

The control constraint is $U = [-0.5, 1]$ and the cost function is given by

$$G_0(v) = \int_P [\tfrac{1}{2}(y - y_d)^2 + \tfrac{1}{2}(v - v_d)^2] d\vec{x} dt,$$

Where $y_d = y_d(\vec{x}, t)$ and $v_d = v_d(\vec{x}, t)$ are the desired state and control and are given by

$y_d(\vec{x}, t) = 0.5\, x_1 x_2 (1 - x_1)(1 - x_2) e^{-t}$, $\forall (\vec{x}, t) \in P$, and

$$v_d(\vec{x}, t) = \begin{cases} 0 & \text{, for } 0 \le t \le 0.5 \\ 0.4 & \text{, for } 0.5 < t \le 1 \end{cases}$$

with the initial control

$v_0(\vec{x}, t) = -0.4 + t$, $\forall (\vec{x}, t) \in P$

Algorithm (7.1) is used here to solve the above problem. The given initial control and its corresponding state are given in the following figures

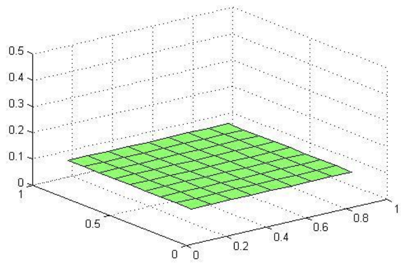

Figure 1a. Initial control at t=0.5

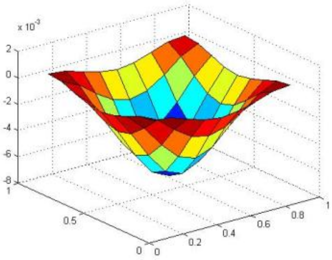

Figure 1b. Corresponding initial state at t=0.5

Depending on the above initial control and its corresponding state), we have the following results according to the

(I) In the GARM: the optimal control and corresponding state are obtained after 12 iterations, the results show with $G_0(v^m)$=5.9048e-08, ζ_m =6.6150e-04, and δ_m=1.2542e-04

Where ζ_m and δ_m are the discrete maximum errors for the state and control respectively.

The optimal control and its corresponding state are shown by the following figures

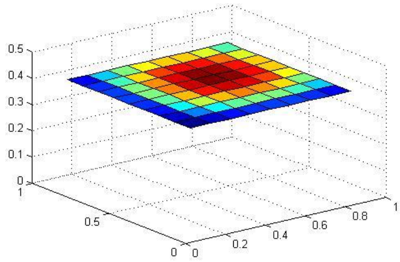

Figure 1c. Optimal control at t=0.5

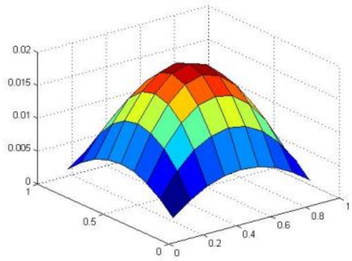

Figure 1d. Corresponding state (of optimal control) at t=0.5

(II) In the FWARM, the optimal control and corresponding state, are obtained after 115 iterations with $G_0(v^m)$=6.4532e-08, ζ_m =6.6150e-04, and δ_m=4.6256e-04

The optimal control and its corresponding state are shown by the following figures

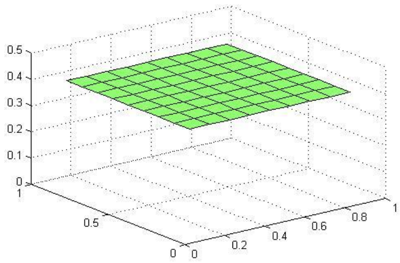

Figure 1e. Optimal control at t=0.5

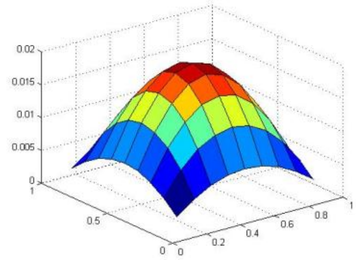

Figure 2f. Corresponding state (of optimal control) at t=0.5

7.2 Example: Consider the COCP, which was considered in example (7.1) but the control constraint is $U = [-1,2]$, and the desired control is given by

$v_d(\vec{x}, t) = -0.5 + 2t$

with the initial control

$$v_0(\vec{x}, t) = \begin{cases} -0.3, & 0 \le t \le 0.3 \\ 0.5, & 0.3 < t \le 0.6 \\ 1.1, & 0.6 < t \le 1 \end{cases}$$

Algorithm (7.1) is used here to solve the above problem. The given initial control and its corresponding state are given in the following figures

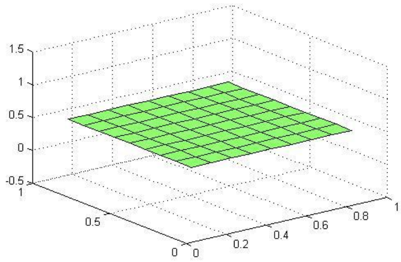

Figure 2a. Initial control at t=0.5

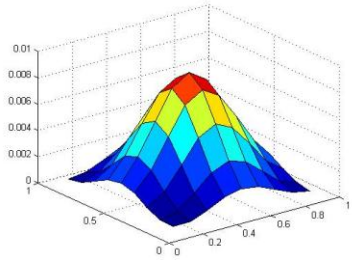

Figure 2b. Corresponding initial state at t=0.5

Depending on the above initial control and its corresponding state), we have the following results:

(I) In the GARM: the optimal control and corresponding state are obtained after 15 iterations, the results show with $G_0(v^m)$ =5.9050e-08, $\zeta_m =$ 6.6150e-04, and δ_m=4.1888e-05

The following figures are obtained in the optimal control and its corresponding state.

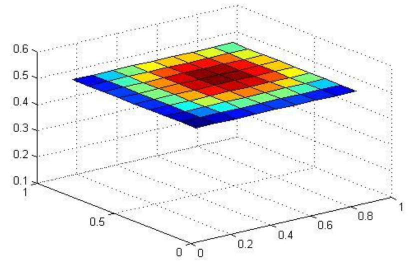

Figure 2c. Optimal control at t=0.5

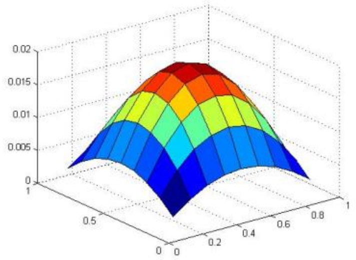

Figure 2d. Corresponding state (of optimal control) at t=0.5

(II) In the FWARM, the optimal control and corresponding state, are obtained after 343 iterations with $G_0(v^m)$=6.8025e-08, ζ_m =6.6150e-04, and δ_m=5.9350e-04 The optimal control and its corresponding state are shown by the following figures

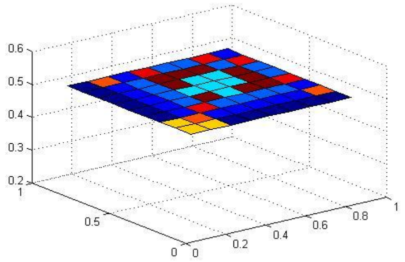

Figure 2e. Optimal control at t=0.5

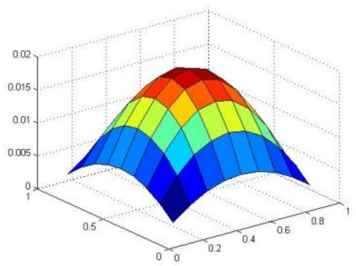

Figure 2f. Corresponding state (of optimal control) at t=0.5

CONCLUSION:

1)The existence theorem of a unique discrete solution for the discrete state equation when the discrete control is fixed is proved by using the GFEIM.

2)The existence theorem of a discrete classical optimal control (is stated and proved. The necessary condition for the optimality of the DCOCP problem is proved.

3) Depending on the results of the above examples, we conclude that (with step length of space variable $h = 0.1$, and step length of time $\Delta t = 0.05$) :

a) The GFEIM, which is used to solve the DSE of hyperbolic boundary value problem as well as the discrete adjoint equation for the state equation. This method is fast and efficient than the finite differences and many other methods.

b) The GARM and the FWARM, which are used to find the minimum value of the cost function. They are suitable and efficient methods to find the DCOC governed by hyperbolic boundary value problem, with parameters $\rho = 0.5$, $b = 0.5$ and $c = 0.5$ in the ARM, but the results which are obtained in the GARM is better than the FWARM and the required time to run the program of the first method is little than the second one.

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