THE SOLUTIONS GROWTH OF 2ND ORDER DES WHERE THE COEFFICIENTS ARE MEROMORPHIC BELONG TO EDREI-FUCHS CLASS

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ABSTRACT: *In this paper, we shall consider the second order linear complex differential equations with meromorphic coefficients. One of the coefficients belongs to Edrei-Fuchs class and the other one satisfy some conditions under which any nontrivial solution of mentioned equation is of infinite order.*

Key words: Edrei-Fuchs class, deficient value, meromorphic function, order, lower order

.1. INTRODUCTION

In our paper, we shall investigate the following complex DE

$$f'' + A(z)f' + B(z)f = 0 \quad (1)$$

where the functions $A(z)$ & $B(z) \not\equiv 0$ are meromorphic.

The fundamental concepts of the theory of value distribution of meromorphic functions are used [1, 2]. By $\rho(f)$ and $\mu(f)$ we meant the order and lower order, respectively, of the meromorphic function $f(z)$.

Some equations like (1) have a finite order nontrivial solution, for instance, $f(z) = e^{z}$ has order one and it is a solution of $f \ + e^{-z} f' - (e^{-z}+1)f = 0$. Hence, we ask the following question: what condition(s) on $A(z)$, $B(z)$ which gives us guarantee that any solution of (1) has order equal to infinite?. Here, there are much works give the answer of this question [3, 4, 5-8].

The authors in [3] proved the following results:

Theorem A Suppose that $A(z)$ is a finite order entire function with finite deficient value, $B(z)$ is a transcendental entire function with $\mu(B) < \frac{1}{2}$. Then any solution $f \neq 0$ of (1) has $\rho(f) = \infty$.

Theorem B Suppose that $A(z)$ satisfy the hypothesis of Theorem A, $B(z) \neq 0$ is an entire function. Assume that a constants $\alpha > 0,\ \beta > 0$ exists, and for every $\varepsilon > 0$, two sets of reals $\{\varphi_k\}$, $\{\theta_k\}$ of finite elements satisfies $\varphi_1 < \theta_1 < \varphi_2 < \theta_2 < \cdots < \varphi_m < \theta_m < \varphi_{m+1} (\varphi_{m+1} = \varphi_1 + 2\pi)$, and

$$\sum_{k=1}^{m} (\varphi_{k+1} - \varphi_k) < \varepsilon$$

for which

$$|B(z)| \geq \exp\{(1 + o(1))\alpha |z|^{\beta}\}$$

when $z \longrightarrow \infty$ in $\varphi_k \leq arg z \leq \theta_k$, $(k = 1,2,\dots,m)$. Then any solution $f \neq 0$ of (1) has $\rho(f) = \infty$.

Theorem C Suppose that $A(z)$ satisfy the hypothesis of Theorem A, $B(z) \neq 0$ is an entire function satisfy that for every $k > 0$

$$\lim_{|z|=r\longrightarrow\infty} \frac{|B(z)|}{r^{k}} = \infty$$

holds for $z \notin G$ where G is a set of r-values satisfying $m_l(G) < \infty$. Then any solution $f \neq 0$ of (1) has $\rho(f) = \infty$.

Definition 1.1 [9] Let $f(z)$ be a finite order meromorphic function in the whole complex plane with $0 < \rho(f) < \infty$. We say that a ray $\arg z = \theta$ that start from origin is a zeropole limit point (ZPL) of $f(z)$, if

$$\lim_{r\longrightarrow\infty} sup \frac{\log n\{\bar{\Omega}(\theta-\varepsilon,\theta+\varepsilon,r), f=0\} + \log n\{\bar{\Omega}(\theta-\varepsilon,\theta+\varepsilon,r), f=\infty\}}{logr} = \rho(f)$$

holds for every $\varepsilon > 0$.

Theorem D [10] Let $f(z)$ be as in Definition 1.1. Suppose, also, that $f(z)$ as q (ZPL) rays and p deficient values where $p \neq 0, \infty$. Then $p \leq q$.

$p \leq q$ mentioned in the previous Theorem is called the Edrei-Fuchs inequality.

Definition 1.2 [9] The meromorphic function $f(z)$ is said to belong to *Edrei-Fuchs (EF)* class if $f(z)$ satisfy the conditions mentioned in the previous Theorem with $p = q \geq 1$, i e, $f(z)$ has order finite and positive and q (ZPL) rays and $p \neq 0$ deficient values.

Definition 1.3 [8] Let $E \subseteq [1, \infty)$. We define the logarithmic measure of E by

$$m_l\,(E) = \int_E \frac{dt}{t}$$

We define the upper and lower logarithmic densities of E by

$$\overline{logdens} E = \lim_{r\to\infty} \sup \frac{m_l\,(E \cap [1, r])}{logr}$$

and

$$\underline{logdens} E = \lim_{r\to\infty} \inf \frac{m_l\,(E \cap [1, r])}{logr}$$

respectively. E has logarithmic density if $\overline{logdens} E = \underline{logdens} E$.

Our results in this paper depend heavily on $(EF)\, class$. Some examples shows that there are a functions belong to this class can be found in [9].

2. SOME NEEDED LEMMAS

In this section, we give some lemmas which will be used to prove our results.

Lemma 2.1 [11] Let (f, Γ) be a pair contains a finite order transcendental, meromorphic function f and

$$\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_q, j_q)\}$$

denote a set of distinct integers order pairs satisfying $k_i > j_i \geq 0$, $i = 1,2,\dots,q$. Let $\varepsilon > 0$ be a constant. Then the following hold:

1. There is $E_1 \subset [0, 2\pi)$ with zero linear measure, such that, when $\psi_0 \in [0, 2\pi) \setminus E_1$, then a real constant $R_0 = R_0(\psi_0) > 1$ exists such that, for z with $arg\, z = \psi_0$ $|z| \geq R_0$, and for each $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\rho-1+\varepsilon)} \quad (2)$$

2. There is $E_2 \subseteq (1,\infty)$ with $m_l(E_2) < \infty$, such that, for each z with $|z| \notin E_2 \cup [0,1]$ and, for each $(k,j) \in \Gamma$, we have (2).

3. There is $E_3 \subset [0,\infty)$ with linear measure is finite, such that

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\rho+\epsilon)} \quad (3)$$

for each z with $|z| \notin E_3$ and $(k,j) \in \Gamma$.

Lemma 2.2 [9] Let $A(z)$ be a finite order meromorphic function with $0 < \rho(A) < \infty$ that has p deficient values, $a_1, a_2, \dots, a_p, (p \geq 1)$, $B(z)$ be a finite order meromorphic function that has a deficient value ∞. Suppose that $\beta > 1$, $0 < \eta < \rho(A)$ are real constants. Then, a sequence $\{t_n\}$ exists satisfying

$$\lim_{n\to\infty} \frac{t_n^{\eta}}{T(t_n, A)} = 0 \quad (4)$$

Moreover, there is $F_n \subseteq [t_n, (\beta+1)t_n]$, with $m(F_n) \leq \frac{(\beta-1)t_n}{4}$ for any sufficiently large n such that for any $R \in [t_n, \beta t_n] \setminus F_n$, the argument θ sets, $E_v(R), (v = 1,2,\dots,p)$, $E_\infty(R)$ such that

$$m\big(E_v(R)\big) =: m\left(\left\{\theta \in [0,2\pi) \left| log \frac{1}{|A(Re^{i\theta}) - a_v|} \geq \frac{\delta_0}{4} T(R,A)\right.\right\}\right) \geq M_1 > 0 \quad (5)$$

and

$$m\big(E_\infty(R)\big) =: m\left(\left\{\theta \in [0,2\pi) \left| log\big|B\big(Re^{i\theta}\big)\big| \geq \frac{\delta_1}{4} T(R,B)\right.\right\}\right) \geq M_2 > 0 \quad (6)$$

holds where $M_1 > 0$, $M_2 > 0$ depending on $A, B, \delta_0 = \min_{1\le v\le p} \delta(a_v, A)$
$\delta_1 = \delta(\infty, B), \beta$ and η only.

Lemma 2.3 [3] Let $g(z)$ be a finite lower order entire function, with $0 < \mu(g) < \frac{1}{2}$, $A(z)$ be a meromorphic function, with finite order. Let a be a finite deficient value of $A(z)$ with deficiency $\delta = \delta(a, A) > 0$. Then, for any $\varepsilon > 0$, a sequence $R_n \longrightarrow \infty$ exists, such that

$$\big|g(R_n e^{i\varphi})\big| > \exp\{R_n^{\mu(g)-\varepsilon}\}, \quad \varphi \in [0,2\pi) \quad (7)$$

$$m(F_n) =: m\left\{\theta \in [0,2\pi): log\big|A\big(R_n e^{i\theta}\big) - a\big| \geq -\frac{\delta}{4} T(R_n, A)\right\} \geq d > 0 \quad (8)$$

hold, for all sufficiently large n, where d is a constant that depend on $\rho(A), \mu(g)$ and δ only.

Lemma 2.4 [4] Let w be an entire function and assume that, $|w'(z)|$ is un bounded on the ray, $arg\, z = \theta$. Then a sequence $z_n = r_n e^{i\theta}$ exists where $r_n \longrightarrow \infty$ such that $w'(z_n) \longrightarrow \infty$, and

$$\left|\frac{w(z_n)}{w'(z_n)}\right| \le \big(1 + o(1)\big)|z_n| \quad (9)$$

as $z_n \longrightarrow \infty$.

Lemma 2.5 [12] Suppose that $g(z)$ is a finite lower order entire function with $0 \leq \mu(g) < 1$. Then, for any $\alpha \in (\mu(g), 1)$, there is $E \subseteq [0,\infty)$ such that $\overline{logdens}E \geq 1 - \frac{\mu(g)}{\alpha}$, where $E = \{r \in [0,\infty) : m(r) > M(r) \cos \pi\alpha\}$, $m(r) = inf_{|z|=r}\, log|g(z)|$, $M(r) = sup_{|z|=r}\, log|g(z)|$

Remark [3] If g is an entire function with zero lower order, in view of Lemma 2.5, we only need to give a slightly modification which make the previous Lemma remains holds.

3. MAIN RESULTS

In this section, we give our main results which are the generalization of the results in [3,9]. In our proofs we shall let $a_1, a_2, \dots, a_p, p \geq 1$ be a deficient values of $A(z)$ with deficiencies $\delta(a_v, A) > 0$, $1 \leq v \leq p$.

Theorem 3.1 Let $A(z)$ be a meromorphic function belongs to EF class and let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2}$. Then every nontrivial solution f of (1) has infinite order.

Proof: Suppose that $f \neq 0$ is a solution of (1) with $\rho(f) < \infty$. From equation (1) we have

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)| \left|\frac{f'(z)}{f(z)}\right| \quad (10)$$

According to Lemma 2.1, a set $E_1 \subseteq (1,\infty)$ exists with $m_l(E_1) < \infty$, such that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, k = 1,2 \quad (11)$$

holds, for all z with $|z| = r \notin E_1 \cup [0, r_0], r_0 > 1$. There are the following two cases:

First Case: $0 < \mu(B) < \frac{1}{2}$. By using Lemma 2.2 and Lemma 2.3, there exists R_n with $R_n < R_{n-1}$, and $R_n \longrightarrow \infty$, such that

$$m\big(E_v(R_n)\big) =: m\left(\left\{\theta \in [0,2\pi) \left| log \frac{1}{|A(R_n e^{i\theta}) - a_v|} \geq \frac{\delta}{4} T(R_n, A)\right.\right\}\right) \geq M_1 > 0 \quad (12)$$

$$\big|B\big(R_n e^{i\theta}\big)\big| > \exp\left\{R_n^{\frac{1}{2}\mu(B)}\right\}, \quad \theta \in [0,2\pi) \quad (13)$$

for every n.

For every $n \geq n_0$ we choose $\theta_n \in E_v(R_n)$. From (10-12), we get

$$|B(R_n e^{i\theta_n})| < |z|^{2\rho(f)}\left(1+\exp\left\{-\frac{\delta}{4}T(R_n,A)\right\} + |a|\right) \quad (14)$$

Thus

$$\exp\left\{R_n^{\frac{1}{2}\mu(B)}\right\} < |z|^{2\rho(f)}\left(1+\exp\left\{-\frac{\delta}{4}T(R_n,A)\right\} + |a|\right) \quad (15)$$

When $n \longrightarrow \infty$, this is a contradiction.

Second Case: $\mu(B)=0$. By Lemma 2.5, a set $E_2 \subset [0,\infty)$ exists with $\overline{logdens}E_2 = 1$, such that

$$log|B(z)| > \frac{\sqrt{2}}{2} logM(r,B) \quad (16)$$

for all z with $|z| = r \in E_2$ where, $M(r,B) = \max_{|z|=r}|B(z)|$. By the above Remark, there is a sequence R_n such that (12), (16) hold. By (10-12) we get (14). Thus, from (14), (16) we obtain

$$M(r,B)^{\frac{\sqrt{2}}{2}} \leq R_n^{2\rho(f)}\left(1+\exp\left\{-\frac{\delta}{4}T(R_n,A)\right\} + |a|\right) \quad (17)$$

Since $B(z)$ is a transcendental, we get

$$\lim_{r\to\infty}\inf\frac{logM(r,B)}{logr} = \infty$$

Thus, the contradiction is obtained from (17). The proof is now completed. □

Theorem 3.2 Let $A(z) \in EF$ be a meromorphic function, $B(z) \neq 0$ be an entire function. Suppose that, two constants $\alpha > 0$ and $\beta > 0$ exists, and for every $\varepsilon > 0$, two of finite collections of real $\{\varphi_k\}$, $\{\theta_k\}$ satisfying $\varphi_1 < \theta_1 < \varphi_2 < \theta_2 < \cdots < \varphi_m < \theta_m < \varphi_{m+1}(\varphi_{m+1} = \varphi_1 + 2\pi)$ and

$$\sum_{k=1"}^{m}(\varphi_{k+1}-\varphi_k) < \varepsilon \quad (18)$$

such that

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\} \quad (19)$$

where $z \longrightarrow \infty$ in $\varphi_k \leq argz \leq \theta_k$, $(k = 1,2,\ldots,m)$ and let

$$|B(z)| \leq \exp\{o(1)|z|^{\beta}\} \quad (20)$$

as $z \longrightarrow \infty$ in $\theta_1 \leq argz \leq \theta_2$. Then every solution $f \neq 0$ of (1) has $\rho(f) = \infty$.

Proof: Suppose there is $f \neq 0$ of (1) has $\rho(f) < \infty$. From (1) we have

$$|A(z)| \leq \left|\frac{f''(z)}{f'(z)}\right| + |B(z)|\left|\frac{f(z)}{f'(z)}\right| \quad (21)$$

By Lemma 2.1 there is $E_1 \subseteq [0,\infty)$ with finite linear measure, such that, for z with $|z| = r \notin E_1 \cup [0,r_1],(r_1 > 1)$ the inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{2\rho(f)} \quad (22)$$

holds.

By Lemma 2.2 and Lemma 2.3 there is a sequence R_n satisfying (12). Take $0 < \varepsilon < \frac{d}{2}$, then for any integer n , choose $\varphi_n \in E_v(R_n) \cap (\cup_{k=1}^{m}[\phi_k,\theta_k])$. Thus there is $k \in \{1,2,\ldots,m\}$ such that, for any n satisfies $\phi_n \in E_v(R_n) \cap [\phi_k,\theta_k]$ (for otherwise ϕ_n is replaced with subsequence ϕ_{n_j}). Hence, from our hypothesis, we get

$$|A(z_n)| \geq \exp\{(1 + o(1))\alpha|z_n|^{\beta}\} \quad (23)$$

as $n \longrightarrow \infty$.

Assume that $|f'(z)|$ is unbounded on $argz = \phi_0$ where $\phi_0 \in [\theta_1,\theta_2] \setminus E$. Thus, by Lemma 2.4, there is a sequence, $z_n = r_n\,exp(i\phi_0)$ where $r_n \longrightarrow \infty$, so that, $f'(z_n) \longrightarrow \infty$ and

$$\left|\frac{f(z_n)}{f'(z_n)}\right| \leq (1+o(1))|z_n| \quad (24)$$

Now combining 20-24 we get a contradiction. The proof is completed. □

Corollary 3.3 Suppose that $A(z)$ be a meromorphic function belongs to EF class and $B(z) \neq 0$ is an entire function. Also, suppose that, there is a constants $\alpha > 0$, $\beta > 0$ and for $\varepsilon > 0$, two sets of reals $\{\varphi_k\}$, $\{\theta_k\}$, satisfies $\varphi_1 < \theta_1 < \varphi_2 < \theta_2 < \cdots < \varphi_m < \theta_m < \varphi_{m+1},(\varphi_{m+1} = \varphi_1 + 2\pi)$, and

$$\sum_{k=1}^{m}(\varphi_{k+1}-\varphi_k) < \varepsilon \quad (25)$$

such that

$$|B(z)| \geq \exp\{(1+o(1)\alpha|z|^{\beta}\} \quad (26)$$

as $z \longrightarrow \infty$ in $\varphi_k \leq argz \leq \theta_k$, $(k = 1,2,\ldots,m)$. Then any solution $f \neq 0$ of (1) has $\rho(f) = \infty$.

Theorem 3.4 Let $A(z)(\neq 0) \in EF$ be a meromorphic function , $B(z) \neq 0$ be an entire function satisfying (20) and suppose that

$$\lim_{|z|=r\to\infty}\frac{|A(z)|}{r^k} = \infty \quad (27)$$

holds for any $k > 0$ and $z \notin G$ of r-values with $m_l(G) < \infty$. Then for any nontrivial solution f of (1) we have $\rho(f) = \infty$.

Proof: Assume that (1) possesses a finite order solution $f \neq 0$. Using Lemma 2.1 there is $E_2 \subseteq (1,\infty)$, with, $m_l(E_2) < \infty$, such that (22) holds for z with $|z| = r \notin E_2 \cup [0,r_2],(r_2 > 1)$. From Lemma 2.4 as in the proof of Theorem 3.2 we have

$$\left|\frac{f(z)}{f'(z)}\right| \leq (1+o(1))|z| \quad (28)$$

For $k > 0$ we have

$$\lim_{n\to\infty}\frac{|A(R_n e^{i\theta})|}{R_n^k} = \infty \quad (29)$$

Combining 20-22, 24 and 29 we get a contradiction. Therefore every nontrivial solution f of (1) has infinite order. □

Corollary 3.5 Suppose that $A(z) \in EF$ be a meromorphic function and $B(z) \neq 0$ is an entire function such that

$$\lim_{|z|=r\to\infty} \frac{|B(z)|}{r^k} = \infty \quad (30)$$

holds for any $k > 0$ and $z \notin G$ of r-values with $m_l(G) < \infty$. Then any nontrivial solution f of (1.1) is of infinite order.

REFERENCES

[1] Hayman W. K: "Meromorphic Functions", *Clarendon Press, Oxford*, (1964).

[2] Laine I: "Nevanlinna Theory and Complex Differential Equations", *Walter de Gruyter Berlin, NewYork,* (1993).

[3] Wu P. C. and ZHU J., "On the growth of solutions to the complex differential equation $f'' + Af' + Bf = 0$", *Science China Press and Springer-Verlag Berlin Heidelberg*, **54** (5): 939–947, (2011).

[4] Gundersen G., "Finite order solution of second order linear differential equations", *Trans American Mathematics Soc.,* **305**: 415–429, (1988).

[5] Chen Z. X., "The growth of solutions of second order linear differential equations with meromorphic coefficients", *Kodai Math Journal*, **22** (2): 208-221, (1999).

[6] Hellenstein S, Miles J and Rossi J, "On the growth of solutions of $f'' + gf' + hf = 0$", *Trans American Mathematical Society*, **324**: 693–705, (1991).

[7] Kwon K.H., "Nonexistence of finite order solution of certain second order linear differential equations", *Kodai Math Journal*, **19**: 378–387, (1996).

[8] Laine, I and Wu P. C., "Growth of solutions of second order linear differential equations", *Proc. American Mathematical Society*, **128** (9): 2693–2703, (2000).

[9] Wu P. C., Shengjian Wu and Jun Zhu, "On the growth of solutions of second order complex differential equation with meromorphic coefficients", *Journal of Inequalities and Applicatins*, **117** (1): 1-13, (2012).

[10] Zhang G. H., "Theory of entire and meromorphic functions, deficient and asymptotic values and singular directions", *American Mathematical Society*, **122**, (1993).

[11] Gundersen G., "Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates", *Journal London Mathematics. Soc.*, **37** (2): 88-104, (1988).

[12] Barry P. D., "Some theorems related to the $cos\pi\rho$ theorem", *Proceedings of the London Mathematics Soc.*, **21**: 334–360, (1970).