MULTIVALENT HARMONIC STAR LIKE FUNCTIONS OF COMPLEX ORDER DEFINED BY A LINEAR OPERATOR

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ABSTRACT: *Some properties of a subclass of harmonic functions defined by multiplier transformations like coefficient inequalities , distortion bounds and Extreme points are investigated in the present paper .*

Keywords. multivalent harmonic functions, multiplier transformation , Salagean operator, Extreme point and Distortion theorem .

1- INTRODUCTION

A complex valued function $f = u + iv$ which is continuous that defined in simple connected domain D is said to be harmonic if u and v are real in D , that is u and v are satisfy , respectively, the Laplace equations . Follows from the above every analytic function is complex-valued harmonic function .

Let $f(z) = h(z) + \overline{g(z)}$ where h and g are analytic in . h is called the analytic part and g is called the co-analytic part of f . For f to be locally univalent and sense-preserving in D it must satisfy the necessary and sufficient condition

$$|h'(z)| > |g'(z)| \text{ [4] in } D ,$$

Let $H(p)$ denote the set of all multivalent harmonic functions $f = h + \overline{g}$ that are sense-preserving in the open unit disk $\{z \in \mathbb{C}: |z| < 1\}$, h and g are of the form

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{(n+p)-1} z^{(n+p)-1} \quad ;$$

$$g(z) = \sum_{n=1}^{\infty} b_{(n+p)-1} z^{(n+p)-1} \; |b_1| = 0 . \qquad \textbf{(1)}$$

Recall that Clunie and Sheil –Small [4] and Jahangiri et al [6] are studied the class $H(1)$ of harmonic univalent functions . For $f \in H(p)$,the differential

operator D^m $(m \in N_0 = N \cup \{0\})$ of f was introduce by Jahangiri et al [5] for fixed positive integer m and for $f = h + \overline{g}$ given by (1) ,the modified Salagean operator $D^m f$ [5] is defined as

$$D^m f(z) = D^m h(z) + \overline{(-1)^m D^m g(z)} ;$$

$$p > m , z \in U \qquad \textbf{(2)}$$

Where

$$D^m h(z) = z^p + \sum_{n=2}^{\infty} \left(\frac{(n+p)-1}{p}\right)^m a_{(n+p)-1} z^{(n+p)-1}$$

And

$$D^m g(z) = \sum_{n=1}^{\infty} \left(\frac{(n+p)-1}{p}\right)^m b_{(n+p)-1} z^{(n+p)-1}$$

Ahuja and Jahangiri [1] are studied the class $H(p)$. Next for function $f = h + \overline{g}$ given by (1) which is belonging to $H(p)$,Cho and Srivastava [3] defined multiplier transformations , the modified multiplier transformation of f given by (1) is defined as

$$I^0_{\gamma,\beta} f(z) = D^0 f(z) = f(z) ,$$

$$I^1_{\gamma,\beta} f(z) = \frac{\gamma D^0 f(z) + \beta D^1 f(z)}{\beta + \gamma} = \frac{\gamma(f(z)) + \beta(f'(z))}{\beta + \gamma}$$

$$I^m_{\gamma,\beta} f(z) = I^1_{\gamma,\beta}\left(I^{m-1}_{\gamma,\beta} f(z)\right) ,$$

$$(m \in N_0) \qquad \textbf{(4)}$$

Where $0 \le \gamma \le \frac{\beta}{2}$ .If f is given by (1) ,then from (3) and (4) we see that

$$I^m_{\gamma,\beta} f(z) = z^p + \sum_{n=2}^{\infty} \left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1} z^{(n+p)-1} + (-1)^m \sum_{n=1}^{\infty} \left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1} z^{(n+p)-1} \quad , \textbf{(5)}$$

Define $SH(p,t,m,\delta,\gamma,\beta,\alpha)$ be a subclass of (p) ,

a functions $f \in SH(p,t,m,\delta,\gamma,\beta,\alpha)$ of the form (1) is in the class $H(p)$ if it is satisfy the following condition

$$R\left(p(1+te^{i\delta})\frac{I^{m+1}_{\gamma,\beta} f(z)}{I^m_{\gamma,\beta} f(z)} - pte^{i\delta}\right) \ge p\alpha \quad ,$$

$$0 \le \alpha < 1 , p \in N \qquad \textbf{(6)}$$

Where $I^m_{\gamma,\beta} f(z)$ is given by **(5)** .

Suppose $\overline{SH}(p,t,m,\delta,\gamma,\beta,\alpha)$ consisting of harmonic functions $f_m(z) = h(z) + \overline{g_m(z)}$

in $H(p)$ such that

$$h(z) = z^p - \sum_{n=2}^{\infty} a_{(n+p)-1} z^{(n+p)-1} \quad ;$$

$$g_m(z) = (-1)^m \sum_{n=1}^{\infty} b_{(n+p)-1} z^{(n+p)-1} ,$$

$$a_{(n+p)-1}, b_{(n+p)-1} \ge 0 \qquad \textbf{(7)}$$

Define

$SH^o(p,t,m,\delta,\gamma,\beta,\alpha) = SH(p,t,m,\delta,\gamma,\beta,\alpha) \cap H(p)$
and
$\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha) = \overline{SH}(p,t,m,\delta,\gamma,\beta,\alpha) \cap H(p)$.

2- MAIN RESULTS

Theorem 1 . Let $f = h + \overline{g}$ be so that h and g are given by (1) with $b_1 = 0$, and

$$\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]|a_{(n+p)-1}|$$

$$+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]|b_{(n+p)-1}|$$

$$\le p(1-\alpha) \qquad \textbf{(8)}$$

Where$0 \le \gamma \le \frac{\beta}{2}, m \in N_0$; $\frac{\gamma}{\beta+\gamma} \le \alpha \le \frac{\gamma}{\beta+\gamma}$
.Then f is sense-preserving ,harmonic univalent in U ,and $f \in SH^o(p,t,m,\delta,\gamma,\beta,\alpha)$.

Proof . If $z_1 \ne z_2$

$$\left|\frac{(f(z_1)-f(z_2))}{(h(z_1)-h(z_2))}\right| \ge 1 - \left|\frac{(g(z_1)-g(z_2))}{(h(z_1)-h(z_2))}\right|$$

$$= 1 - \left|\frac{\sum_{n=2}^{\infty} b_{(n+p)-1}\left(z_1^{(n+p)-1}-z_2^{(n+p)-1}\right)}{z_1^p+\sum_{n=2}^{\infty} a_{(n+p)-1}z_1^{(n+p)-1} - z_2^p + \sum_{n=2}^{\infty} a_{(n+p)-1}z_2^{(n+p)-1}}\right|$$

$$= 1 - \left|\frac{\sum_{n=2}^{\infty} b_{(n+p)-1}\left(z_1^{(n+p)-1}-z_2^{(n+p)-1}\right)}{(z_1^p-z_2^p)+\sum_{n=2}^{\infty} a_{(n+p)-1}\left(z_1^{(n+p)-1}-z_2^{(n+p)-1}\right)}\right|$$

$$> 1 - \frac{\sum_{n=2}^{\infty}((n+p)-1)|b_{(n+p)-1}|}{1-\sum_{n=2}^{\infty}((n+p)-1)|a_{(n+p)-1}|}$$

$$\ge 1 - \frac{\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}-\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]|b_{(n+p)-1}|}{\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]|a_{(n+p)-1}|}$$

$$\ge 0 .$$

Which proves univalence . Since

$$|h'(z)| = \left|pz^{p-1}+\sum_{n=2}^{\infty}((n+p)-1)a_{(n+p)-1}z^{(n+p)-2}\right| \ge |z|^{p-1}+\sum_{n=2}^{\infty}\left(\frac{(n+p)-1}{p}\right)|a_{(n+p)-1}||z|^{(n+p)-2}$$

$$\ge 1-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]|a_{(n+p)-1}|$$

$$\ge \sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]|b_{(n+p)-1}|$$

$$\ge \sum_{n=2}^{\infty}((n+p)-1)|b_{(n+p)-1}|\,|z|^{(n+p)-2} \ge |g'(z)|$$

This shows that f is sense-preserving in .
Using the fact that $R(w) \ge p\alpha$ if and only if $|(1-\alpha)+w| \ge |(1+\alpha)-w|$, it is suffices to show that

$$\left|p(1-\alpha)+\frac{p(1+te^{i\delta})I_{\gamma,\beta}^{m+1}f(z)-pte^{i\delta}I_{\gamma,\beta}^{m}f(z)}{I_{\gamma,\beta}^{m}f(z)}\right| - \left|p(1+\alpha)-\frac{p(1+te^{i\delta})I_{\gamma,\beta}^{m+1}f(z)-pte^{i\delta}I_{\gamma,\beta}^{m}f(z)}{I_{\gamma,\beta}^{m}f(z)}\right| \ge 0$$

$$= \left|p(1-\alpha)I_{\gamma,\beta}^{m}f(z)+p(1+te^{i\delta})I_{\gamma,\beta}^{m+1}f(z)-pte^{i\delta}I_{\gamma,\beta}^{m}f(z)\right|$$

$$-\left|p(1+\alpha)I_{\gamma,\beta}^{m}f(z)-p(1+te^{i\delta})I_{\gamma,\beta}^{m+1}f(z)+pte^{i\delta}I_{\gamma,\beta}^{m}f(z)\right|$$

$$= \left|p(1-\alpha)\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}+(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)+\right.$$

$$p\left(1+te^{i\delta}\right)\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}-(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)-$$

$$pte^{i\delta}\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}+(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)\Bigg|$$

$$-\Bigg|p(1+\alpha)\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}+(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)-$$

$$p\left(1+te^{i\delta}\right)\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}-(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)+$$

$$pte^{i\delta}\left(z^p+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}z^{(n+p)-1}+(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}z^{(n+p)-1}\right)\Bigg|$$

$= \left|p(1-\alpha)+p\left(1+te^{i\delta}\right)-pte^{i\delta}z^p+\right.$

$\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p\left(1+te^{i\delta}\right)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)+\right.$

$\left.p(1-\alpha)-pte^{i\delta}\right]a_{(n+p)-1}z^{(n+p)-1}-$

$(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p\left(1+te^{i\delta}\right)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)\right.$

$\left.p(1-\alpha)+pte^{i\delta}\right]b_{(n+p)-1}z^{(n+p)-1}\Big|$

$-\left|\left(p(1+\alpha)-p\left(1+te^{i\delta}\right)+pte^{i\delta}\right)z^p\right.$

$-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p\left(1+te^{i\delta}\right)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)\right.$

$\left.-p(1+\alpha)-pte^{i\delta}\right]a_{(n+p)-1}z^{(n+p)-1}$

$$+(-1)^m\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m\left[p\left(1+te^{i\delta}\right)\right.$$

$$\left.\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(1+\alpha)+pte^{i\delta}\right]b_{(n+p)-1}z^{(n+p)-1}\Big|$$

$$\geq (p(2-\alpha)|z|^p-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m$$

$$\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)+p-p\alpha-pt\right]|a_{(n+p)-1}||z|^{(n+p)-1}$$

$-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)\right.$

$\left.-p+p\alpha+pt\right]|b_{(n+p)-1}||z|^{(n+p)-1}-p\alpha|z|^p$

$-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)\right.$

$\left.-p-p\alpha-pt\right]|a_{(n+p)-1}||z|^{(n+p)-1}$

$-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)\right.$

$\left.+p+p\alpha+pt\right]|b_{(n+p)-1}||z|^{(n+p)-1}$

$$\geq 2p(1-\alpha)|z|^p-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m$$

$$\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p\alpha-pt\right]|a_{(n+p)-1}|$$

$-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p\alpha+\right.$

$\left.pt\right]|b_{(n+p)-1}|$

$\geq p(1-\alpha)|z|^p$

$$p(1-\alpha)\left\{1-\sum_{n=2}^{\infty}\frac{1}{p(1-\alpha)}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]|a_{(n+p)-1}|-\sum_{n=2}^{\infty}\frac{1}{p(1-\alpha)}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}+p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)|b_{(n+p)-1}|\right\}\geq 0 \qquad \text{(by 8)}$$

Putting $(i)p=1,t=0$ $(ii)p=1,t=0,\beta=1$ in the above Theorem , we obtain

Corollary 1 . Let $f=h+\overline{g}$ be so that h and g are given by (1) with $b_1=0$. furthermore , let

$$\sum_{n=2}^{\infty}\left(\frac{\beta n+\gamma}{\beta+\gamma}\right)^{m}\left[\left(\frac{\beta n+\gamma}{\beta+\gamma}\right)-\alpha\right]|a_n|+$$

$$\sum_{n=2}^{\infty}\left(\frac{\beta n-\gamma}{\beta+\gamma}\right)^{m}\left[\left(\frac{\beta n-\gamma}{\beta+\gamma}\right)+\alpha\right]|b_n|\leq 1-\alpha$$

Where $0\leq\gamma\leq\frac{\beta}{2}, m\in N_0$; $\frac{\gamma}{\beta+\gamma}\leq\alpha\leq\frac{\gamma}{\beta+\gamma}$ .Then f is sense-preserving ,harmonic univalent in U .

Remark 1. We note that the result is obtained by Hasan Bayram and Sibel Yalcin [2] .

$$+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}+p(\alpha+t)\right]b_{(n+p)-1}$$

$$\leq p(1-\alpha) \qquad \textbf{(9)}$$

Where $0\leq\gamma\leq\frac{\beta}{2}, n\in N_0$; $\frac{\gamma}{\beta+\gamma}\leq\alpha\leq\frac{\gamma}{\beta+\gamma}$.

Corollary 2 . Let $f=h+\overline{g}$ be so that h and g are given by (1) with $b_1=0$. furthermore , let

$$\sum_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{m}\left[\left(\frac{n+\gamma}{1+\gamma}\right)-\alpha\right]|a_n|$$

$$+\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{m}\left[\left(\frac{n-\gamma}{\beta+\gamma}\right)+\alpha\right]|b_n|\leq 1-\alpha$$

Where $0\leq\gamma\leq\frac{1}{2}, m\in N_0$; $\frac{\gamma}{1+\gamma}\leq\alpha\leq\frac{1}{1+\gamma}$.Then f is sense-preserving ,harmonic univalent in U .

Remark 2. We note that the result is obtained by Yasar and Sibel Yalcin [7] .

Theorem 2. Let $f_m=h+\overline{g_m}$ be so that h and g are given by **(7)** with $b_1=0$ .Then $f\in\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ if and only if

Proof . The " if " part functions holds immediately by Theorem 1 also noting that $\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)\subset SH^o(p,t,m,\delta,\gamma,\beta,\alpha)$. For the "only if " part , if the condition **(9)** is not hold we show that $f_n\notin\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$. Note that a necessary and sufficient condition for $f_m=h+\overline{g_m}$ given by **(9)** to be in $\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ is that the condition **(8)** to be satisfied . This is equivalent to

$$R\left(\frac{p(1-\alpha)z^p\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}+\gamma}{\beta+\gamma}\right)^{m}}{z^p-\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}+\gamma}{\beta+\gamma}\right)^{m}a_{(n+p)-1}z^{(n+p)-1}+\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}+\gamma}{\beta+\gamma}\right)^{m}b_{(n+p)-1}z^{(n+p)-1}}\right.$$

$$\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]a_{(n+p)-1}z^{(n+p)-1}$$

$$+\frac{\sum_{n=2}^{\infty}\left(\frac{\beta\frac{(n+p)-1}{p}-\gamma}{\beta+\gamma}\right)^{m}}{z^p-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^{m}a_{(n+p)-1}z^{(n+p)-1}+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}b_{(n+p)-1}z^{(n+p)-1}}$$

$$\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right] b_{(n+p)-1}z^{(n+p)-1}\Bigg) \geq 0$$

This hold for all values of , such that $|z| = r < 1$. Choosing z on the positive real axies such that $0 \leq z = r < 1$ we obtain

$$\frac{p(1-\alpha)-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]a_{(n+p)-1}r^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}r^{n-1}+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}r^{n-1}}$$

$$-\frac{\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]a_{(n+p)-1}r^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m a_{(n+p)-1}r^{n-1}+\sum_{n=2}^{\infty}\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m b_{(n+p)-1}r^{n-1}}$$

$$\geq 0 \qquad \textbf{(10)}$$

The numerator in **(10)** is negative for r sufficiently close to 1 if condition **(9)** does not hold ,therefore there exist $z_0 = r_0$ in (0,1) for which the quotient in **(10)** is negative .But this contradicts the required condition for $f_m \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ and so the proof is complete .

Theorem 3. Let f_m is agiven by **(7)** .Then $f_m \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ if and only if

$$f_m(z) = \sum_{n=1}^{\infty} X_{(n+p)-1}h_{(n+p)-1}(z) + Y_{(n+p)-1}g_{m_{((p+n)-1)}}(z)$$

$$h_p(z) = z^p,\ h_{(n+p)-1}(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]} z^{(n+p)-1};$$

$(n = 2,3, \ldots.)$

$$g_{m_p}(z) = z^p,$$

$$g_{m_{((p+n)-1)}}(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]} z^{(n+p)-1};\ (n = 1,2, \ldots.)$$

$\sum_{n=1}^{\infty}\left(X_{(n+p)-1} + Y_{(n+p)-1}\right) = 1$, $X_{(n+p)-1} \geq 0$ and $Y_{(n+p)-1} \geq 0$; $0 \leq \gamma \leq \frac{\beta}{2}, n \in N_0$;

$\frac{\gamma}{\beta+\gamma} \leq \alpha \leq \frac{\gamma}{\beta+\gamma}$ .In particular the extreme points of $\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ are $\{h_{(n+p)-1}\}$ and

$\{g_{m_{(p+n-1)}}\}$.

Proof . suppose that f_m functions of the form **(7)** , we note that

$$f_m(z) = \sum_{n=1}^{\infty} X_{(n+p)-1}h_{(n+p)-1}(z) + Y_{(n+p)-1}g_{m_{((p+n)-1)}}(z)$$

$$= \sum_{n=1}^{\infty}\left(X_{(n+p)-1} + Y_{(n+p)-1}\right)z^p -$$

$$\sum_{n=2}^{\infty}\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}$$

$$X_{(n+p)-1}z^{(n+p)-1} + (-1)^m\sum_{n=2}^{\infty}\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m}$$

$$\frac{1}{\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}Y_{n+p-1}\,\overline{z^{n+p-1}}$$

Then $\sum_{n=2}^{\infty}\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}{p(1-\alpha)}$

$$\left(\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}X_{(n+p)-1}\right)$$

$$+\sum_{n=2}^{\infty}\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]}{p(1-\alpha)}\left(\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]}Y_{(n+p)-1}\right)$$

$$=\sum_{n=2}^{\infty}X_{(n+p)-1}+\sum_{n=2}^{\infty}Y_{(n+p)-1}=1-X_p-Y_p\leq 1$$

And so $f_m\in\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$. Conversely ,if $f_m\in\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$, then

$$a_{(n+p)-1}\leq\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}$$

And

$$b_{(n+p)-1}\leq\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]}$$

Set

$$X_{(n+p)-1}=\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}{p(1-\alpha)}a_{(n+p)-1}\ ,$$

$(n=2,3,\ldots.)$,

and

$$Y_{n+p-1}=\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]}{p(1-\alpha)}b_{n+p-1}\ ,$$

$(n=1,2,\ldots.)$,and

$X_p+Y_p=1-\sum_{n=2}^{\infty}\left(X_{(n+p)-1}+Y_{(n+p)-1}\right)$

Where $X_{(n+p)-1}\geq 0$ and $Y_{(n+p)-1}\geq 0$. Therefore, we obtain $f_m(z)=\left(X_p+Y_p\right)z^p+\sum_{n=2}^{\infty}X_{(n+p)-1}h_{(n+p)-1}(z)+Y_{(n+p)-1}g_{m_{((p+n)-1)}}(z)$.

Theorem 4. Let $f_m\in\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ .Then for $|z|=r<1$ and $0\leq\gamma\leq\frac{\beta}{2}, n\in N_0$; $\frac{\gamma}{\beta+\gamma}\leq\alpha\leq\frac{\gamma}{\beta+\gamma}$ we have

$$|f_m(z)|\leq r^p+\frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)^{m}\left[p(1+t)\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}r^{p+1}\ ,$$

and

$$|f_m(z)| \geq r^p - \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{p+1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{p+1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]} r^{p+1}$$

Proof . Let $f_m \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$. sinc $|f_m(z)| \leq r^p + \sum_{n=2}^{\infty}(a_{(n+p)-1} + b_{(n+p)-1}) r^{p+1}$

$$\leq r^p + \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]} r^{p+1}$$

$$\sum_{n=2}^{\infty}\left\{\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]}{p(1-\alpha)} a_{(n+p)-1} + \frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)+p(\alpha+t)\right]}{p(1-\alpha)} b_{(n+p)-1}\right\}$$

$$\leq r^p + \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]} r^{p+1}$$

The proof for the left hand inequality is similar and will be omitted .

Corollary 1. Let f_m is given by **(7)** such that $f_m \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ where

$0 \leq \gamma \leq \frac{\beta}{2}, n \in N_0 ; \frac{\gamma}{\beta+\gamma} \leq \alpha \leq \frac{\gamma}{\beta+\gamma}$ .Then

$$\left\{ w: |w| < 1 - \frac{p(1-\alpha)}{\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\left(\frac{\beta\left(\frac{p+1}{p}\right)+\gamma}{\beta+\gamma}\right)-p(\alpha+t)\right]} \right\}$$

$\subset f_m(U)$

Theorem 5. The class $\overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ is closed under the convex combinations .

Proof . Let $f_{m_i} \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$ for $= 1,2, \ldots \ldots$, where f_{m_i} is given by

$$f_{m_i}(z) = z^p - \sum_{n=2}^{\infty} a_{((n+p)-1)_i} z^{(n+p)-1}$$

$$+(-1)^m \sum_{n=1}^{\infty} b_{((n+p)-1)_i} \overline{z}^{(n+p)-1} \quad .$$

Then by **(9)**

$$\sum_{n=2}^{\infty}\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}-p(\alpha+t)\right]}{p(1-\alpha)} a_{((n+p)-1)_i} + \sum_{n=2}^{\infty}\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}+p(\alpha+t)\right]}{p(1-\alpha)} b_{((n+p)-1)_i}$$

$$\leq 1 \qquad , \qquad \textbf{(11)}$$

fo

r $\sum_{i=1}^{\infty} t_i = 1$, $0 < t_i < 1$,the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z^p + \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{((n+p)-1)_i}\right) z^{(n+p)-1}$$

$$+(-1)^m \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{((n+p)-1)_i}\right) \overline{z}^{(n+p)-1}$$ Then by **(11)** ,

$$\sum_{n=2}^{\infty}\frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}-p(\alpha+t)\right]}{p(1-\alpha)}$$

$\left(\sum_{i=1}^{\infty} t_i\, a_{((n+p)-1)_i}\right) +$

$$(-1)^m \sum_{n=2}^{\infty} \frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma} + p(\alpha+t)\right]}{p(1-\alpha)}$$

$$\left(\sum_{i=1}^{\infty} t_i\, b_{((n+p)-1)_i}\right)$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} \frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma}\right)^m}{p(1-\alpha)} \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)+\gamma}{\beta+\gamma} - p(\alpha+t)\right] a_{((n+p)-1)_i} + \frac{\left(\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma}\right)^m \left[p(1+t)\frac{\beta\left(\frac{(n+p)-1}{p}\right)-\gamma}{\beta+\gamma} + p(\alpha+t)\right]}{p(1-\alpha)} b_{((n+p)-1)_i} \right\}$$

$\leq \sum_{i=1}^{\infty} t_i = 1$ This is the condition required by **(9)** and so $\sum_{i=1}^{\infty} t_i\, f_{m_i}(z) \in \overline{SH^o}(p,t,m,\delta,\gamma,\beta,\alpha)$.

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