JS-SEMIPRIME SUBMODULES AND SOME RELATED CONCEPTS

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ABSTRACT: *Let R be a commutative ring with identity and N is a proper submodule of an R-module M. A submodule N is said to be JS-semiprime if whenever $f^n(m) \in N + J(M)$ for some $f \in S = End(M)$, $m \in M$ and $n \in \mathbb{Z}^+$, implies that $f(m) \in N$ where $J(M)$ is the Jacobson radical of M . The goal of this paper is to study this new class of submodules. Some of the properties and characterizations for this concept are considered and proved.*

1. INTRODUCTION

In this paper, all rings are commutative and all modules are unitary. A submodule of an R- module M which Dauns [2] was named semiprime defined as follows : A proper submodule N of an R- module M is called semiprime, if whenever $r^n x \in \mathrm{N}$, $r \in \mathrm{R}$, $x \in \mathrm{M}$ and $n \in \mathbb{Z}^+$, implies that $rx \in \mathrm{N}$. In [5] was given the most important result that study gets , if N is a proper submodule of an R- module M , then N is semiprime , if and only, if for each $r \in \mathrm{R}$, $x \in \mathrm{M}$ such that $r^2 x \in \mathrm{N}$, then $rx \in \mathrm{N}$. The concept of S-semiprime submodules was introduced in [7] as follows: A proper submodule N of an R- module M is said to be S-semiprime, if whenever $f^2(m) \in \mathrm{N}$; $f \in \mathrm{End(M)}$ and $m \in \mathrm{M}$, implies that $f(m) \in \mathrm{N}$. The notion of J-semiprime submodule was defined in [6], where a proper submodule N of M is called J-semiprime, if whenever $r^n x \in \mathrm{N} + J(\mathrm{M})$, $r \in \mathrm{R}$, $x \in \mathrm{M}$ and $n \in \mathbb{Z}^+$, then $rx \in \mathrm{N}$; $J(\mathrm{M})$ refers to the Jacobson radical of M, where it has been defined as the intersection of all maximal submodules of M. In this article we will give a new class of submodules named JS- semiprime submodules, where a proper submodule N of an R-module M is called JS- semiprime, if whenever $f^n(x) \in \mathrm{N} + J(\mathrm{M})$; $f \in \mathrm{End(M)}$, $x \in \mathrm{M}$ and $n \in \mathbb{Z}^+$, implies that $f(x) \in \mathrm{N}$. We study this type of submodules and prove some new results that are useful in our scientific knowledge.

2. JS-Semiprime submodules

Recall that a proper submodule N of an R- module M is said to be S-semiprime submodule of M, if whenever $f^2(m) \in \mathrm{N}$, for some $f \in S = \mathrm{End(M)}$ and $m \in \mathrm{M}$, then $f(m) \in \mathrm{N}$, [7].

Now, we give the definition of the concept of JS-semiprime submodule.

Definition (2.1):

Let N be a proper submodule of an R- module M, then N is called JS-semiprime, if whenever $f^n(m) \in \mathrm{N} + J(\mathrm{M})$, for some $f \in S = \mathrm{End(M)}$, $m \in \mathrm{M}$ and $n \in \mathbb{Z}^+$, implies that $f(m) \in \mathrm{N}$.

Remarks and examples (2.2):

1) Every JS-semiprime submodule of an R-module M is J-semiprime.

Proof:

Suppose that N is an JS-semiprime submodule of M and let $r^n x \in \mathrm{N} + J(\mathrm{M})$ for some $r \in \mathrm{R}$, $x \in \mathrm{M}$ and $n \in \mathbb{Z}^+$. Define $f : \mathrm{M} \to \mathrm{M}$ by $f(m) = rm$, for all $m \in \mathrm{M}$, $f \in \mathrm{End(M)}$ and $f^n(x) = r^n x \in \mathrm{N} + J(\mathrm{M})$. But N is JS-semiprime submodule , thus $f(x) = r\, x \in \mathrm{N}$. This means that N is an J- semiprime submodule of M.

The converse of the previous remark is not true. For example, let $\mathrm{M} = \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module and $\mathrm{N} = 6\mathbb{Z} \oplus \mathbb{Z}$, then N is an J-semiprime submodule of M but it is not JS-semiprime, since if we define, $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, by $f(n,m) = (m,n)$,for all $n, m \in \mathbb{Z}$. It is clear that $f \in \mathrm{End(M)}$. Now $f^2(0,2) = f\big(f(0,2)\big) = f(2,0) = (0,2) \in \mathrm{N} + J(\mathrm{M})$, but $f(0,2) = (2,0) \notin \mathrm{N}$, this means that N is not JS-semiprime.

2) It is known that every J-semiprime submodule is semiprime, [6], therefore every JS-semiprime submodule is semiprime.

3) Every JS-semiprime submodule N of an R-module M is S-semiprime.

Proof:

Let $f^n(m) \in \mathrm{N}$ where $f \in \mathrm{End(M)}$, $m \in \mathrm{M}$ and $n \in \mathbb{Z}^+$. Since N is JS-semiprime submodule and $f^n(m) \in \mathrm{N} + J(\mathrm{M})$, therefore $f(m) \in \mathrm{N}$. Thus, N is an S-semiprime submodule .

The converse of the previous remark is not true in general. For example, the module $\mathrm{M} = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ as $\mathbb{Z}$-module, the submodule $\mathrm{N} = \{(0,0), (1,0)\}$ is an S-semiprime submodule of M, but it is not JS-semiprime, since N it is not J-semiprime, [6].

4) Let P be a prime number the module $\mathbb{Z}_{P^\infty}$ as $\mathbb{Z}$-module has no JS-semiprime submodule.

5) In the module $\mathrm{M} = \mathbb{Q}$ as $\mathbb{Z}$-module, the submodule 0_{M} is the only JS-semiprime submodule of $\mathbb{Q}$.

6) Let M be an R-module with $J(\mathrm{M}) = 0$ and N is a proper submodule of M, then N is S-semiprime submodule of M, if and only, if it is JS-semiprime.

The following proposition introduce a characterization of JS-semiprime submodules which can be proved easily.

Proposition (2.3):

Let N be a proper submodule of an R-module M, then N is an JS-semiprime submodule of M, if and only, if whenever $f^2(m) \in \mathrm{N} + J(\mathrm{M})$, for some $f \in \mathrm{End(M)}$ and $m \in \mathrm{M}$, implies that $f(m) \in \mathrm{N}$.

Proposition (2.4):

Let N be a submodule of an R-module M, then N is an JS-semiprime submodule if and only if for every submodule K of M such that $f^2(\mathrm{K}) \subseteq \mathrm{N} + J(\mathrm{M})$; $f \in \mathrm{End(M)}$ implies that $f(\mathrm{K}) \subseteq \mathrm{N}$.

Proof:-

Suppose that $f^2(\mathrm{K}) \subseteq \mathrm{N} + J(\mathrm{M})$; K is a submodule of M. If $f(\mathrm{K}) \nsubseteq \mathrm{N}$, then there exists $f(\mathrm{k}) \notin \mathrm{N}$; $\mathrm{k} \in \mathrm{K}$. But N is JS-semiprime submodule and $f^2(\mathrm{k}) \in \mathrm{N} + J(\mathrm{M})$, thus we get a contradiction. Therefore $f(\mathrm{K}) \subseteq \mathrm{N}$. Conversely let $f^2(m) \in \mathrm{N} + J(\mathrm{M})$; $m \in \mathrm{M}$ and $f \in \mathrm{End(M)}$, then $f^2(< m >) \subseteq \mathrm{N} + J(\mathrm{M})$, by assumption $f(< m >) \subseteq \mathrm{N}$. Therefore $f(m) \in \mathrm{N}$ and hence N is an JS-semiprime submodule of M.

A submodule N of an R-module M is said to be fully invariant if $f(\mathrm{N}) \subseteq \mathrm{N}$, for each $f \in \mathrm{End(M)}$, [3].

Proposition (2.5):

Let M be a nonzero R-module and N is a proper fully invariant submodule of M, then N is an JS-semiprime submodule of M, if and only, if $[\mathrm{N} + J(\mathrm{M}) : f^2(\mathrm{K})] \subseteq [\mathrm{N} : f(\mathrm{K})]$ for all $\mathrm{N} \subsetneq \mathrm{K}$ and for all $f \in \mathrm{End}(\mathrm{M})$.

Proof:-

Suppose that N is an JS-semiprime submodule of M, K is a submodule of M; $\mathrm{N} \subsetneq \mathrm{K}$ and $f \in \mathrm{End}(\mathrm{M})$. Let $r \in [\mathrm{N} + J(\mathrm{M}) : f^2(\mathrm{K})]$, thus $rf^2(\mathrm{K}) \subseteq \mathrm{N} + J(\mathrm{M})$, thus $f^2(r\mathrm{K}) \subseteq \mathrm{N} + J(\mathrm{M})$. By assumption we get that $f(r\mathrm{K}) \subseteq \mathrm{N}$, and hence $rf(\mathrm{K}) \subseteq \mathrm{N}$, then $r \in [\mathrm{N}: f(\mathrm{K})]$. Therefore $[\mathrm{N} + J(\mathrm{M}) : f^2(\mathrm{K})] \subseteq [\mathrm{N}: f(\mathrm{K})]$. Conversely, assume that $f^2(m) \in \mathrm{N} + J(\mathrm{M})$ where $f \in \mathrm{End}(\mathrm{M})$, $m \in \mathrm{M}$. If $m \in \mathrm{N}$, then we are done. In case $m \notin \mathrm{N}$, then $[\mathrm{N} + J(\mathrm{M}) : f^2(\mathrm{N} + < m >)] \subseteq [\mathrm{N}: f(\mathrm{N}+< m >)]$. But $1 \in [\mathrm{N} + J(\mathrm{M}): f^2(\mathrm{N} + < m >)]$, hence $1 \in [\mathrm{N}: f(\mathrm{N}+< m >)]$. This implies that $f(\mathrm{N}+< m >) \subseteq \mathrm{N}$ and hence $f < m > \subseteq \mathrm{N}$, therefore $f(m) \in \mathrm{N}$. This means that N is an JS-semiprime submodule of M.

Corollary (2.6):

Let M be a nonzero R-module , then $\{0_\mathrm{M}\}$ is an JS-semiprime submodule of M, if and only, if $[J(\mathrm{M}): f^2(\mathrm{K})] \subseteq \mathrm{Ann}\, f(\mathrm{K})$ for all nonzero submodule K of M and $f \in \mathrm{End}(\mathrm{M})$.

An R-module M is said to be multiplication if for each submodule N of M, there exists an ideal I of R such that $\mathrm{N} = \mathrm{IM}$, [4].

The next proposition gives conditions for which J-semiprime submodules of M coincide with JS-semiprime submodules of M.

Proposition (2.7):

Let M be a nonzero multiplication module, then $\{0_\mathrm{M}\}$ is an J-semiprime submodule of M, if and only, if it is JS-semiprime submodule of M.

Proof:-

Let $m \in \mathrm{M}$ and $f \in \mathrm{End}$ (M), such that $f^2(m) \in J(\mathrm{M})$. If $f(m) = 0$, then we are done. If $f(m) \neq 0$, then $< f(m) > \neq 0$, but M is multiplication , therefore $< f(m) > = \mathrm{IM}$, for some ideal I of R. Now $\mathrm{I} < f(m) > = \mathrm{I}^2\mathrm{M}$. Thus $\mathrm{I} < f^2(m) > = \mathrm{I}^2 f(\mathrm{M})$. But $\mathrm{I} < f^2(m) > \subseteq J(\mathrm{M})$, hence $\mathrm{I}^2 f(\mathrm{M}) \subseteq J(\mathrm{M})$. Since $\{0_\mathrm{M}\}$ is J-semiprime submodule of M, therefore $\mathrm{I} < f(m) > = 0$, then $\mathrm{I}^2\mathrm{M} = 0$. By (remark (2) in (2.2)) we get that $\mathrm{IM} = 0$ and hence $< f(m) > = 0$, which is a contradication, therefore $f(m) = 0$. Thus $\{0_\mathrm{M}\}$ is an JS-semiprime submodule of a multiplication module M. The converse side from (remark (1) in (2.2)).

Now, we show some properties of JS-semiprime submodules and give there proofs.

Proposition (2.8):

Let N and K be JS-semiprime submodules of an R-module M, then $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule in M.

Proof:-

First, we see that $\mathrm{N} \cap \mathrm{K}$ is a proper submodule in M, since $\mathrm{N} \cap \mathrm{K} \subseteq \mathrm{N}$ and N is proper in M. Now, let $x \in \mathrm{M}$ and $f \in \mathrm{End}$ (M) such that $f^2(x) \in (\mathrm{N} \cap \mathrm{K}) + J(\mathrm{M})$. Therefore $f^2(x) \in (\mathrm{N} + J(\mathrm{M})) \cap (\mathrm{K} + J(\mathrm{M}))$, hence $f^2(x) \in \mathrm{N} + J(\mathrm{M})$ and $f^2(x) \in \mathrm{K} + J(\mathrm{M})$, since both N and K are JS- semiprime , therefore $f(x) \in \mathrm{N} \cap \mathrm{K}$. This implies that $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule of M.

Proposition (2.9):

Let N and K be proper submodules of an R- module M such that $\mathrm{N} \subseteq \mathrm{K}$. If $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule of K , then N is an JS-semiprime submodule of K. Conversely if N is an JS-semiprime of K, then $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime of K.

Proof:-

Suppose that $f^2(x) \in \mathrm{N} + J(\mathrm{K})$, where $x \in \mathrm{K}$ and $f \in \mathrm{End}$ (K). Now $f^2(x) \in (\mathrm{N} + J(\mathrm{K})) \cap \mathrm{K}$, then $f^2(x) \in (\mathrm{N} \cap \mathrm{K}) + J(\mathrm{K})$, thus $f(x) \in \mathrm{N} \cap \mathrm{K}$ and hence $f(x) \in \mathrm{N}$. Therefore N is an JS-semiprime submodule of K. For the converse direction assume that $f^2(m) \in (\mathrm{N} \cap \mathrm{K}) + J(\mathrm{K})$, for $m \in \mathrm{K}$ and $f \in \mathrm{End}$ (K). Now, $f^2(m) \in (\mathrm{N} + J(\mathrm{K})) \cap (\mathrm{K} + J(\mathrm{K}))$, then $f^2(m) \in \mathrm{N} + J(\mathrm{K})$, but N is an JS-semiprime submodule of K, thus $f(m) \in \mathrm{N}$ and it is clear that $f(m) \in \mathrm{K}$, hence $f(m) \in \mathrm{N} \cap \mathrm{K}$, and we conclude that $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule of K.

Proposition (2.10):

Let N be an JS-semiprime submodule of an R-module M and K is a proper submodule of M such that $\mathrm{N} \subseteq \mathrm{K}$, $J(\mathrm{K}) = J(\mathrm{M})$ and $\mathrm{End}(\mathrm{M}) = \mathrm{End}(\mathrm{K})$ then $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule of K.

Proof:-

Since $\mathrm{N} \subseteq \mathrm{K}$, then $\mathrm{N} \cap \mathrm{K}$ is proper submodule of K. Now let $f^2(m) \in (\mathrm{N} \cap \mathrm{K}) + J(\mathrm{K})$ for some $m \in \mathrm{K}$ and $f \in \mathrm{End}$ (K). Therefore $f^2(m) \in \mathrm{N} + J(\mathrm{M})$, but N is an JS-semiprime of M, therefore $f(m) \in \mathrm{N}$. Also $f(m) \in \mathrm{K}$. This implies that $f(m) \in \mathrm{N} \cap \mathrm{K}$, and hence $\mathrm{N} \cap \mathrm{K}$ is an JS-semiprime submodule of K.

Definition (2.11): [1]

Let M, N and K be R-modules. Then M is said to be N-projective, if for each epimorphism $f: \mathrm{N} \to \mathrm{K}$ and each homomorphism $g: \mathrm{M} \to \mathrm{K}$ there is a homomorphism $h: \mathrm{M} \to \mathrm{N}$ such that the following diagram:

$$\begin{array}{ccccc} & & \mathrm{M} & & \\ & h \swarrow & \downarrow g & & \\ \mathrm{N} & \xrightarrow{f} & \mathrm{K} & \to & 0 \end{array}$$

Commutes, i.e. $f \circ h = g$.

Proposition (2.12):

Let f: $\mathrm{M} \to \mathrm{M}'$ be an epimorphism. If N is JS-semiprime submodule of M, such that $\ker f \subseteq \mathrm{N}$ and $\ker f \ll \mathrm{M}$, then $f(\mathrm{N})$ is an JS-semiprime submodule of M', where M' is M-projective module.

Proof:-

$f(\mathrm{N})$ is a proper of M', since if $f(\mathrm{N}) = \mathrm{M}'$,then $f(\mathrm{N}) = f(\mathrm{M})$, this implies that $\mathrm{M} = \mathrm{N}$, which is a contradiction. Therefore $f(\mathrm{N})$ is a proper submodule of M'. Now, let $h^2(m') \in f(\mathrm{N}) + J(\mathrm{M}')$ where $h \in \mathrm{End}$ (M') and $m' \in \mathrm{M}'$, we have to show that $h(m') \in f(\mathrm{N})$. Since f is an epimorphism and $m' \in \mathrm{M}'$, then there exists $m \in \mathrm{M}$ such that $f(m) = m'$.

Consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{M}' & & \\ & \mathrm{k} \swarrow & \downarrow h & & \\ \mathrm{M} & \xrightarrow{f} & \mathrm{M}' & \rightarrow & \mathrm{o} \end{array}$$

Since M' is M - projective module, then there exists a homomorphism K such that $f \circ \mathrm{K} = h$. Now, $h^2(m') = h\,(h\,(m')) \in f(\mathrm{N}) + J\,(\mathrm{M}')$. Thus $(f \circ \mathrm{k} \circ f \circ \mathrm{k} \circ f\,)(m) \in f(\mathrm{N}) + J(\mathrm{M}')$, and hence $f((\mathrm{k} \circ f)^2(m)) \in f(\mathrm{N}) + J(\mathrm{M}')$. But $\ker f \subseteq \mathrm{N}$ and $\ker f \ll \mathrm{M}$, therefore $((\mathrm{k} \circ f)^2(m)) \in \mathrm{N} + J(\mathrm{M})$. By assumption N is an JS-semiprime submodule of M, then $(\mathrm{k} \circ f)(m) \in \mathrm{N}$, and hence $h(f(m)) \in f(\mathrm{N})$. This implies that $h(m') \in f(\mathrm{N})$. Hence $f(\mathrm{N})$ is an JS-semiprime submodule of M'.

Corollary (2.13):

Let N be an JS-semiprime submodule of M and K is a submodule of M with $\mathrm{K} \subseteq \mathrm{N}$ and $\mathrm{K} \ll \mathrm{M}$, then $\frac{\mathrm{N}}{\mathrm{K}}$ is an JS-semiprime submodule of $\frac{\mathrm{M}}{\mathrm{K}}$, where $\frac{\mathrm{M}}{\mathrm{K}}$ is an M-projective module.

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