GALERKIN'S METHOD TO SOLVE THE LINEAR SECOND ORDER DELAY MULTI-VALUE PROBLEMS

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ABSTRACT: *In this work, the linear second order delay multi-value problems are studied. A special type of the above problems namely Sturm-Liouville problems has been introduced and the Galerkin's method is used to solve this type of problems.*

Keywords: Galerkin's method, sturm-Liouville problem, delay differential equation, delay eigen-value problem, delay multi-value problem

.INTRODUCTION

The concepts of the linear second order multi-value problems are generalized to be hold for the linear second order delay multi-value problems. This study is made especially for the Sturm-Liouville problems that coined in this paper as linear second order delay multi-value problems. Also one of the most common numerical methods named the Galerkin's is developed to solve this type of problems.

2. Basic definitions and facts

In this section, the basic definitions and facts related to this work are recalled, which starts with the following definition that given by Driver [1].

Definition 2.1

A delay differential equation is an equation in which the unknown function and some of its derivatives, evaluated at arguments which are different by any of fixed number or function of values.

Consider the n-th order delay differential equation:

$$K(x, y(x), y(x-\tau_1),\dots, y(x-\tau_m), y'(x), y'(x-\tau_1),\dots, y'(x-\tau_m), y^{(n)}(x),\dots, y^{(n)}(x-\tau_m)) = f(x) \quad (1)$$

where K is a given function and $\tau_1, \tau_2, \dots, \tau_m$ are given fixed positive numbers called the time delays, [2].

We say that equation (1) is homogenous delay differential equation in case $f(x)=0$, which we handle in this paper, otherwise this equation is called non-homogenous delay differential equation,(3).

For simplicity, we can write a first order linear delay differential equation with constant coefficients and one time delay as follows:

$$a_0 y'(x) + a_1 y'(x-\tau) + a_2 y(x) + a_3 y(x-\tau) = f(x) \quad (2)$$

where $f(x)$ is a given continuous function and τ is a positive constant and a_0, a_1, a_2 and a_3 are constants.

Now it is important to recall the following definition which is given by Halanay [5].

Definition 2.2

A delay Eigen-value problem is a problem in which the unknown Eigen-function and some of its derivatives, evaluated at arguments which are different by any of fixed number or function of values.

Consider the following linear second order delay Eigen-value Sturm- Liouville problem:

$$-(P(x)\, y'(x))' + (q(x) - \lambda\, r(x))y(x-\tau) = 0 \quad (3)$$

together with the boundary conditions:

$$\begin{aligned} a_1 y(a) &= a_2 y'(a) \qquad ,x \in [a-\tau, a] \\ b_1 y(b) &= b_2 y'(b) \qquad ,x \in [b-\tau, b] \qquad (4) \\ y(x-\tau) &= \varphi(x-\tau), \text{if } \; x-\tau < a \end{aligned}$$

where a_i, b_i $(i=1,2)$are prescribed constants such that not both in each case equal to zero, $\tau > 0$ is the time delay, p ,p', q and r are given real continuous functions in $[a, b]$, p and r are positive in $[a, b]$, λ is a constant plays the role of the eigen-value. φ is the initial function defined on $x \in [x_0 - \tau, x_0]$.

Hence, the new concept of this work is given by the following definition.

Definition 2.3

A delay multi-value problem is a problem in which the unknown multi-function and some of its derivatives, evaluated at arguments which are different by any of fixed number or function of values.

Consider the following linear second order delay boundary multi-value Sturm- Liouville problem:

$$(p_i(x)y_i'(x))' + (q_i(x) - \sum\lambda_j r_{ij}(x))y_i(x-\tau) = 0 \text{ , } i, j=1,2,\dots,n \quad (5)$$

together with the boundary conditions:

$$\begin{aligned} a_{i1}\, y_i(a) &= a_{i2}\, y_i'(a) \qquad ,x \in [a-\tau, a] \\ b_{i1}\, y_i(b) &= b_{i2}\, y_i'(b) \qquad ,x \in [b-\tau, b] \qquad (6) \\ y_i(x-\tau) &= \varphi_i(x-\tau), \text{if } \; x-\tau < a \end{aligned}$$

where a_{ik}, b_{ik} $(k=1,2)$are prescribed constants such that not both in each case equal to zero, $\tau > 0$ is the time delay, p_i, p_i', q_i and r_{ij} are given real continuous functions in $[a, b]$, p_i and r_{ij} are positive in $[a, b]$, λ_{ij} are constants play the role of the multi-values. φ_i are the initial functions defined on $x \in [x_0 - \tau, x_0]$.

Thus, all the facts that satisfied for the linear second order multi-value problems given by Bhattacharyya and others [6, 7, 8], are also hold for the problem given by equations (5) and (6). That is, the following remarks are hold:

Remark 2.1

All the delay eigen-values are real.

Remark 2.2

The delay eigen-functions of the problem given by equations (5) and (6) are orthogonal to the weight functions r_{ij} where $i=j$.

Remark 2.3

The delay eigen-functions are complete and normalized in $L^2[a,b]$.

Remark 2.4

There are infinite number of delay eigen-values forming a monotone increasing sequence with $\lambda_{ij} \to \infty$ as $j \to \infty$. Moreover, the delay eigen-functions corresponding to the delay eigen-values λ_{ij} has exactly j roots on the interval (a,b) for each $i=1,2,\dots,n$.

Remark 2.5

Each delay Eigen-value of the problem given by equations (5) and (6) correspond only one delay Eigen-function in $L^2[a,b]$.

More generally, if we rewrite the problem given by equations (5) and (6) as L $y=0$ where $y=[y_1, y_2, \ldots, y_n]$, L= [$L_1$, L_2, …,L_n].

$$L_i = -\frac{d}{dx^2} p_i(x) - \frac{d}{dx} p_i'(x) + A_i(x) q_i(x), \quad i = 1,2,\ldots,n \tag{7}$$

where $A_i(x)$ are operators defined by:

$A_i(x)y_i(x) = y_i(x-\tau)$

Then, the operator given by equation (7) is self-adjoint, (9). Therefore L satisfies the following property:

Proposition 2.1

The operator L is self-adjoint. Hence, the qualitative proof of the above proposition is the same as, the qualitative proof of the multi-value problems given by Volkmer and Binding, [10].

3. Galerkin's method

In this section we use the Galerkin's method to solve the problem given by equations (5) and (6). This method is one of the important methods that used to approximate the solution of differential and integral equations. The method is described as a special case of the weighted residual methods, [11].

For simplicity, fix $n=2$, therefore the problem given by equation (5) and (6) reduces to the following one:

$(p_1(x)y_1'(x))' + (q_1(x) - \sum\lambda_j r_{1j}(x))y_1(x-\tau) = 0,\ j=1, 2$ (8 a)

$(p_2(x)y_2'(x))' + (q_2(x) - \sum\lambda_j r_{2j}(x))y_2(x-\tau) = 0$ (8 b)

together with the boundary conditions

$a_{i1}\, y_i(a) = a_{i2}\, y_i'(a)$ $\quad ,x \in [a-\tau, a]$, $i=1, 2$ (9 a)

$b_{i1}\, y_i(b) = b_{i2}\, y_i'(b)$ $\quad ,x \in [b-\tau, b]$ (9 b)

$y_i(x-\tau) = \varphi_i(x-\tau)$, if $x-\tau < a$

where the same assumptions given for the problem given by equations (5) and (6) hold.

This method based on approximating the unknown function y_1 as a linear combination of n linearly independent functions $\{\alpha_t\}^n_{t=1}$. That is , write

$y_1 = \sum c_t\, \alpha_t,\ t=1,2,\ldots,n$ (10)

where c_t are the unknown parameters that must be determined. But this approximated solution must satisfy the boundary conditions given by equation (9 a). Therefore by substituting the new approximated solution into equation (8 a) one can get:

$R_1(x,\lambda_1,\lambda_2,\vec{c}\,) = -(p_1(x)\,(\sum c_t\alpha'_t(x))' + (q_1(x) - \sum\lambda_j r_{ij}(x))(\sum c_t\alpha_t(x-\tau))$ (11)

where R_1 is the error in the approximation of equation (11) and $\vec{c}$ is the vector of $n-2$ elements of c_t, $i,j=1,2$.

To find $\vec{c}$ and λ_1,λ_2 , choose n linearly independent $\{\beta_f\}^n_{f=1}$ which are orthogonal to R_1.

That is,

$$\int_a^b \beta_f(x) R_1(x,\lambda_1,\lambda_2,\vec{c})dx = 0 \qquad , f = 1,2,\ldots,n \tag{12}$$

From equation (10) we obtained that equation (12) can be expressed as:

$$\int_a^b \beta_f(x)\Big[-\Big(p_i(x)\sum_{t=1}^n c_t\alpha_t'(x)\Big)' + \Big(q_i(x) - \sum_{j=1}^n \lambda_j r_{ij}(x)\Big)\sum_{t=1}^n c_t\alpha_t(x-\tau)\Big]dx = 0 \tag{13}$$

where $f=1,2,\ldots,n$.

By evaluating the above equation at each f one can obtain a system of n nonlinear equations with n unknowns that can be solved easily.

To illustrate this method see the following example:

Example 3.1

Consider the following linear second order delay two-value Sturm- Liouville problem:

$$-y_1''(x) + (1+3x-\lambda_1-\lambda_2 x)y_1(x-1) = 0 \tag{14 a}$$

$$-y_2''(x) + (x+3-\lambda_1 x-\lambda_2)y_2(x-1) = 0 \tag{14 b}$$

together with the boundary conditions (15 a)

$y_2(1) = 0 \quad y_2(2) - y_2'(2) = 0 \qquad , x \in [1,2]$ (15 b)

$y_i(x-1) = x-1 \quad , i = 1,2.$

First, approximate the unknown function y_1 as a polynomial of degree three, i.e.,

$y_1(x) = \sum c_t\, x^{t-1}$, $x \in [0,1]$.

But, this approximated solution must satisfy the boundary conditions given by equation (15 a), therefore this solution reduces to:

$y_1(x) = c_2\, x - 2\, c_4\, x^2 + c_4\, x^3, \quad x \in [0,1]$.

That is,

$y_1(x-1) = c_2\,(x-1) - 2\,c_4\,(x-1)^2 + c_4\,(x-1)^3,\ x \in [1,2]$.

By substitute this approximated solution into equation (14 a) one can get:

$R_1(x,\lambda_1,\lambda_2,c_2,c_4) = 6c_4x - 10\,c_4 + (1+3x-\lambda_1-\lambda_2x)$
$.(c_2(x-1) + 6c_4(x-1) + c_4x^3 - 5c_4x^2)$ (16)

Choose the functions $1,x,x^2$ and x^3 to be orthogonal to the error function R_1 to get the following system of equations:

$$\int_1^2 R_1 dx = 0$$

$$\int_1^2 R_1 x dx = 0$$

$$\int_1^2 R_1 x^2 dx = 0$$

$$\int_1^2 R_1 x^3 dx = 0$$

Thus, the nontrivial solution of the above system is $\lambda_1=1$ $\lambda_2=3$, $c_4=0$ and $c_2\neq 0$.

Second, substitute the values of λ_1 and λ_2 into equation (14 b) to get: $-y_2''=0$. The solution of this equation is $y_2(x) = d_1 + d_2x$, $x\in [0,1]$, therefore $y_2(x-1) = d_1 + d_2(x-1)$ is the approximated solution for equation (14 b) where $x\in [1,2]$.

But, this approximated solution must satisfy the boundary conditions given by equation (15 b).Thus $d_1=0$ and $d_2\neq 0$.

Hence, $((1,3),(\ c_2x,\ d_2x))$ is the double Eigen-pair of the problem given by equations (14) and (15) where $x\in [0,1]$.

That is, $((1,3),(\ c_2(x-1),\ d_2(x-1)))$ is the nontrivial solution of equation (14) where $x\in [1,2]$.

Hint

Math-Cad software package has been used for help to solve the above problems and the programs are so easy that omitted.

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