SOLUTION SECOND–ORDER OF PARTIAL DIFFERENTIAL EQUATIONS BY LIE GROUP

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ABSTRACT: *In the recent task, solution of linear and nonlinear PDE's was established by Lie group the calculations of 2nd order.*

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1. INTRODUCTION

Symmetry methods for PDEs were first developed in the late 19th century by Marius Sophus Lie [1]. They are important in the study of PDEs arising in mathematics, physics, engineering and many other disciplines since they can be used to obtain special reduced solutions of the PDEs. Also PDEs have a wide range of applications in many fields, such as physics, engineering, and chemistry, which are fundamental for the mathematical formulation of continuum models [10, 11]. The Lie symmetry analysis method functions an remarkable turn in studying the PDEs [2-4].Over the past 120 years, the use of groups based on local symmetries originally due to Lie [8] has played an important role in obtaining invariant/similarity solutions of differential equations [3, 14–18]. Based on the symmetries, many useful properties of PDEs, such as symmetry reductions, similarity solutions, group classification, nonlocal symmetries, and conservation laws, can be investigated successively [5–9]. One of the heart subjects of Lie symmetry analysis is to obtain the group invariant solutions, it is very well known that this method is the most successful manner for area employment mathematics to explore accurate solutions of ODEs and PDEs [12, 13]. Furthermore author in [20] confirmed definition for (symmetry group and determining equation).In [21] author realize(invariant for 2nd order of PDE). Likewise in [22] proved (infinitesimal criterion for invariance of PDE).

2.Invariant of Scaler Partial Differential Equations, [19, 20]:

we concentrated the 2nd order case given by:

$$u_t = \Delta := \Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \quad \ldots(1)$$

Consider evolutionary PDEs of the second order with 2-independent variables (x,t) ,1-dependent variable u introduced by:

$$X^{[2]}\left(u_t - \Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})\right)\Big|_{u_t=\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})} = 0 \quad \ldots(2)$$

In terms of the coordinates $x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}$, for (1) becomes an algebraic equation that defines a hyper surface in $x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}$ - *space* , for any solution $u=G(x,t)$ of (1)write as:

$$\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = \Upsilon(x,t,G(x,t),\partial_x G,\partial_t G(x,t),\partial_{xx}G(x,t),\partial_{xt}G(x,t),\partial_{tt}G(x,t)) \quad \ldots(3)$$

2.1.Prolongation Formulas in Multidimension , [20]:

This following with respect to multidimensional situation (with independent $x^i, i=1,...,n$, dependent variables u^α , $\alpha=1,...,m$

Now, write the vector field as :

$$X = \zeta^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \ldots(4)$$

In the form

$$X^{(1)} = X + \zeta_i^\alpha \frac{\partial}{u_i^\alpha}$$

$$X^{(2)} = X^{(1)} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} \quad (5)$$

Wherever

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\zeta^j)$$

$$\zeta_{i.j}^\alpha = D_j(\zeta_i^\alpha) - u_{i.k}^\alpha D_j(\zeta^k) \quad \ldots(6)$$

And

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + ... \quad \ldots(7)$$

2.2.Extended Infinitesimal Transformations of 1-Dependent and 2- Independent variables, [14, 20]:

Determine a 1-parameter Lie group of point transformations

$v^* = V(t,x,u;\epsilon) = v + \epsilon\, \tau(t,x,u) + O(\epsilon^2)$

$n^* = N(t,x,u;\epsilon) = n + \epsilon\, \xi(t,x,u) + O(\epsilon^2)$ …(8)

$u^* = U(t,x,u;\epsilon) = u + \epsilon\, \eta(t,x,u) + O(\epsilon 2)$

Now , we presented the total derivation in this case is given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x}$$

...(9)

The extended infinitesimals is given by:

$$\begin{aligned}\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\zeta)\\ &= \eta_t + u_t(\eta_u - \tau_t) - u_t^2 \tau_u - u_x \zeta_t - u_x u_t \zeta_u\end{aligned}$$

…(10)

$$\begin{aligned}\zeta_x &= D_t(\eta) - u_t D_x(\tau) - u_x D_x(\zeta)\\ &= \eta_x + u_x(\eta_u - \zeta_x) - u_x^2 \zeta_u - u_t \tau_x - u_x u_t \tau_u\end{aligned}$$

…(11)

$$\begin{aligned}\zeta_{tx} = \zeta_{xt} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\zeta)\\ &= \eta_{xt} + u_x \eta_{tu} + u_{tx}(\eta_t - \tau_t) + u_t(\eta_{xt} + u_x \eta_{ut} - \tau_{tx} - u_x \tau_{ut})\\ &\quad - 2u_t u_{xt} \tau_u - u_t^2(\tau_{ux} + u_x \tau_{uu}) - u_{xx} \zeta_t - u_x(\zeta_{tx} + \zeta_{tu} u_x)\\ &\quad - u_{xx} u_t \zeta_u - u_x u_{tx} \zeta_u - u_x u_t(\zeta_{ux} + \zeta_{uu} u_x) - u_{tt}(\tau_x + u_x \tau_u)\\ &\quad - u_{tx}(\zeta_x + \zeta_u u_x)\end{aligned}$$

…(12)

$$\begin{aligned}\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\zeta)\\ &= \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + u_x^2 \eta_{uu} - 2u_{xx} \zeta_x - u_x \zeta_{xx}\\ &\quad - u_x^2 \zeta_{xu} - \tau_u(u_t u_{xx} + 2u_x u_{xt}) - u_x^3 \zeta_{uu} - 2u_{xt} \tau_x - u_t \tau_{xx}\\ &\quad - 2u_x u_t \tau_{xu} - u_x^2 u_t \tau_{uu} - 3u_x u_{xx} \zeta_u\end{aligned}$$

…(13)

$$\begin{aligned}\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{tx} D_x(\zeta)\\ &= \eta_{tt} + 2u_t \eta_{tu} + u_{tt} \eta_u + u_t^2 \eta_{uu} - 2u_{xt} \zeta_t - u_x \zeta_{tt}\\ &\quad - 2u_x u_t \zeta_{tu} - \zeta_u(u_x u_{tt} + 2u_t u_{xt}) - u_x u_t^2 \zeta_{uu} - 2u_{tt} \tau_t\\ &\quad - u_t \tau_{tt} - 2u_t^2 \tau_{tu} - 3u_t u_{tt} \tau_u - u_t^3 \tau_{uu}\end{aligned}$$

… (14)

$$\begin{aligned}\zeta_{xxt} &= D_t(\zeta_{xx}) - u_{xxt} D_t(\tau) - u_{xxx} D_t(\zeta)\\ &= \eta_{xxt} + \eta_{xxu} u_t + 2u_{xt} \eta_{xu} + 2u_x(\eta_{xtu} + \eta_{xuu} u_t) + u_{xxt} \eta_u\\ &\quad + u_{xx}(\eta_{tu} + \eta_{uu} u_t) + 2u_x u_{xt} \eta_{uu} + u_x^2(\eta_{tuu} + \eta_{uuu} u_t) - 2u_{xxt} \zeta_x\\ &\quad - 2u_{xx}(\zeta_{xt} - \zeta_{xu} u_t) - u_{xt} \zeta_{xx} - u_x(\zeta_{xxt} - \zeta_{xxu} u_t) - 2u_x u_{xt} \zeta_{xu}\\ &\quad - u_x^2(\zeta_{xtu} - \zeta_{xuu} u_t) - \tau_{ut}(u_t u_{xx} + 2u_x u_{xt}) - \tau_u(u_{tt} u_{xx} + u_t u_{xxt}\\ &\quad + 2u_{xt}^2 + 2u_x u_{xtt}) - 3u_x^2 u_{xt} \zeta_{uu} - u_x^3(\zeta_{tuu} - \zeta_{uuu} u_t) - 2u_{xtt} \tau_x\\ &\quad - 2u_{xt} u_t \tau_{xu} - 2u_{xt}(\tau_{xt} + \tau_{xu} u_t) - u_{tt} \tau_{xx} - u_t(\tau_{xxt} + \tau_{xxu} u_t)\\ &\quad - 2u_{xt} u_t \tau_{xu} - 2u_x u_{tx} \tau_{xu} - 2u_x u_t(\tau_{xtu} + \tau_{xuu} u_t) - 2u_x u_{xt} u_t \tau_{uu}\\ &\quad - u_x^2 u_{tt} \tau_{uu} - u_x^2 u_t(\tau_{tuu} + \tau_{uuu} u_t) - 3u_{xt} u_{xx} \zeta_u - 3u_x u_{xxt} \zeta_u\\ &\quad - 3u_x u_{xx}(\zeta_{ut} + \zeta_{uu} u_t) - u_{xxt}(\tau_t + \tau_u u_t) - u_{xxx}(\zeta_t + \zeta_u u_t)\end{aligned}$$

...(15)

3.Invariant solutions:

Consider a second order of PDE (1) that admits a 1-parameter Lie group of point transformations with generator introduced :

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \zeta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

...(16)

we assume that $\zeta(t,x,u) \neq 0$.

Definition (3.1),[3, 23]: $u = G(x,t)$ is an invariant solution of PDE (1) resulting from its admitted point symmetry with the infinitesimal generator of (16) iff :**1-** $u = G(x,t)$ is an **invariant surface** of (16) that is:

$X\left(u - G(t,x)\right)\Big|_{u=G(t,x)} = 0$ when X is defined by (16) ,i.e

$$\zeta_1(t,x,u)\frac{\partial G(t,x)}{\partial x} + \zeta_2(t,x,u)\frac{\partial G(t,x)}{\partial t} = \eta(x,t,u)$$

...(17)

2- $u = G(x,t)$ solves (1) that is

$\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0$ when $u = G(x,t)$, i.e

$\Upsilon(x,t,G(x,t),\partial_x G,\partial_t G(x,t),\partial_{xx} G(x,t),\partial_{xt} G(x,t),\partial_{tt} G(x,t)) = 0$

...(18)

Equation (17) is called the **invariant surface condition** for the invariant solutions resulting from invariance under the point symmetry (16).

Algorithm: We calculate the Lie point symmetry of PDE's (linear and non-linear) are given in the following steps:

Step1: Write all terms of equation (2) left-handed direction.

Step2: Write generator for symmetry with unknown ζ , τ *and* η in specific:

$$X = \tau(t,x,u).\frac{\partial}{\partial t} + \zeta(t,x,u).\frac{\partial}{\partial x} + \eta(t,x,u).\frac{\partial}{\partial u}$$

Step3: We need 2-prolongation in style:

$$X^{[2]}=X+\zeta^{t}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t}+\zeta^{x}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}+\zeta^{tt}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}}$$
$$+\zeta^{tx}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tx}}+\zeta^{xx}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{xx}}$$

Step4: Writ

$$X^{[2]}\left(u_t-\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})\right)\Big|_{u_t=\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})}=0$$

Step5: By using expansion given in equation (10-14)

Step6: Now replacing u_t by .

$$\Upsilon(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})$$

Step7: Find general solution.

4.Applications: The following are some examples of PDE's solved by using above algorithm.

Application (1): Consider the non-linear PDE given:

$$\frac{\partial u(x,t)}{\partial t}=u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+\left(\frac{\partial u(x,t)}{\partial x}\right)^2-\frac{\partial u(x,t)}{\partial x} \quad ...(1)$$

Write the generator given in Algorithm for ***step2*** and we needed the 2-prolongation introduced in ***step3*** .
Now ,apply the formula in ***step3*** to (1) we obtain:

$$X^{[2]}\left(\frac{\partial u(x,t)}{\partial t}-u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}-\left(\frac{\partial u(x,t)}{\partial x}\right)^2+\frac{\partial u(x,t)}{\partial x}\right)\Bigg|_{(1)=0}=0 \quad ...(2)$$

We resulted:

$$\zeta^{t}-u\zeta^{xx}+\zeta^{x}(1-2u_x)-\eta u_{xx}=0 \quad ...(3)$$

By using (25-29) we introduced the following:

$$\eta_t+u_t\eta_u-\tau_t u_t-u_t^2\tau_u-u_x\zeta_t-u_x u_t\zeta_u-u[\eta_{xx}+2u_x\eta_{xu}+u_{xx}\eta_u+u_x^2\eta_{uu}-2u_{xx}\zeta_x-u_x\zeta_{xx}-u_x^2\zeta_{xu}-\tau_u u_t u_{xx}+2u_x u_{xt})u_x^3\zeta_{uu}-2u_{xt}\tau_x-u_t\tau_{xx}-2u_x u_t\tau_{xu}-u_x^2u_t\tau_{uu}-3u_x u_{xx}\zeta_u]+\left[\eta_x+u_x\eta_u-\zeta_x u_x-u_x^2\zeta_u-u_t\tau_x-u_x u_t\tau_u\right](1-2u_x)-\eta u_{xx}=0 \quad ...(4)$$

Now, replacing u_t by left-handed direction for (1)we get:

$$\eta_t+(uu_{xx}+u_x^2-u_x)(\eta_u-\tau_t)-(uu_{xx}+u_x^2-u_x)^2\tau_u-u_x\zeta_t-(uu_{xx}+u_x^2-u_x)u_x\zeta_u-u[\eta_{xx}+2u_x\eta_{xu}+u_{xx}\eta_u+u_x^2\eta_{uu}-2u_{xx}\zeta_x-u_x\zeta_{xx}-u_x^2\zeta_{xu}-\tau_u[(uu_{xx}+u_x^2-u_x)u_{xx}+2u_x u_{xt}]-u_x^3\zeta_{uu}-2u_{xt}\tau_x-(uu_{xx}+u_x^2-u_x)\tau_{xx}-2(uu_{xx}+u_x^2-u_x)u_x\tau_{xu}-(uu_{xx}+u_x^2-u_x)u_x^2\tau_{uu}-3u_x u_{xx}\zeta_u]+[\eta_x+u_x\eta_u-\zeta_x u_x-u_x^2\zeta_u-(uu_{xx}+u_x^2-u_x)\tau_x-u_x(uu_{xx}+u_x^2-u_x)\tau_u](1-2u_x)-\eta u_{xx}=0 \quad ...(5)$$

By separation of the coefficient we get the following system:

$$u_{xx}: -u\eta_u+2u\zeta_x=0 \quad ...(6)$$

$$u_x: u(\eta_u-\tau_t-2\eta_{xu}+\zeta_{xx}-\tau_{xx})+u^2\tau_{xx}+\tau_t-\zeta_t=0 \quad ...(7)$$

$$u_x^2: \eta_u-\tau_t+\tau_u+u(-2\tau_u-\zeta_u-\eta_{uu}+\zeta_{xu}+\tau_{xx})+u^2(\tau_u+2\tau_{xu})=0 \quad ...(8)$$

$$u_x^3: -2\tau_u-\zeta_u+\zeta_{uu}+u(2\tau_u+2\tau_{xu}-\tau_{uu})+u^2\tau_{uu}=0 \quad ...(9)$$

$$u_x^4: \tau_u+u\tau_u=0 \quad ...(10)$$

$$u_x u_{xx}: u^2\tau_u+u(-\tau_u+3\zeta_u)=0 \quad ...(11)$$

$$u_x^2u_{xx}: u\tau_u=0 \quad ...(12)$$

$$u_x u_{xt}: 2u\tau_u=0 \quad ...(13)$$

$$u_{xt}: 2\tau_x=0 \quad ...(14)$$

$$1: \eta_t-\eta_x-u\eta_{xx}=0 \quad ...(15)$$

Then the general solution of above polynomial is introduced as:

$$\zeta(t,x,u)=\frac{1}{2}(x+t)c_1+\frac{1}{2}(x-t)c_3+c_4 \quad ..(16)$$

$$\tau(t,x,u)=c_1t+c_2 \quad ...(17)$$

$$\eta(t,x,u)=uc_3 \quad ...(18)$$

We get the following Lie point symmetry :

$$X_1=\frac{1}{2}(x+t)\frac{\partial}{\partial x}+t\frac{\partial}{\partial t} \quad ..(19)$$

$$X_2=\frac{\partial}{\partial t} \quad ...(20)$$

$$X_3=\frac{1}{2}(x-t)\frac{\partial}{\partial x}+u\frac{\partial}{\partial u} \quad ...(21)$$

$$X_4=\frac{\partial}{\partial x} \quad ...(22)$$

Application (2): Consider the PDE given in style:

$$\frac{\partial^2 u(x,t)}{\partial X^2}=\ln(\sin t)\frac{\partial u}{\partial t} \quad ..(1)$$

Record the generator decided in Algorithm for ***step2*** and we needed the 2-prolongation introduced in ***step3***. Now ,apply the formula in ***step3*** to (1) we result:

$$X^{[2]}\left(\frac{\partial^2 u(x,t)}{\partial^2 x}-\ln(\sin t)\frac{\partial u(x,t)}{\partial t}\right)\Bigg|_{(1)=0}=0 \quad ...(2)$$

We performed the following:

$$\zeta_{xx}-\ln(\sin t)\zeta_t-\cot t\,\tau u_t=0 \quad ...(3)$$

By using (25-29), we informed the following:

$$\eta_{xx}+2\eta_{xu}u_x-\zeta_{xx}u_x-\tau_{xx}u_t+\eta_u u_{xx}-2\zeta_x u_{xx}-2\tau_x u_{tx}+\eta_{uu}u_x^2-2\zeta_{xu}u_x^2-2\tau_{xu}u_t u_x-\zeta_{uu}u_x^3-\tau_{uu}u_t u_x^2-3\zeta_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_x u_{tx}-\ln(\sin t)\left[\eta_t+\eta_u u_t-\tau_t u_t-\zeta_t u_x-\tau_u u_t^2-\zeta_u u_t u_x\right]-\cot t\,\tau u_t=0 \quad ...(4)$$

Now , replacing $u_t = \frac{1}{\ln(\sin t)} u_{xx}$ we obtain the following result:

$$\eta_{xx} + 2\eta_{xu} u_x - \zeta_{xx} u_x - \tau_{xx} \frac{1}{\ln(\sin t)} u_{xx} + \eta_u u_{xx} - 2\zeta_x u_{xx} - 2\tau_x u_{tx} + \eta_{uu}(u_x)^2$$
$$-2\zeta_{xu}(u_x)^2 - 2\left(\frac{1}{\ln(\sin t)} u_{xx}\right)\tau_{xu} u_x - \zeta_{uu}(u_x)^3 - \tau_{uu}\left(\frac{1}{\ln(\sin t)} u_{xx}\right)(u_x)^2 - 3\zeta_u u_x u_{xx}$$
$$-\left(\frac{1}{\ln(\sin t)} u_{xx}\right)\tau_u u_{xx} - 2\tau_u u_x u_{tx} - \ln(\sin t)\begin{bmatrix} \eta_t + \eta_u\left(\frac{1}{\ln(\sin t)} u_{xx}\right) - \tau_t\left(\frac{1}{\ln(\sin t)} u_{xx}\right) \\ -\zeta_t u_x - \tau_u\left(\frac{1}{\ln(\sin t)} u_{xx}\right)^2 - \zeta_u\left(\frac{1}{\ln(\sin t)} u_{xx}\right) u_x \end{bmatrix}$$
$$-\cot t\, \tau \left(\frac{1}{\ln(\sin t)} u_{xx}\right)$$
...(5)

By separation of the coefficient we get the Lie symmetries:

$$u_{xx} : -\frac{1}{\ln(\sin t)} \tau_{xx} - 2\zeta_x + \tau_t + \cot\tau = 0 \quad ...(6)$$

$$u_x : 2\eta_{xu} - \zeta_{xx} + \ln(\sin t)\zeta_t = 0 \quad ...(7)$$

$$u_{xx} u_x : -\frac{2}{\ln(\sin t)} \tau_{xu} - 2\zeta_u = 0$$
...(8)

$$u_{tx} : -2\tau_x = 0 \quad ...(9)$$

$$u_x^2 : \eta_{uu} - 2\zeta_{xu} - \frac{1}{\ln(\sin t)} \tau_u = 0 \quad ...(10)$$

$$u_x^3 : -\zeta_{uu} = 0 \quad ...(11)$$

$$u_x u_{tx} : -2\tau_u = 0 \quad ...(12)$$

$$u_{xx}^2 : \frac{1}{\ln(\sin t)} \tau_u = 0 \quad ...(13)$$

$$u_{xx} u_x^2 : -\frac{1}{\ln(\sin t)} \tau_u = 0 \quad (14)$$

$$1 : \eta_{xx} - \ln(\sin t)\eta_t = 0 \quad .(15)$$

Then the general solution of above system given in the following:

$$\zeta = \frac{1}{2}(c_1 t + c_2)x - \frac{2c_4 t}{\ln(\sin t)} + c_6 \quad ...(16)$$

$$\tau = \frac{1}{2}c_1 t^2 + c_2 t + c_3 \quad ...(17)$$

$$\eta = \left[-c_1\left(\ln(\sin t)x^2 - 2t\right) + 8c_4 x + 8c_5\right]u + \alpha(x,t)$$
...(18)

We obtain the Lie symmetries:

$$X_1 = \frac{1}{2}t\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \left(\ln(\sin t)x^2 - 2t\right)u\frac{\partial}{\partial u}$$
...(19)

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \quad ...(20)$$

$$X_3 = \frac{\partial}{\partial t} \quad ...(21)$$

$$X_4 = \frac{-2t}{\ln(\sin t)}\frac{\partial}{\partial x} + 8xu\frac{\partial}{\partial u} \quad ...(22)$$

$$X_5 = 8u\frac{\partial}{\partial u} \quad ...(23)$$

$$X_6 = \frac{\partial}{\partial x} \quad ...(24)$$

$$X_\alpha = \alpha(x,t)\frac{\partial}{\partial u} \quad ...(25)$$

Application (3): Consider the PDE given introduced.

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\mu_0^2 x^2 \frac{\partial u(x,t)}{\partial x^2} + 2\mu_0^2 x \frac{\partial u(x,t)}{\partial x} + \mu_0 u(x,t)$$
...(1)

Where μ_0 is constant.

Now , to find the determining equation of (1) allow the generator decided in Algorithm for ***step2*** and we needed the 2-prolongation introduced in ***step3***. Now ,apply the formula in ***step3*** to (1) we result:

$$X^{[2]}\left(\frac{\partial u(x,t)}{\partial t} - \frac{1}{2}\mu_0^2 x^2 \frac{\partial u(x,t)}{\partial x^2} - 2\mu_0^2 x \frac{\partial u(x,t)}{\partial x} - \mu_0 u(x,t)\right)\Bigg|_{(1)=0} = 0$$
...(2)

Then ,the determining equations write of type:

$$\zeta_t - 2\mu_0^2 x \zeta_x - \frac{1}{2}\mu_0^2 x^2 \zeta_{xx} - \mu_0 \eta - \left(\mu_0^2 x u_{xx} + 2\mu_0^2 u_x\right)\zeta = 0$$
...(3)

Now ,to find the Lie point symmetries of (1) we must need ζ_t, ζ_x *and* ζ_{xx} we procure the next:

$$\eta_t + u_t(\eta_u - \tau_t) - u_t^2\tau_u - u_x\zeta_t - u_x u_t \zeta_u - 2\mu_0^2 x\left[\eta_x + u_x(\eta_u - \zeta_x) - u_x^2\zeta_u - u_t\tau_x - u_x u_t \tau_u\right]$$
$$-\frac{1}{2}\mu_0^2 x^2\begin{bmatrix} \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\zeta_x - u_x\zeta_{xx} \\ -u_x^2\zeta_{xu} - \tau_u\left(u_t u_{xx} + 2u_x u_{xt}\right) - u_x^3\zeta_{uu} - 2u_{xt}\tau_x \\ -u_t\tau_{xx} - 2u_x u_t \tau_{xu} - u_x^2 u_t \tau_{uu} - 3u_x u_{xx}\zeta_u \end{bmatrix} - \mu_0\eta - (\mu_0^2 x u_{xx} + 2\mu_0^2 u_x)\zeta = 0$$
...(4)

Now , replace u_t by left-handed direction of (1) we get:

$$\eta_t + \left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right)(\eta_u - \tau_t) - \left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right)^2\tau_u$$
$$-u_x\zeta_t - \left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right)u_x\zeta_u - 2\mu_0^2 x\begin{bmatrix} \eta_x + u_x(\eta_u - \zeta_x) - u_x^2\zeta_u \\ -\tau_x\left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x \right. \\ \left. + \mu_0 u(x,t)\right) - \tau_u u_x\left(\frac{1}{2}\mu_0^2 x^2 u_{xx} \right. \\ \left. + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right) \end{bmatrix}$$
$$-\frac{1}{2}\mu_0^2 x^2\begin{bmatrix} \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\zeta_x - u_x\zeta_{xx} \\ -u_x^2\zeta_{xu} - \tau_u u_{xx}\left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right) + 2u_x u_{xt} \end{bmatrix} - u_x^3\zeta_{uu} - 2u_{xt}\tau_x$$
$$-\tau_{xx}\left(\frac{1}{2}\mu_0^2 x^2 u_{xx} + 2\mu_0^2 x u_x + \mu_0 u(x,t)\right) - \mu_0\eta - (\mu_0^2 x u_{xx} + 2\mu_0^2 u_x)\zeta = 0$$
.(5)

By separation of coefficient yields:

$$u_{xx}: \frac{1}{2}\mu_0^2 x^2(\eta_u-\tau_t)- \mu_0^3 x^2 u\,\tau_u-\mu_0 u\,\tau_u+\mu_0^4 x^3\tau_x -\frac{1}{2}\mu_0^2 x^2\eta_u-\mu_0^2 x^2\zeta_x-\frac{1}{2}\mu_0^2 x^2\tau_{xx}=0 \quad ...(6)$$

$$u_{xx}^2: -\frac{1}{4}\mu_0^4 x^4\tau_u-\frac{1}{2}\mu_0^2 x^2\tau_u=0 \quad ...(7)$$

$$u_x: 2\mu_0^2 x(\eta_u-\tau_t)-4\mu_0^3 x\,u\,\tau_u-2\mu_0^2 x(\eta_u-\zeta_x)-2\mu_0^2 x\,\tau_x -\mu_0^2 x^2\eta_{xu}+\frac{1}{2}\mu_0^2 x^2\zeta_{xx}-2\mu_0^2 x\,\tau_{xx}-\mu_0 u\,\tau_u-\mu_0 u\,\zeta_u=0 \quad ...(8)$$

$$u_x^2: -4\mu_0^4 x^2\tau_u-2\mu_0^2 x\,\zeta_u+2\mu_0^2 x\,\zeta_u-2\mu_0^2 x\,\tau_u-\frac{1}{2}\mu_0 x^2\eta_{uu}+\frac{1}{2}\mu_0^2 x^2\zeta_{xu} \quad ...(9)$$

$$u_x u_{xx}: -\mu_0^4 x^3\tau_u-\frac{1}{2}\mu_0 x^2\zeta_u-\frac{1}{2}\mu_0^2 x^2\tau_u-2\mu_0^2 x\,\tau_u=0 \quad ...(10)$$

$$u_x^3: -\zeta_{uu}=0 \quad ...(11)$$

$$u_x u_{xt}: -2\tau_u=0 \quad ...(12)$$

$$u_{xt}: -2\tau_x=0$$

$$1: \eta_t-\mu_0\eta(\eta_u-\tau_t)-2\mu_0^3 u^3\tau_u-2\mu_0^2 x\,\eta_x-\mu_0 u\,\tau_x -\frac{1}{2}\mu_0^2 x^2\eta_{xx}-\mu_0 u\,\tau_{xx}=0 \quad ...(13)$$

Then the general solution given as:

$$\zeta(t,x,u)=\frac{1}{2}x\left[(c_1 t+c_2)\ln(x)+2c_4 t+2c_5\right] \quad ...(14)$$

$$\tau(t,x,u)=\frac{1}{2}c_1 t^2+c_2 t+c_3 \quad ...(15)$$

$$\eta(t,x,u)=-\begin{bmatrix}4\ln(x)^2 c_1+\left((-12c_1 t-12c_2)\mu_0^2\right)+16c_4\\ +9t(c_1 t+2c_2)\mu_0^2-8t(c_1 t+2c_2)\mu_0\\ +(4c_1-24c_4)t-16c_6)\mu_0^2\end{bmatrix}u+\alpha(t,x) \quad ...(16)$$

The Lie symmetries of (1) given as:

$$X_1=\frac{1}{2}x\,t\ln(x)\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}+\begin{pmatrix}-4\ln(x)^2+12t\,\mu_0^2\ln(x)\\ -9t^2\mu_0^4+8t^2\mu_0^3+4\mu_0^2\end{pmatrix}u\frac{\partial}{\partial u} \quad ...(17)$$

$$X_2=\frac{1}{2}x\ln(x)\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+\left(12\mu_0^2\ln(x)-18t\,\mu_0^4+16t\,\mu_0^3\right)u\frac{\partial}{\partial u} \quad ...(18)$$

$$X_3=\frac{\partial}{\partial t} \quad ...(19)$$

$$X_4=x\,t\frac{\partial}{\partial x}+\left(-16\ln(x)+24t\,\mu_0^2\right)u\frac{\partial}{\partial u} \quad ...(20)$$

$$X_5=x\frac{\partial}{\partial x} \quad ...(21)$$

$$X_6=16\mu_0^2 u\frac{\partial}{\partial u} \quad ...(22)$$

$$X_\alpha=\alpha(t,x)\frac{\partial}{\partial u} \quad ...(23)$$

5. DISCUSSION AND CONCLUSION

The goal of this assignment, established algorithm of Lie group method to solved the 2nd order of PDEs and likewise exercised of more variance applications of PDEs.

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