NS-PRIME SUBMODULES

Iman A. Athab
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
E-mail: iarqaqir310@gmail.com

ABSTRACT: Let R be a commutative ring with identity and M is a unitary R-module. A proper submodule N of M is said to be nearly S-prime (written as NS-prime) if whenever \( f(m) \in N \) for some \( f \in S = \text{End}(M) \) and \( m \in M \), implies that either \( m \in N + J(M) \) or \( f(M) \subseteq N + J(M) \), where \( J(M) \) is the Jacobson radical of M. The aim of this article is to study this class of submodules and interpret various properties for this type of submodule.

1. INTRODUCTION

In our research all rings will be commutative with identity and all modules are unital. A proper submodule \( N \) of an R-module M is called prime, if for each \( r \in R \) and \( x \in M \) with \( rx \in N \) implies that either \( x \in N \) or \( r \in [N:M] = \{ s \in R; sM \subseteq N \} \). In [6] it was introduced the concept of S-prime submodule as follows: A proper submodule \( N \) of an R-module M is said to be S-prime, if whenever \( f(m) \in N \) for some \( f \in \text{End}(M) \) and \( m \in M \) implies that either \( m \in N \) or \( f(N) \subseteq N \). In [6] it was stated that every S-prime submodule is a prime submodule.

The notion of nearly prime submodule (for short N-prime submodule) was given in [9], where a proper submodule \( N \) of an R-module M is said to be N-prime, if whenever \( r \in R \) and \( m \in M \) with \( rm \in N \), then either \( m \in N + J(M) \) or \( r \in [N + J(M):M] \), where \( J(M) \) is the Jacobson radical of M. In this article, we study a new class of submodules, which is called NS-prime submodules, where a proper submodule \( N \) of a module M is NS-prime, if for all \( f \in \text{End}(M) \) and for all \( m \in M \) with \( f(x) \in N \), implies that either \( m \in N + J(M) \) or \( f(M) \subseteq N + J(M) \). Many results concerned with this concept is presented. We give some properties and characterization about this type of submodules.

2. NS-prime submodules

Let R be a commutative ring with identity and let M be a unitary R-module. A proper submodule \( N \) of M is called S-prime, if whenever \( f(m) \in N \), for some \( f \in \text{End}(M) \) and \( m \in M \), implies that either \( m \in N \) or \( f(M) \subseteq N \). We introduce the following definition:

Definition (2.1):
A proper submodule \( N \) of an R-module M is said to be nearly S-prime \( (\text{short NS-prime}) \), if whenever \( f(m) \in N \), for some \( f \in \text{End}(M) \) and \( m \in M \), implies that either \( m \in N + J(M) \) or \( f(M) \subseteq N + J(M) \), where \( J(M) \) is the Jacobson radical of M.

Remarks and examples (2.2):
1) Every NS-prime submodule of an R-module M is N-prime.
2) Every proper submodule \( N \) of M is NS-prime if and only if \( f(N) \subseteq N + J(M) \) for all \( f \in \text{End}(M) \) and \( f(m) \in N \) for some \( m \in N + J(M) \).
3) Every NS-prime submodule of M is N-prime.

Proof:
Let \( N \) be an NS-prime submodule of \( M \) and Suppose that \( rm \in N \) for \( r \in R \) and \( m \in M \). Assume that \( m \in N + J(M) \).

Define \( f:M \to M \) by \( f(x) = rx \), \( x \in M \). Now, \( rm = f(m) \in N \), but \( N \) is NS-prime submodule of \( M \) and \( m \in N + J(M) \). Thus \( f(M) \subseteq N + J(M) \). Therefore \( rM \subseteq N + J(M) \) and hence \( N \) is N-prime.

The following example shows that the converse of the previous remark is not true in general.

Let \( M = \mathbb{Z}_2 + \mathbb{Z} \) as \( \mathbb{Z} \)-module, and let \( N = (0) + 2\mathbb{Z} \).

\( J(M) = 0 \). \( N \) is N-prime. Now, define \( f:M \to M \) by \( f(n,m) = (\bar{n},m) \) for all \( (n,m) \in \mathbb{Z}_2 + \mathbb{Z} \). \( f \in \text{End}(M) \) and \( f(\bar{1},2) = (0,2) \in (0) + 2\mathbb{Z} \).

But \( (\bar{1}, 2) \in N + J(M) \) and \( f(\mathbb{Z}_2 + \mathbb{Z}) = (0) + \mathbb{Z} \not\subseteq (0) + 2\mathbb{Z} \).

Therefore \( N \) is N-prime, but it is not NS-prime submodule of M.

3) Every NS-prime submodule of M is NS-prime submodule.

Proof:
Let \( m \in M \) and \( f \in S = \text{End}(M) \) with \( f(m) \in N \). Since N is S-prime submodule of M, thus either \( m \in N \) or \( f(M) \subseteq N \). Therefore either \( m \in N + J(M) \) or \( f(M) \subseteq N + J(M) \).

The converse of the remark (2) is not true in general. As the following example shows. For the module \( \mathbb{Z}_{p} \) as \( \mathbb{Z} \)-module, we have \( (0) \) is NS-prime submodule, but it is not S-prime submodule.

Recall that a proper submodule \( N \) of an R-module M is called NS-primary submodule of M, if whenever \( f(m) \in N \) for some \( f \in S = \text{End}(M) \) and \( m \in M \), implies that either \( m \in N + J(M) \) or \( f^n(M) \subseteq N + J(M) \) for some \( n \in \mathbb{Z}^+ \).

4) The rational number \( \mathbb{Q} \) as \( \mathbb{Z} \)-module has the \( \{ 0 \} \) submodule as the only NS-prime submodule of \( \mathbb{Q} \).

The following proposition gives a characterization of NS-prime submodules.

Proposition (2.3):
Let \( N \) be a submodule of an R-module M then N is NS-prime submodule if and only if for every submodule K of M such that \( f(K) \subseteq N \), \( f \in \text{End}(M) \) implies that either \( K \subseteq N + J(M) \) or \( f(M) \subseteq N + J(M) \).

Proof:
Assume that \( f(K) \subseteq N \), where K is a submodule of M. Suppose \( K \not\subseteq N + J(M) \), then there exists \( k \in K \setminus K \subseteq N + J(M) \). It is clear that \( f(k) \in N \), but N is NS-prime submodule, thus \( f(M) \subseteq N + J(M) \). Conversely let \( m \in M \) with \( f(m) \in N \), then \( f(m) \subseteq N + J(M) \) by assumption either \( m \subseteq N + J(M) \) or \( f(M) \subseteq N + J(M) \) this implies that N is NS-prime submodule.

If R is an integral domain and M is an R-module. An element \( x \in M \) is called a torsion element of M if \( ann(x) \neq 0 \). The set of all torsion elements of M which is denoted by T(M) is a submodule of M, \( \{ 2 \} \).
Proposition (2.4): \[10\]
Let $M$ be a module over an integral domain $R$, if $T(M) \neq M$ and $\ker f \subseteq T(M)$ for all $0 \neq f \in \text{End}(M)$, then $T(M)$ is an $S$-prime submodule of $M$.

From the previous proposition and (remark (2) in (2.2)) we can prove the following corollary.

Corollary (2.5):
Let $M$ be a module over an integral domain $R$, if $T(M) \neq M$ and $\ker f \subseteq T(M)$ for all $0 \neq f \in \text{End}(M)$, then $T(M)$ is an $NS$-prime submodule of $M$.

The following result gives some properties of $NS$-prime submodules.

Proposition (2.6):
Let $K$ be an $NS$-prime submodule of an $R$-module $M$ and let $N$ be a submodule of $M$ with $J(N) = J(M)$ and $N$ is $M$-injective then either $N \subseteq K$ or $K \cap N$ is $NS$-prime submodule of $N$.

Proof:-
Suppose that $N \nsubseteq K$, then $K \cap N$ is a proper submodule of $N$.

Now, let $f(x) \in K \cap N$ where $f \in \text{End}(N)$ and $x \in N$. If $x \notin (K \cap N) + J(N)$, we have to show that $f(N) \nsubseteq (K \cap N) + J(N)$. Consider the diagram.

Now the following proposition gives a characterization of $NS$-prime submodule.

Proposition (2.8):
If $M$ is a nonzero multiplication $R$-module, then $\{0_M\}$ is $N$-prime submodule of $M$, if and only, if it is $NS$-prime submodule of $M$.

Proof:-
Suppose that $f(m) = 0$, where $f \in \text{End}(M)$, and $m \in M$. If $m \notin \{0_M\} + J(M)$, then $m \neq 0$. We have to show that $f(M) \nsubseteq J(M)$, since $m \neq 0$, then $0 \neq (m) = IM$, for some ideal $I$ of $R$. Now, if $f(M) = 0$, then we are done. Suppose that $f(M) \neq 0$, since $M$ is multiplication module thus there exists a nonzero ideal $J$ of $R$ such that $f(M) = JM$. Now, $0 = f((im)) = f(M) = I(J(M)) \subseteq IM$, which implies that $J \subseteq \text{Ann}(IM)$. From proposition (2.7), we obtain $\text{Ann}(IM) \subseteq J(M)$, and hence $J \subseteq J(M)$, therefore $J \subseteq J(M)$, this implies that $f(M) \subseteq J(M)$. Therefore $\{0_M\}$ is $NS$-prime submodule of $M$. The converse side is clear from (remark (1) in (2.2)).

We introduce the concept of $NS$-prime module and prove some characterization of this concept.

Definition (2.9):
Let $M$ be an $R$-module. If $\{0_M\}$ is $NS$-prime submodule of $M$, then we say that $M$ is $NS$-prime module.

See the following:

Proposition (2.10):
If $M$ is a multiplication module, and $N$ is a submodule of $M$, then $N$ is $N$-prime submodule, if and only, if it is $NS$-prime submodule of $M$.

Proof:-
From [1, Corollary (3.22)], we get that $M \nsubseteq N$ is a multiplication module. Thus $N$ is $N$-prime submodule of $M$, if and only, if $N$ is $NS$-prime submodule of $M$.

Definition (2.11):[3]
Let $M$ and $M'$ be $R$-modules, the module $M'$ is called $M$-projective, if for every homomorphism $f: M' \rightarrow M$, where $K$ is a submodule of $M$, can be lifted to a homomorphism $g: M' \rightarrow M$.

Now, we can prove the following.

Proposition (2.12):
Let $f: M \rightarrow M'$ be an $R$-module epimorphism. If $N$ is $NS$-prime submodule of $M$ such that $\ker f \subseteq N$, then $f(N)$ is $NS$-prime submodule of $M'$, where $M'$ is $M$-projective module.

Proof:-
Now consider the following diagram:

$$
\begin{align*}
& M' \\
& \downarrow k \\
& M \\
& \downarrow f \\
& M' \rightarrow M\backslash \rightarrow 0
\end{align*}
$$
Since \( M' \) is \( M \)-projective, then there exists a homomorphism \( k \), such that \( f \circ k = h \). But \( h(m') \in f(N) \), thus \( (f \circ k)(m') \in f(N) \), and hence \( (f \circ k)(f(m)) \in f(N) \). Since \( \ker{f} \subseteq N \), therefore \((k \circ f)(m) \in N \). But \( N \) is \( NS \)-prime submodule of \( M \) and \( m \notin N + J(M) \), then \((k \circ f)(M) \subseteq N + J(M) \). This implies that \( f((k \circ f)(M)) \subseteq f(N) + J(M) \), and hence \( (f \circ k) \circ f(M) \subseteq f(N) + J(M) \). Therefore \( h(M) \subseteq f(N) + J(M) \). This means that \( f(N) \) is an \( NS \)-prime submodule of \( M' \).

**Definition (2.14):**
A submodule \( N \) of an \( R \)-module \( M \), is said to be fully invariant if \( f(N) \subseteq N \), for all \( f \in \text{End}(M) \).

**Proposition (2.15):**
Let \( N \) be a proper fully invariant submodule of an \( R \)-module \( M \). If \([N : f(k)] \subseteq [N + J(M) : f(M)]\) for all \( N \subseteq K \) and for all \( f \in \text{End}(M) \), then \( N \) is an \( NS \)-prime submodule of \( M \).

**Proof:**
Let \( f(m) \in N \) where \( f \in \text{End}(M) \), and \( m \in M \). Suppose that \( m \notin N + J(M) \). Thus \( N \nsubseteq N + m \). Therefore by assumption \([N : f(N + m)] \subseteq [N + J(M) : f(M)]\). But \( 1 \in [N : f(N + m)] \), hence \( 1 \in [N + J(M) : f(M)] \), which implies that \( f(M) \subseteq N + J(M) \). Thus \( N \) is an \( NS \)-prime submodule of \( M \).

**Proposition (2.16):**
Let \( N \) be an \( NS \)-prime submodule of an \( R \)-module \( M \) then, \([N : f(K)] \subseteq [N + J(M) : f(M)]\) for all submodule \( K \) of \( M \) with \( N + J(M) \subseteq K \).

**Proof:**
Let \( N \) be an \( NS \)-prime submodule of \( M \) and \( K \) is a submodule of \( M \) with \( N + J(M) \subseteq K \), thus there exists \( x \in K \) and \( x \notin N + J(M) \). Suppose that \( r \in [N : f(K)] \) where \( r \in R \), this implies that \( r f(x) \in N \). Now, define \( h: M \to M \), by \( h(m) = r f(m) \), for all \( m \in M \). It is clear that \( h \in \text{End}(M) \), also \( h(x) = r f(x) \in N \). But \( N \) is \( NS \)-prime submodule of \( M \) and \( x \notin N + J(M) \), therefore \( h(M) \subseteq N + J(M) \), this implies that \( r f(M) \subseteq N + J(M) \), hence \( r \in [N + J(M) : f(M)] \). Therefore \([N : f(K)] \subseteq [N + J(M) : f(M)]\).

**Combining proposition (2.15) and proposition (2.16) we have the following characterization.**

**Proposition (2.17):**
Let \( N \) be a proper fully invariant submodule of an \( R \)-module \( M \), then \( N \) is \( NS \)-prime submodule, if and only if, \([N : f(K)] \subseteq [N + J(M) : f(M)]\) for all submodule \( K \) of \( M \) with \( N + J(M) \subseteq K \) and for all \( f \in \text{End}(M) \).

**We will prove the following result.**

**Proposition (2.18):**
Let \( \emptyset : M \to M \) be an epimorphism. If \( N \) is fully invariant \( NS \)-prime submodule of \( M \) with \( \ker{\emptyset} \subseteq J(M) \), then \( \emptyset^{-1}(N) \) is also \( NS \)-prime submodule of \( M \).

**Proof:**

It is clear that \( \emptyset^{-1}(N) \) is proper in \( M \). Now, let \( f(m) \in \emptyset^{-1}(N) \), where \( m \in M \) and \( f \in \text{End}(M) \). If \( m \notin \emptyset^{-1}(N) + J(M) \), we have to show that \( f(M) \subseteq \emptyset^{-1}(N) + J(M) \). Since \( f(m) \in \emptyset^{-1}(N) \), then \( \emptyset \circ f(m) \in N \), but \( N \) is \( NS \)-prime submodule of \( M \) and one can show that \( m \notin N + J(M) \), therefore \( \emptyset \circ f(M) \subseteq N + J(M) \). Thus \( f(M) \subseteq \emptyset^{-1}(N + J(M)) \). But \( \emptyset^{-1}(N + J(M)) \subseteq \emptyset^{-1}(N) + J(M) \), and hence \( f(M) \subseteq \emptyset^{-1}(N) + J(M) \). This implies that \( \emptyset^{-1}(N) \) is an \( NS \)-prime submodule of \( M \).

**REFERENCES:**


