*N*S-PRIME SUBMODULES

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ABSTRACT: *Let R be a commutative ring with identity and M is a unitary R-module. A proper submodule N of M is said tobe nearly S-prime (written as NS-prime) if whenever $f(m) \in N$ for some $f \in S = End(M)$ and $m \in M$, implies that either $m \in N + J(M)$ or $f(M) \subseteq N + J(M)$, where $J(M)$ is the Jacobson radical of M. The aim of this artical is to study this class of submodules and interpret various properties for this type of submodule.*

1. INTRODUCTION

In our research all rings will be commutative with identity and all modules are unital. A proper submodule N of an R-module M is called prime, if for each $r \in \mathrm{R}$ and $x \in \mathrm{M}$ with $rx \in \mathrm{N}$ implies that either $x \in \mathrm{N}$ or $r \in [\mathrm{N:M}] = \{s \in \mathrm{R}; s\mathrm{M} \subseteq \mathrm{N}\}$, [8]. In [6] was introduced the concept of S-prime submodule as follows: A proper submodule N of an R-module M is said to be S-prime , if whenever $f(m) \in \mathrm{N}$ for some $f \in \mathrm{End(M)}$ and $m \in \mathrm{M}$ implies that either $m \in \mathrm{N}$ or $f(\mathrm{M}) \subseteq \mathrm{N}$. In [6] was stated that every S-prime submodule is a prime submodule.

The notion of nearly prime submodule (for short *N*-prime submodule) was given in [9], where a proper submodule N of an R-module M is said to be *N*-prime , if whenever $r \in \mathrm{R}$ and $m \in \mathrm{M}$ with $rm \in \mathrm{N}$, then either $m \in \mathrm{N} + J(\mathrm{M})$ or $r \in [\mathrm{N} + J(\mathrm{M}) : \mathrm{M}]$, where $J(\mathrm{M})$ is the Jacobson radical of M. In This article, we study a new class of submodules, which is called *N*S-prime submodules, where a proper submodule N of an R-module M is *N*S-prime, if for all $f \in \mathrm{End(M)}$ and for all $m \in \mathrm{M}$ with $f(x) \in \mathrm{N}$, implies that either $m \in \mathrm{N} + J(\mathrm{M})$ or $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. Many results concerned with this concept is presented. We give some properties and characterization about this type of submodules.

2. *N*S-prime submodules

Let R be a commutative ring with identity and let M be a unitary R-module. A proper submodule N of M is called S-prime, if whenever $f(m) \in \mathrm{N}$, for some $f \in \mathrm{End(M)}$ and $m \in \mathrm{M}$, implies that either $m \in \mathrm{N}$ or $f(\mathrm{M}) \subseteq \mathrm{N}$, [6].

We introduce the following definition:

Definition (2.1):

A proper submodule N of an R-module M is said to be nearly S-prime (for short *N*S-prime), if whenever $f(m) \in \mathrm{N}$, for $m \in \mathrm{M}$ and $f \in S = \mathrm{End(M)}$, then either $m \in \mathrm{N} + J(\mathrm{M})$ or $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, where $J(\mathrm{M})$ is the Jacobson radical of M.

Remarks and examples (2.2):

1) Every *N*S-prime submodule of an R-module M is *N*-prime.

Proof:-

Let N be an *N*S-prime submodule of M and Suppose that $rm \in \mathrm{N}$ for $r \in \mathrm{R}$ and $m \in \mathrm{M}$. Assume that $m \notin \mathrm{N} + J(\mathrm{M})$. Define $f: \mathrm{M} \longrightarrow \mathrm{M}$ by $f(x) = rx$, $x \in \mathrm{M}$. Now, $rm = f(m) \in \mathrm{N}$, but N is *N*S-prime submodule of M and $m \notin \mathrm{N} + J(\mathrm{M})$, thus $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. Therefore $r\mathrm{M} \subseteq \mathrm{N} + J(\mathrm{M})$ and hence N is an *N*-prime.

The following example shows that the converse of the previous remark, is not true in general.

Let $\mathrm{M} = \mathbb{Z}_2 + \mathbb{Z}$ as $\mathbb{Z}$-module, and let $\mathrm{N} = \{\bar{0}\} + 2\mathbb{Z}$. $J(\mathrm{M}) = 0$. N is *N*-prime. Now, define, $f: \mathrm{M} \longrightarrow \mathrm{M}$ by $f(\bar{n}, m) = (\bar{0}, m)$ for all $(\bar{n}, m) \in \mathbb{Z}_2 + \mathbb{Z}$. $f \in \mathrm{End(M)}$and $f(\bar{1}, 2) = (\bar{0}, 2) \in \{\bar{0}\} + 2\mathbb{Z}$. But $(\bar{1}, 2) \notin \mathrm{N} + J(\mathrm{M})andf(\mathbb{Z}_2 + \mathbb{Z}) = \{\bar{0}\} + \mathbb{Z} \nsubseteq \{\bar{0}\} + 2\mathbb{Z}$.

Therefore N is *N*-prime, but it is not *N*S-prime submodule of M.

2) Every S-prime submodule N of an R-module M is *NS*-prime submodule.

Proof:-

Let $m \in \mathrm{M}$ and $f \in S = \mathrm{End(M)}$ with $f(m) \in \mathrm{N}$. Since N is S-prime submodule of M, thus either $m \in \mathrm{N}$ or $f(\mathrm{M}) \subseteq \mathrm{N}$.Therefore either $m \in \mathrm{N} + J(\mathrm{M})$ or $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$.

The converse of the remark (2) is not true in general. As the following example shows. For the module $\mathbb{Z}_{\rho^\infty}$ as $\mathbb{Z}$-module are have ($\bar{0}$) is *NS*-prime submodule, but it is not S-prime submodule.

Recall that a proper submodule N of an R-module M is called *NS*-primary submodule of M, if whenever $f(m) \in \mathrm{N}$ for some $f \in S = \mathrm{End(M)}$ and $m \in \mathrm{M}$, implies that either $m \in \mathrm{N} + J(\mathrm{M})$ or $f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$ for some $n \in \mathbb{Z}^+$, [7].

3) Every *NS*-prime submodule of an R-module M is *NS*-primary.

The converse of (3) is not true, for example the $\{\bar{0}\}$ submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module is an *NS*-primary submodule of $\mathbb{Z}_4$ which is not *NS*-prime submodule of $\mathbb{Z}_4$.

4) The rational number $\mathbb{Q}$ as $\mathbb{Z}$-module has the { 0 } submodule as the only *NS*-prime submodule of $\mathbb{Q}$.

The following proposition gives a characterization of *N*S-prime submodules.

Proposition (2.3):

Let N be a submodule of an R- module M then N is *N*S-prime submodule if and only, if for every submodule K of M such that $f(\mathrm{K}) \subseteq \mathrm{N}$; $f \in \mathrm{End(M)}$ implies that either $K \subseteq \mathrm{N} + J(\mathrm{M})$ or $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$.

Proof:-

Assume that $f(\mathrm{K}) \subseteq \mathrm{N}$, where K is a submodule of M. Suppose $K \nsubseteq \mathrm{N} + J(\mathrm{M})$, then there exists $k \in K$; $k \notin \mathrm{N} + J(\mathrm{M})$. It is clear that $f(k) \in \mathrm{N}$, but N is *N*S-prime submodule, thus $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. Conversely let $m \in \mathrm{M}$ with $f(m) \in \mathrm{N}$, then $f(\langle m \rangle) \subseteq \mathrm{N}$, by assumpition either $\langle m \rangle \subseteq \mathrm{N} + J(\mathrm{M})$ or $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$ this implies that N is *N*S-prime submodule.

If R is an integral domain and M is an R- module. An element $x \in \mathrm{M}$ is called a torsion element of M if $ann(x) \neq 0$. The set of all torsion elements of M which is denoted by T(M) is a submodule of M, [2].

Proposition (2.4): [10]

Let M be a module over an integral domain R, if $T(\mathrm{M}) \neq \mathrm{M}$ and $\ker f \subseteq T(\mathrm{M})$ for all $0 \neq f \in \mathrm{End}(\mathrm{M})$, then T(M) is an S-prime submodule of M.

From the previous proposition and (remark (2) in (2.2)) we can prove the following corollary.

Corollary (2.5):

Let M be a module over an integral domain R, if $T(\mathrm{M}) \neq \mathrm{M}$ and $\ker f \subseteq T(\mathrm{M})$ for all $0 \neq f \in \mathrm{End}(\mathrm{M})$, then T(M) is an *N*S-prime submodule of M.

The following result gives some properties of *N*S-prime submodules.

Proposition (2.6):

Let K be an *N*S-prime submodule of an R-module M and let N be a submodule of M with $J(\mathrm{N}) = J(\mathrm{M})$ and N is M-injective then either $\mathrm{N} \subseteq \mathrm{K}$ or $\mathrm{K} \cap \mathrm{N}$ is *N*S-prime submodule of N.

Proof:-

Suppose that $\mathrm{N} \nsubseteq \mathrm{K}$, then $\mathrm{K} \cap \mathrm{N}$ is a proper submodule of N. Now, let $f(x) \in \mathrm{K} \cap \mathrm{N}$ where $f \in \mathrm{End}(\mathrm{N})$ and $x \in \mathrm{N}$. If $x \notin (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N}) = (\mathrm{K} + J(\mathrm{N})) \cap \mathrm{N}$, therefore $x \notin \mathrm{K} + J(\mathrm{N})$. We have to show that $f(\mathrm{N}) \subseteq (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N})$. Consider the following diagram.

$$\begin{array}{ccccc} O \to & \mathrm{N} & \overset{i}{\to} & \mathrm{M} \\ & f \downarrow & \swarrow & h \\ & \mathrm{N} & & \end{array}$$

Where i is the inclusion map. Since N is M-injective, therefore there exists a homomorphism $h: \mathrm{M} \to \mathrm{N}$ such that $h \circ i = f$. Clearly that $h \in \mathrm{End}(\mathrm{M})$. But $f(x) = h \circ i(x) = h(x) \in \mathrm{K}$. Since K is *N*S-prime submodule of M and $x \notin \mathrm{K} + J(\mathrm{M})$, thus $h(\mathrm{M}) \subseteq \mathrm{K} + J(\mathrm{M})$. Also we have $f(\mathrm{N}) = (h \circ i)(\mathrm{N}) = h(\mathrm{N}) \subseteq \mathrm{N}$ and $f(\mathrm{N}) = h(\mathrm{N}) \subseteq h(\mathrm{M}) \subseteq (\mathrm{K} + J(\mathrm{N})) \cap \mathrm{N}$. This implies that $f(\mathrm{N}) \subseteq (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N})$, and hence $\mathrm{K} \cap \mathrm{N}$ is *N*S-prime submodule of N.

Let us prove the following an important characterization.

Proposition (2.7):

Let M be a nonzero R-module, then $\{0_\mathrm{M}\}$ is *N*-prime submodule of M, if and only, if $\mathrm{Ann}(\mathrm{N}) \subseteq [J(\mathrm{M}):\mathrm{M}]$, for all nonzero submodule N of M.

Proof:-

Assume that N is a nonzero submodule of M and $\{0_\mathrm{M}\}$ is *N*-prime submodule of M. Let $r \in \mathrm{Ann}(\mathrm{N})$, $r \in \mathrm{R}$. Since $\mathrm{N} \neq 0$, thus there exists $0 \neq x \in \mathrm{N}$. But $rx = 0$, since $\{0_\mathrm{M}\}$ is *N*-prime submodule of M and $x \neq 0$ therefore $r\mathrm{M} \subseteq J(\mathrm{M})$, this means that $r \in [J(\mathrm{M}):\mathrm{M}]$, and hence $\mathrm{Ann}(\mathrm{N}) \subseteq [J(\mathrm{M}):\mathrm{M}]$. Conversely, let $rx = 0$, for $r \in \mathrm{R}$ and $x \in \mathrm{M}$. Suppose that $x \notin J(\mathrm{M})$, then $x \neq 0$. Therefore $\langle x \rangle$ is a nonzero submodule of M. By assumption $\mathrm{Ann}(\langle x \rangle) \subseteq [J(\mathrm{M}):\mathrm{M}]$. Since $r \in \mathrm{Ann}(\langle x \rangle)$, thus $r \in [J(\mathrm{M}):\mathrm{M}]$, this implies that, $r\mathrm{M} \subseteq J(\mathrm{M})$ and hence $\{0_\mathrm{M}\}$ is *N*-prime submodule of M.

An R-module M is said to be multiplication if for each submodule N of M, there exists an ideal I of R such that $\mathrm{N} = \mathrm{IM}$, [5].

Now the following proposition gives a characterization of *N*S-prime submodule.

Proposition (2.8):

If M is a nonzero multiplication R-module, then $\{0_\mathrm{M}\}$ is *N*-prime submodule of M, if and only, if it is *N*S-prime submodule of M.

Proof:-

Suppose that $f(m) = 0$, where $f \in \mathrm{End}(\mathrm{M})$, and $m \in \mathrm{M}$. If $m \notin \{0_\mathrm{M}\} + J(\mathrm{M})$, then $m \neq 0$. We have to show that $f(\mathrm{M}) \subseteq J(\mathrm{M})$, since $m \neq 0$, then $0 \neq \langle m \rangle = \mathrm{IM}$, for some ideal I of R. Now, if $f(\mathrm{M}) = 0$, then we are done. Suppose that $f(\mathrm{M}) \neq 0$, since M is multiplication module thus there exists a nonzero ideal J of R such that $f(\mathrm{M}) = \mathrm{JM}$. Now, $0 = f(\langle m \rangle) = \mathrm{I}f(\mathrm{M}) = \mathrm{I}(\mathrm{J}(\mathrm{M})) = \mathrm{J}(\mathrm{IM})$, which implies that $\mathrm{J} \subseteq \mathrm{Ann}(\mathrm{IM})$. From proposition (2.7), we obtain $\mathrm{Ann}(\mathrm{IM}) \subseteq [J(\mathrm{M}):\mathrm{M}]$ and hence $\mathrm{J} \subseteq [J(\mathrm{M}):\mathrm{M}]$, therefore $J\mathrm{M} \subseteq J(\mathrm{M})$, this implies that $f(\mathrm{M}) \subseteq J(\mathrm{M})$. Therefore $\{0_\mathrm{M}\}$ is *N*S-prime submodule of M. The converse side is clear from (remark (1) in (2.2)).

We introduce the concept of *N*S-prime module and prove some characterization of this concept.

Definition (2.9):

Let M be an R-module. If $\{0_\mathrm{M}\}$ is *N*S-prime submodule of M, then we say that M is *N*S-prime module.

See the following:

Proposition (2.10):

If M is a multiplication module, and N is a submodule of M, then N is *N*-prime submodule, if and only, if it is *N*S-prime submodule of M.

Proof:-

From [1, Corollary (3.22)], we get that $\frac{\mathrm{M}}{\mathrm{N}}$ is a multiplication module. Thus N is *N*-prime submodule of M, if and only, N is *N*S-prime submodule of M.

Definition (2.11):[3]

Let M and M\` be R-modules, the module M\` is called M-projective, if for every homomorphism $f: \mathrm{M}` \to \frac{\mathrm{M}}{\mathrm{K}}$, where K is a submodule of M, can be lifted to a homomorphism $g: \mathrm{M}` \to \mathrm{M}$.

Now, we can prove the following .

Proposition (2.12):

Let $f: \mathrm{M} \to \mathrm{M}`$ be an R-module epimorphism. If N is *N*S-prime submodule of M such that $\ker f \subseteq \mathrm{N}$, then $f(\mathrm{N})$ is *N*S-prime submodule of M\`, where M\` is M-projective module.

Proof:-

$f(\mathrm{N})$ is a proper submodule of M\`. Since if $f(\mathrm{N}) = \mathrm{M}`$. Thus $\mathrm{M} = \mathrm{N}$, which is a contradiction. Now, let $h(m`) \in f(\mathrm{N})$, where $m` \in \mathrm{M}$, $h \in \mathrm{End}(\mathrm{M}`)$. Suppose that, $m` \notin f(\mathrm{N}) + J(\mathrm{M}`)$. We have to show that $h(\mathrm{M}`) \subseteq f(\mathrm{N}) + J(\mathrm{M}`)$. $f$ is an epimorphism and $m` \in \mathrm{M}`$, therefore there exists $m \in \mathrm{M}$ such that $f(m) = m`$.

Now consider the following diagram:

$$\begin{array}{ccc} & \mathrm{M}` & \\ k \swarrow & \downarrow h & \\ \mathrm{M} \underset{f}{\to} & \mathrm{M}` \to o & \end{array}$$

Since M` is M-projective, then there exists, a homomorphism $k$ , such that $f \circ k = h$. But $h(m`) \in f(\mathrm{N})$, thus $(f \circ k)(m`) \in f(\mathrm{N})$, and hence $(f \circ k)(f(m)) \in f(\mathrm{N})$. Since $\ker f \subseteq \mathrm{N}$, therefore $(k \circ f)(m) \in \mathrm{N}$. But N is *N*S-prime submodule of M and $m \notin \mathrm{N} + J(\mathrm{M})$, then $(k \circ f)(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. This implies that $f((k \circ f)(\mathrm{M})) \subseteq f(\mathrm{N}) + J(\mathrm{M`})$, and hence $(f \circ k) \circ f(\mathrm{M}) \subseteq f(\mathrm{N}) + J(\mathrm{M`})$. Therefore $h(\mathrm{M`}) \subseteq f(\mathrm{N}) + J(\mathrm{M`})$. This means that $f(\mathrm{N})$ is an *N*S-prime submodule of M`.

Corollary (2.13):

Let N be an *N*S-prime submodule of M and K , is a submodule of M with $\mathrm{K} \subseteq \mathrm{N}$, then $\frac{\mathrm{N}}{\mathrm{K}}$ is *N*S-prime submodule of $\frac{\mathrm{M}}{\mathrm{K}}$, where $\frac{\mathrm{M}}{\mathrm{K}}$ is an M-projective module.

Definition (2.14):[4]

A submodule N of an R-module M, is said to be fully invariant if $f(\mathrm{N}) \subseteq \mathrm{N}$, for all $f \in \mathrm{End(M)}$.

Proposition (2.15):

Let N be a proper fully invariant submodule of an R-module M. If $[\mathrm{N}: f(\mathrm{K})] \subseteq [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$ for all $\mathrm{N} \subsetneq \mathrm{K}$ and for all $f \in \mathrm{End(M)}$, then N is an *N*S-prime submodule of M.

Proof:-

Let $f(m) \in \mathrm{N}$ where $f \in \mathrm{End(M)}$, and $m \in \mathrm{M}$. Suppose that $m \notin \mathrm{N} + J(\mathrm{M})$. Thus $\mathrm{N} \subsetneq \mathrm{N} + \langle m \rangle$. Therefore by assumption $[\mathrm{N}: f(\mathrm{N} + \langle m \rangle)] \subseteq [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$.

But $1 \in [\mathrm{N}: f(\mathrm{N} + \langle m \rangle)]$, hence $1 \in [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$, which implies that $f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. Thus N is an *N*S-prime submodule of M.

Proposition (2.16):

Let N be an *N*S– prime submodule of an R-module M then, $[\mathrm{N}: f(\mathrm{K})] \subseteq [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$, for all submodule K of M with $\mathrm{N} + J(\mathrm{M}) \subsetneq \mathrm{K}$.

Proof:-

Let N be an *N*S - prime submodule of M and K is a submodule of M with $\mathrm{N} + J(\mathrm{M}) \subsetneq \mathrm{K}$, thus there exists $x \in \mathrm{K}$ and $x \notin \mathrm{N} + J(\mathrm{M})$. Suppose that $r \in [\mathrm{N}: f(\mathrm{K})]$ where $r \in \mathrm{R}$, this implies that $rf(x) \in \mathrm{N}$. Now,define $h: \mathrm{M} \to \mathrm{M}$, by $h(m) = rf(m)$, for all $m \in \mathrm{M}$. It is clear that $h \in \mathrm{End(M)}$, also $h(x) = rf(x) \in \mathrm{N}$. But N is *N*S – prime submodule of M and $x \notin \mathrm{N} + J(\mathrm{M})$, therefore $h(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, this implies that $r f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, hence $r \in [\mathrm{N} + J(\mathrm{M}) : f(\mathrm{M})]$. Therefore $[\mathrm{N}: f(\mathrm{K})] \subseteq [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$.

Combining proposition (2.15) and proposition (2.16) we have the following characterization.

Proposition (2.17):

Let N be a proper fully invariant submodule of an R-module M, then N is *N*S-prime submodule, if and only, if $[\mathrm{N}: f(\mathrm{K})] \subseteq [\mathrm{N} + J(\mathrm{M}): f(\mathrm{M})]$, for all submodule K of M with $\mathrm{N} + J(\mathrm{M}) \subsetneq \mathrm{K}$ and for all $f \in \mathrm{End(M)}$.

We will prove the following result.

Proposition (2.18):

Let $\emptyset: \mathrm{M} \to \mathrm{M}$ be an epimorphism. If N is fully invariant *N*S-prime submodule of M with $\ker \emptyset \subseteq J(\mathrm{M})$, then $\emptyset^{-1}(\mathrm{N})$ is also *N*S-prime submodule of M.

Proof:-

It is clear that $\emptyset^{-1}(\mathrm{N})$ is proper in M. Now, let $f(m) \in \emptyset^{-1}(\mathrm{N})$, where $m \in \mathrm{M}$ and $f \in \mathrm{End(M)}$. If $m \notin \emptyset^{-1}(\mathrm{N}) + J(\mathrm{M})$, we have to show that $f(\mathrm{M}) \subseteq \emptyset^{-1}(\mathrm{N}) + J(\mathrm{M})$. Since $f(m) \in \emptyset^{-1}(\mathrm{N})$, then $\emptyset \circ f(m) \in \mathrm{N}$, but N is *N*S-prime submodule of M and one can show that $m \notin \mathrm{N} + J(\mathrm{M})$, therefore $\emptyset \circ f(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$. Thus $f(\mathrm{M}) \subseteq \emptyset^{-1}(\mathrm{N} + J(\mathrm{M}))$. But $\emptyset^{-1}(\mathrm{N} + J(\mathrm{M})) \subseteq \emptyset^{-1}(\mathrm{N}) + J(\mathrm{M})$, and hence $f(\mathrm{M}) \subseteq \emptyset^{-1}(\mathrm{N}) + J(\mathrm{M})$. This implies that $\emptyset^{-1}(\mathrm{N})$ is an *N*S-prime submodule of M.

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