EXACT SERIES SOLUTIONS OF ONE-DIMENSIONAL FINITE AMPLITUDE SOUND WAVES

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ABSTRACT. *An effective application of the recently developed semi-analytic technique, the generalized residual power series method (shortly GRPSM), is presented. The generalization provides the solution in closed-form series solution with easily computable components. The reported results, with application to fully developed shock wave equation that describes the flow of most gases and compressible, reveal the reliability of the proposed algorithm for handling diverse types of nonlinear partial differential equations in mathematical physics.*

Keywords: Generalized residual power series method; Shock wave equation; Conservation law; Series solution.

1. INTRODUCTION

Linear and Nonlinear partial differential equations arise in enormous number of problems in physics and engineering, as well as in a variety of science fields. One of the most famous applications is the conservation law that comes from the formulation of familiar physical laws of conservation of energy, mass and momentum. Such models are of foremost importance in mathematical physics.

In this study, the one-dimensional scalar conservation law, without resources, governed by the initial-value problem

$$D_t V + D_x F(V) = 0, \qquad V(x,t_0) = f(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^{+}, \tag{1}$$

where $D_t = \dfrac{\partial}{\partial t}$ and $D_x = \dfrac{\partial}{\partial x}$, is considered. This equation describes the propagation of the finite amplitude sound waves in terms of the particle velocity $V(x,t)$ with respect to space x and time t [1].

The closed-form exact-analytic solutions for most of such nonlinear phenomena are not derived. So, many works interested in approximating numeric-analytic solutions that allow physicists to draw conclusions in an efficient way. In the last three decades, many authors devoted their attention to solve shock wave and wave equations; for instance, the Adomian decomposition method (ADM) [2-5], homotopy perturbation method (HPM) [6-7], homotopy analysis method [8-9], variational iteration method (VIM) [9-10] and the reduced differential transform method (RDTM) [11-14]. Also, see references therein.

In this work, the generalized residual power series method(GRPSM) [15] is improved to tackle the one-dimensional scalar conservation law Eq.(1). In section two, we begin with some basic facts and explain the use of the proposed method for Eq.(1). Section three is devoted to apply the GRPSM for some shock wave type equations. Numerical experiments are included in Section four.

2. THE GENERALISED RESIDUAL POWER SERIES METHOD

The residual power series method (RPSM) was first proposed by Abu Arqub [16] for the treatment of linear and nonlinear ordinary differential equations of integer and fractional orders, see [17-20] and references therein. This method based on constructing term-by-term series solutions in a form of Taylor series expansion without need of linearization, discretization, perturbation or unrealistic assumptions.

An extension of the RPSM for solving Eq.(1) is presented. The analytic $V(x,t)$ can be expanded into a Taylor series about $t = t_0$:

$$V(x,t) = V(x,0) + \sum_{n=1}^{\infty} \xi_n(x)(t-t_0)^n = f(x) + \sum_{n=1}^{\infty} \xi_n(x)(t-t_0)^n, \tag{2}$$

where, $\xi_i(x)$ are to be determined consecutively by solving the algebraic equation

$$\lim_{t \to t_0} \mathrm{Res}_m(x,t) = 0, \; m = 1,2,\ldots$$

where $\mathrm{Res}_m(x,t)$ is the analytic m th- residual function defined by

$$\mathrm{Res}_m(x,t) = D_t^{(m-1)}\left(D_t V + D_x F(V)\right).$$

Az-Zo'bi [15] proved the following related facts:

Theorem 1. *The residual function* $\mathrm{Res}_m(x,t)$ *vanishes as* m *approaches the infinity.*

Theorem 2. *For* $k > 0$, *the* k *th-order approximate solution*

$$V_k(x,t) = f(x) + \sum_{n=1}^{k} \xi_n(x)(t-t_0)^n,$$

obtained by the GRPSM, to the exact solution $V(x,t)$ *of Eq.(1) coincides the Taylor series expansion of* $V(x,t)$ *about* $t = t_0$.

Corollary 1. *Suppose that the truncated series* $V_k(x,t)$ *is used as an approximation to the solution* $V(x,t)$ *of problem Eq.(1) on a rectangle*

$$R = \{(x,t) : x_0 - h \le x \le x_0 + h, t_0 - \rho \le t \le t_0 + \rho, h > 0, \rho > 0\} \subset D$$

,then positive numbers $\eta(t)$, *satisfies* $|\eta(t) - t_0| \le \rho$, *and* μ_k *exist with*

$$\left|V(x,t)-V_k(x,t)\right| \le \frac{\mu_k}{(k+1)!}\rho^{k+1}.$$

Corollary 2. *Suppose that the solution $V(x,t)$ is a polynomial of t, then the GRPSM results the exact solution.*

3. SERIES SOLUTION OF THE SHOCK WAVE EQUATION

Example 3.1 Consider Eq.(1) for propagation of an isentropic perfect gas with infinitely small waves, i.e., with

$$F(V)=\frac{1}{c_0+\frac{1}{2}(\gamma+1)V}.$$

where c_0, γ are constants and γ is the specific heat. Banta [1] derives semi-analytic solution for this model in the form

$$V(x,t)=V(x,t_0)+\sum_{n=1}^{\infty}(-1)^n D_x^{n-1}\left[F^n(V)D_xV\right]_{t=t_0}\frac{(t-t_0)^n}{n!}. \tag{3}$$

To illustrate the technique discussed in Sections 2 and starting with $V(x,t_0)=f(x)$, the unknown coefficients of series solution Eq.(2) were obtained with aid of symbolic computation software, *Mathematica*. First few coefficients are listed to be:

$$\xi_1(x)=-\frac{D_xf(x)}{c_0+\frac{1}{2}(1+\gamma)f(x)},$$

$$\xi_2(x)=\frac{1}{2}\left(-\frac{(1+\gamma)(D_xf(x))^2}{\left(c_0+\frac{1}{2}(1+\gamma)f(x)\right)^3}+\frac{D_x^2f(x)}{\left(c_0+\frac{1}{2}(1+\gamma)f(x)\right)^2}\right),$$

$$\xi_3(x)=-\frac{1}{6}\left(\frac{3(1+\gamma)^2(D_xf(x))^3}{\left(c_0+\frac{1}{2}(1+\gamma)f(x)\right)^5}-\frac{3(1+\gamma)D_xf(x)D_x^2f(x)}{\left(c_0+\frac{1}{2}(1+\gamma)f(x)\right)^4}+\frac{D_x^3f(x)}{\left(c_0+\frac{1}{2}(1+\gamma)f(x)\right)^3}\right)$$

The series solution coincides that's one derived in Eq.(3).

Example 3.2 The propagation of low-amplitude plane sound waves is governed by Eq.(1) with flux

$$F(V)=\frac{1}{c_0}-\frac{\gamma+1}{2c_0^2}V. \tag{4}$$

The series solution can be obtained if $c_0 >> \frac{1}{2}(\gamma+1)V$, as shown in [1]. Converting to a moving coordinate system, through replacing the space x by $x-\frac{t}{c_0}$, we get

$$F(V)=-\frac{\gamma+1}{2c_0^2}V.$$

Proceeding as in Example 1, unknown coefficients of series solution Eq.(2) were obtained recursively by the GRPSM. The first few coefficients are

$$\xi_1(x)=-\frac{(1+\gamma)}{2c_0^2}f(x)D_xf(x),$$

$$\xi_2(x)=\frac{1}{2}\left(\frac{1+\gamma}{2c_0^2}\right)^2 D_x\left[f(x)D_xf(x)\right],$$

$$\xi_3(x)=\frac{1}{6}\left(\frac{1+\gamma}{2c_0^2}\right)^3 D_x^2\left[f(x)D_xf(x)\right],$$

$$\vdots$$

Continuing this process, we get the series solution obtained in [1].

4. NUMERICAL ILLUSTRATION

For the study case under consideration Eq.(1), numerical implementation of our generalization is presented with flux given in Eq.(4). Let $c_0=2$, $\gamma=\frac{3}{2}$ and the initial condition is assumed to be

$$V(x,0)=e^{-x^2/2},\ x \varepsilon \mathrm{R}$$

Up to 100 terms of the series solution Eq.(2) were calculated, with aid of Mathematica. Below, we list the first few terms

$$\xi_1(x)=\frac{1}{16}e^{-x^2}\left(-5+8e^{\frac{x^2}{2}}\right)x\ ,$$

$$\xi_2(x)=\frac{1}{512}e^{-\frac{3x^2}{2}}\left(-5+8e^{\frac{x^2}{2}}\right)\left(5-8e^{\frac{x^2}{2}}-15x^2+8e^{\frac{x^2}{2}}x^2\right),$$

$$\xi_3(x)=\frac{e^{-2x^2}\left(-5+8e^{\frac{x^2}{2}}\right)x\left(-75+150e^{\frac{x^2}{2}}-48e^{x^2}+100x^2-110e^{\frac{x^2}{2}}x^2+16e^{x^2}x^2\right)}{6144}$$

$$\vdots$$

Best obtained approximation is with 37 terms, i.e.,

$$V_{appr}(x,t)=\sum_{k=0}^{37}\xi_k(x)t^k. \qquad (5)$$

Figure 1.a shows the surface plot of the approximated solution Eq.(5). The corresponding absolute error defined at any point as

$$E_{Abs}=\left|D_tV_{app}+D_xF\left(V_{app}\right)\right|, \qquad (6)$$

is shown in Figure 1.b.

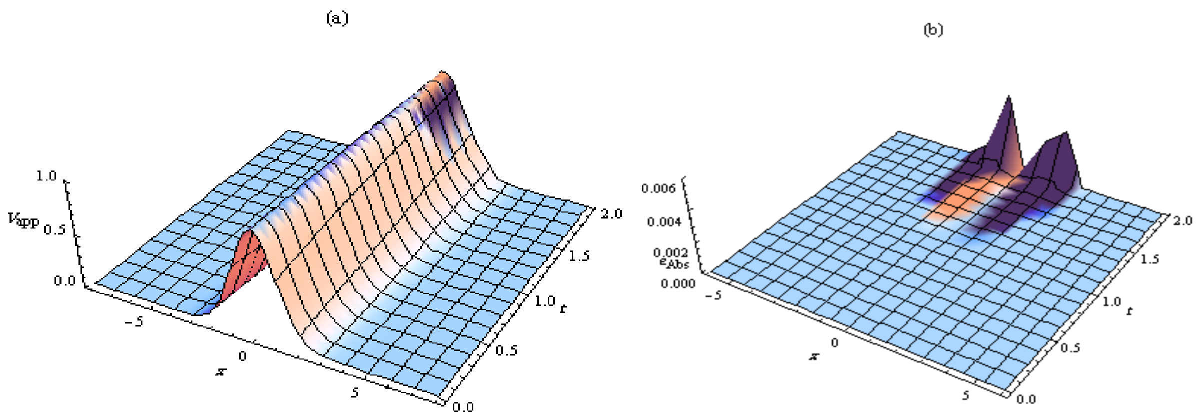

Fig.1. Behavior of: (a) approximate solution V_{App} using 21 iterations of GRPSM, (b) corresponding absolute error E_{Abs} for $-8\le x\le 8,\ 0\le t\le 2.$

Table 1. The absolute errors for solving Shock wave equation Eq.(4) using 37-terms of GRPSM

x	t	E_{Abs}	x	E_{Abs}
-3	0	3.46945×10^{-18}	1	2.77556×10^{-17}
	0.4	3.46945×10^{-18}		2.77556×10^{-17}
	0.8	0.		8.32667×10^{-17}
	1.2	3.747×10^{-16}		4.59681×10^{-8}
	1.6	1.69196×10^{-11}		1.99576×10^{-7}
	2	6.48935×10^{-8}		7.12756×10^{-4}
-1	0	2.77556×10^{-17}	2	2.77556×10^{-17}
	0.4	2.77556×10^{-17}		1.38778×10^{-17}
	0.8	2.77556×10^{-17}		4.16334×10^{-17}
	1.2	1.34584×10^{-11}		1.45319×10^{-11}
	1.6	6.61778×10^{-7}		5.591×10^{-7}
	2	2.85427×10^{-3}		1.97536×10^{-3}
0	0	0.	3	3.46945×10^{-18}
	0.4	5.20417×10^{-18}		0.
	0.8	6.93889×10^{-18}		3.46945×10^{-18}
	1.2	6.93889×10^{-18}		3.60822×10^{-16}
	1.6	3.69149×10^{-15}		1.3355×10^{-11}
	2	1.58138×10^{-11}		4.63813×10^{-8}

To demonstrate the accuracy of GRPSM numerically, absolute errors for some values of x and t, rounding to 3 significant digits, are listed in Table.

5. DISCUSSION AND CONCLUSION

The residual power series method is reformulated to tackle one-dimensional scalar conservation laws. Models, with applications to the finite amplitude sound waveform, were considered to illustrate the efficiency, simplicity and convenience of the generalized residual power series method. A semi-analytic solution in closed form is obtained. Numerical results reveal that the generalized scheme is more accurate than other existing methods. It may be concluded that the GRPSM is very powerful and efficient technique in finding analytic solutions for wide classes of problems and can be also easy to be extended to other non-linear evaluation equations, with the aid of *Mathematica*.

REFERENCES

1. E.D. Banta, Lossless propagation of one-dimensional, finite amplitude sound waves, J. Math. Anal. Appl. 10 (1965) 166-173.
2. K. Al-Khaled and F. M. Allan, Construction of solutions for the shallow water equations by the decomposition method, Math. . Comput. Simul. 66 (2004) 479–486.
3. E. Az-Zo’bi and K. Al Khaled, A new convergence proof of the Adomian decomposition method for a mixed hyperbolic elliptic system of conservation laws, Appl. Math. Comput. 217 (2010) 4248–4256.
4. E. A. Az-Zo'bi, An Approximate Analytic Solution for Isentropic Flow by An Inviscid Gas Equations, Archives of Mechanics, 66 (3) (2014) 203-212.
5. E. A. Az-Zo'bi, Construction of Solutions for Mixed Hyperbolic Elliptic Riemann Initial Value System of Conservation Laws, Applied Mathematical Modeling, 37 (2013) 6018-6024.
6. J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Soliton. Fract. 26 (2005) 695.
7. M. E. Berberler and A. Yildirim, He's homotopy perturbation method for solving the shock wave equation, Applicable Analysis, 88 (7) (2009) 997-1004.
8. D. Kumar, J. Singh, S. Kumar, Sushila and B.P. Singh, Numerical computation of nonlinear shock wave equation of fractional order, Ain Shams Engineering Journal 6 (2015) 605–611
9. I. Khatami, N. Tolou, J. Mahmoudi and M. Rezvani, Application of Homotopy Analysis Method and Variational Iteration Method for Shock Wave Equation, J. of Applied Analysis 8 (5) (2008) 848-853.
10. E. A. Az-Zo`bi, On the Convergence of Variational Iteration Method for Solving Systems of Conservation Laws, Trends in Applied Sciences Research 10 (3) (2015) 157-165.
11. E. A. Az-Zo'bi and M. F. Marashdeh & Kamal Al Dawoud, Numerical Simulation of One-Dimensional Shallow Water Equations, International Journal of Sciences: Basic and Applied Research 23 (2) (2015) 196-203.
12. E. A. Az-Zo'bi, Analytic-Numeric Simulation of Shock Wave Equation Using Reduced Differential Transform Method, Science International (Lahore) 27 (3) (2015) 1749-1753.
13. E. A. Az-Zo'bi, K. Al Dawoud and M. F. Marashdeh, Numeric-analytic solutions of mixed-type systems of balance laws, Applied Mathematics and Computation 265 (2015) 133–143.
14. Emad A. Az-Zo'bi, On the Reduced Differential Transform Method and its Application to the Generalized Burgers-Huxley Equation, Applied Mathematical Sciences, 8 (177) (2014) 8823–8831.
15. E. A. Az-Zo'bi, Exact Analytic Solutions for Nonlinear Diffusion Equations via Generalized Residual Power Series Method, International Journal of Mathematics and Computer Science, 14 (1) (2019).
16. O. Abu Arqub, A. El-Ajou, A. Bataineh, I. Hashim, A representation of the exact solution of generalized Lane-Emden equations using a new analytical method, Abstract Appl. Anal., Article ID 378593, 10 pages, doi:10.1155/2013/378593, (2013).
17. O. Abu Arqub, A. El-Ajou, S. Momani, Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations, IJ. Comput. Phys., 293, 385–399, (2015).
18. A. Kumar, S. Kumar, Residual power series method for fractional Burger types equations, Nonlinear Engineering, Modeling and Application, 5 (4), 683–708, doi:10.1515/nleng-2016-0028, (2016).
19. H. Tariq, G. Akram, Residual power series method for solving timespace-fractional Benney-Lin equation arising in falling film problems, J. Appl. Math. Comput., 55 (1-2), 683–708, (2017). 10
20. A. Arafa1 and G. Elmahdy, Application of Residual Power Series Method to Fractional Coupled Physical Equations Arising in Fluids Flow, International Journal of Differential Equations 2018, Article ID 7692849, 10 pages https://doi.org/10.1155/2018/769284