COFINITE-GENERALIZED-HOLLOW $LIFTING_g$ MODULES

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ABSTRACT: *Let R be any ring with identity , and let M be a unitary left R-module . A submodule N of M is called generalized small submodule of M denoted by* $(N \ll_G M)$*, if for every essential submodule K of M with M= N+K implies K=M . A submodule K of M is called G-coessential of N in M if* $\frac{N}{K} \ll_G \frac{M}{K}$*. M is called cofinite generalized* $lifting_g$ *module , if every cofinite submodule N of M , N has a generalized coessential submodule in M which is a direct summand of M . in this paper we introduce a cofinite generalized hollow* $lifting_g$ *module . M is called cofinite generalized hollow* $lifting_g$ *module (for short C-G-hollow* $lifting_g$ *module , if for every cofinite submodule N of M with* $\frac{M}{N}$ *is G-hollow , N has a generalized coessential submodule that is a direct summand of M. and we study some properties of this type of modules.*

Keywords : generalized small submodule , cofinite-generalized-hollow module , cofinite-generalized-lifting module.

1-INTRODUCTION:

Throughout this paper R is a ring with identity , and every R-module is a unitary left R-module , N⊆ M denotes N is a submodule of M . Let M be an R-module , and let N⊆M , N is called essential submodule of M (denoted by $N \subseteq_e M$) if every non zero submodule K of M , we have $N \cap K \neq 0$ [1]. A submodule N of M is called small submodule of M (denoted by $N \ll M$) , if for every K⊆M , M= N+K implies K = M [1] . A non zero module M is called hollow if every proper submodule of M is small , [1] . Rad(M) is the sum of all small submodules of M [1]. A submodule N of M is called generalized–small submodule of M (for short G-small) and (denoted by $N \ll_G M$), if for every $K \subseteq_e M$, M= N+K implies K=M [2].$Rad_g(M)$ is the sum of all G-small submodules of M [2] , It clear that Rad(M) ⊆ $Rad_g(M)$, but the converse in general is not true . A nonzero module M is called generalized-hollow (for short , G-hollow) , if every proper submodule of M G-small (in [3] , it is denoted by (e-hollow) .

A Submodule K of M is called coessential submodule of N in M (denoted by $K \subseteq_{Ce} N$) if $\frac{N}{K} \ll \frac{M}{K}$, [4] . A submodule K of M is called G-coessential submodule of N in M (denoted by $K \subseteq_{Gce} N$) , if $\frac{N}{K} \ll_G \frac{M}{K}$. an R-module M is called generalized lifting or satisfies (GD1) , if for every submodule N of M , there exists a direct summand K of M , such that $K \subseteq_{GCe} N$ in M [3]. It is clear that every lifting module is a generalized lifting module . In [6] Orhan and Tribak are introduce hollow lifting module , A module M is called hollow lifting , if for every submodule N of M with $\frac{M}{N}$ is hollow , N has coessential submodule of M that is a direct summand of M . In this paper we introduce a cofinite generalized hollow module (for short C-G-hollow) . We give the some basic properties of C-G-hollow modules . Also we introduce cofinite generalized $lifting_g$ module as a generalization of hollow lifting module . we prove some results similar to results of hollow lifting modules .

2. Cofinite generalized hollow module

It is know that a non zero R-module M is called G-hollow module , if every proper submodule of M is G-small . in this section we define a cofinite generalized hollow module (in short C-G-hollow) and we study some properties of this type of modules .

Definition 2.1[2]: A submodule N of M is called generalized small submodule of M (for short , G-small) and (denoted by $N \ll_G M$) , if for every $K \subseteq_e M$, M=N+K implies K = M .

And a nonzero module M is called generalized-hollow (for short , G-hollow) , if every proper submodule of M G-small .(in [3] it is denoted by e-hollow) .

Now we introduce the following :-

Definition 2.2 : A non zero R-module is called cofinite-generalized hollow module (for short C-G-hollow) , if for every proper cofinite submodule of M is G-small .

Remarks and Examples 2.3 :-

1- It is clear that every semisimple module is C-G-hollow module .
2- Every hollow is C-G-hollow module.
3- The converse of (2) is not true in general , Q as Z-module is not hollow , and the only cofinite submodule of Q is Q which is not proper hence Q is C-G-hollow module.
4- If M is finitely generated and every submodule of M is closed , then M is C-G-hollow module. Since M is finitely generated then every submodule of M is cofinite in M , and since every submodule of M is closed hence every submodule is G-small in M .

5- It is clear that Z_4 as Z-module is C-G-hollow module , since $\{\bar{0},\bar{2}\}$ is cofinite , G-small in Z_4.

6- Z as Z-module is not C-G-hollow. to see that , consider $3Z \subset Z$, $3Z$ is cofinite submodule of Z . but 3Z+2Z=Z and $2Z \subset_e Z$, $2Z \neq Z$. hence 3Z is not G-small in Z .

Remark 2.4: A direct sum of C-G-hollow modules need not C-G-hollow as the following example shows:-

The Z-modules $Z_4, Z_3$are C-G-hollow , but $Z_4 \oplus Z_3 \cong Z_{12}$ is not C-G-hollow Z-module. Since $<\bar{3}> + <\bar{2}> = Z_{12}$, $<\bar{2}> \subset_e Z_{12}$, $<\bar{2}> \neq Z_{12}$, that is $<\bar{3}>$ is not G-small in Z_{12} .

Recall that a submodule N of M is called fully invariant if $f(N) \subseteq N$ for every f∈ $End(M)$. and an R-module M is called duo module, if every submodule of M is fully invariant , [6] .

Proposition 2.5: Let M = $M_1 \oplus M_2$ be a duo R-modules . if M_1 and M_2 are C-G-hollow of M , provided N∩ $M_i \neq M_i$ for all i = 1, 2, then M is C-G-hollow .

Proof: Let N be cofinite proper submodule of M , then N=($N \cap M_1) \oplus (N \cap M_2)$, $N \cap M_1 \subset M_1$ and $N \cap M_2 \subset M_2$, [6].

Now$\frac{M}{N}=\frac{tM_1 \oplus M_2}{N}=\frac{tM_1+N}{N} \oplus \frac{M_2+N}{N}$
$=\frac{tM_1+(N \cap tM_1)+(N \cap tM_2)}{(N \cap tM_1)\oplus(N \cap tM_2)} \oplus \frac{tM_2+(N \cap tM_1)+(N \cap tM_2)}{(N \cap tM_1)\oplus(N \cap tM_2)}$

$$\cong \frac{M_1}{(N \cap M_1)} \oplus \frac{M_2}{((N \cap M_2)}$$

Now $\frac{\frac{M}{N}}{\frac{M_1+N}{N}} \cong \frac{M}{M_1+N} = \frac{M_1 \oplus M_2}{M_1+N} = \frac{(M_1+N)+M_2}{(M_1+N)} \cong \frac{M_2}{(M_1+N) \cap M_2} = \frac{M_2}{N \cap M_2}$, therefore

$$\frac{M_2}{N \cap M_2} \text{ is finitely generated.}$$

Similarly $\frac{M_1}{N \cap M_1}$ *is finitely generated* , henc $N \cap M_1$and $N \cap M_2$ are cofinite submodules of M_1 and M_2 respectively .Since M_1 and M_2 are C-G-hollow , then $N \cap M_1$and $N \cap M_2$ are G-small submodules of M_1 and M_2 respectively. Thus N=($N \cap M_1) \oplus (N \cap M_2)$ is G-small , [7] , therefore M is C-G-hollow.

Recall that an R-module M is called distributive if for all N, W, K ⊆M, N∩(K + W) = (N ∩ K) + (N ∩ W) .

Equivalently, N + (K ∩ W) = (N + K) ∩ (N + W) [8].

Proposition 2.6: Let M = $M_1 \oplus M_2$ be R-module with M_1, M_2 ⊆M and M is distributive , provided N ∩ $M_i \neq M_i$ for all i = 1, 2 and N⊂M . if M_1, M_2 are C-G-hollow then M is C-G-hollow .

Proposition 2.7: Let N be Proper submodule of M , if M is C-G-hollow then M/N is C-G-hollow.

Proof: Let $\frac{K}{N} \subset \frac{M}{N}$ and K $\neq M$. Such that $\frac{K}{N}$ is cofinite submodule of $\frac{M}{N}$, then $\frac{M/N}{K/N} \cong \frac{M}{K}$ is finitely generated , then K is a cofinite submodule of M , since M is C-G-hollow then K$\ll_G M$. Hence $\frac{K}{N} \ll_G \frac{M}{N}$ [7].

Corollary 2.8: The nonzero homomorphic image of C-G-hollow module is C-G-hollow.

Proof :- since every homomorphic image is isomorphic to a quotient module .

Corollary 2.9: The direct summand of C-G-hollow is again C-G-hollow .

Proposion 2.10: Let M be an R-module, Let N⊆ M, if M/N is C-G-hollow , and N$\ll_G M$, then M is C-G-hollow.

Proof: Let L⊂ Msuch that M/L is finitely generated and let M=L+K , K$\subset_e M$, then

$\frac{M}{N} = \frac{L+N}{N} + \frac{K+N}{N}$, $\frac{L+N}{N} \neq \frac{M}{N}$, if $\frac{L+N}{N} = \frac{M}{N}$, then $\frac{K+N}{N} \subseteq \frac{L+N}{N}$ hence $K \subseteq L$ but M=L+K then M=L , which is a contradiction , thus $\frac{L+N}{N} \neq \frac{M}{N}$.

Now $\frac{M}{L} = \frac{M+N}{L+N} = \frac{M}{L+N}$ then $\frac{M}{L+N}$ is finitely generated , but $\frac{\frac{M}{N}}{\frac{L+N}{N}} \cong \frac{M}{L+N}$, thus $\frac{L+N}{N}$ is cofinite in $\frac{M}{N}$. And $\frac{K+N}{N} \subseteq_e \frac{M}{N}$, since$\frac{M}{N}$ is C.G.hollow , then$\frac{K+N}{N}=\frac{M}{N}$, then K+N=M , by assumption N$\ll_G M$, hence K=M.

Proposion 2.11: Let M be an C-G-hollow R-module . If M has a cofinite proper essential submodule N of M with every submodule of N is cofinite in N , then M is finitely generated.

Proof: Let N⊂ M, N$\subset_e$M with M/N is finitely generated , then

$\frac{M}{N}$=R(x_1, N) + R$(x_2, N) + \cdots$ R(x_n, N) , for $x_1, x_2, \cdots, x_n \in N$, hence $\frac{M}{N}$=Rx_1 +R$x_2 + \cdots +$R$x_n + N$

Then m + N= $r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$,for m $\in M, r_1, r_2, \dots, r_n \in R$.

If m-$r_1x_1 + r_2x_2 + \cdots + r_nx_n \in N$, hence

m-$r_1x_1 + r_2x_2 + \cdots + r_nx_n = n$

m=$r_1x_1 + r_2x_2 + \cdots + r_nx_n + n$

m=$< x_1, x_2, \cdots, x_n > +N$

Let K=$< x_1, x_2, \cdots, x_n >$

Now M=K+N , if $K \neq M$, then $\frac{M}{K} = \frac{N+k}{K} \cong \frac{N}{N \cap K}$.

Since $\frac{N}{N \cap K}$ is finitely generated (by assumption) then $\frac{M}{K}$ is finitely generated , thus K is cofinite proper submodule of

M but M is C-G-hollow , and $N \subset_e M$, then M=N wich is a contradiction ,thus M=K=$< x_1, x_2, \cdots, x_n >$

Then M is finitely generated.

3- C-G-Hollow Modules and C-G-Lifting Modules

As it is known that every hollow is lifting . we define C-G-lifting and show that every C-G-hollow module is C-G-lifting.

Definition 3.1: An R-module M is called C-G-Lifting module , if for every cofinite submodule A of M , there exists a direct summand B of M such that $\frac{A}{B} \ll_G \frac{M}{B}$ in M .

The following theorem gives a characterization of C-G-Lifting modules.

Theorem 3.2: Let M be an R-module. Then the following statements are equivalent :-

1- M is C-G-Lifting .
2- For every cofinite submodule A in M , there is a decomposition M = $M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll_G M_2$.
3- Every cofinite submodule A of M can be written as A = B ⊕ S, where B is a direct summand of M and $S \ll_G M$.

Proof: 1→2) Suppose M is C-G-Lifting and let A cofinite submodule of M , there exists a decomposition M = $M_1 \oplus M_2$ with $M_1 \subseteq$ A and $\frac{A}{M_1} \ll_G \frac{M}{M_1}$. Now A =A∩M = A $\cap(M_1 \oplus M_2)$, hence by modular law , A = $M_1 \oplus$(A $\cap M_2$) . Define $\emptyset: \frac{M}{M_1} \to M_2$ by $\emptyset(m + M_1) = m_2$ where m = $m_1 + m_2$, $m_1 \in M_1, m_2 \in M_2$. It is clear that ∅ is an isomorphism. As $\frac{A}{M_1} \ll_G \frac{M}{M_1}$, then $\emptyset(\frac{A}{M_1}) \ll_G M_2$, [7] . But $\emptyset\left(\frac{A}{M_1}\right) =$ A $\cap M_2$. Hence A $\cap M_2 \ll_G M_2$.

2→3) Let A be a cofinite submodule of M . By (2) , there is a decomposition M = $M_1 \oplus M_2$ such that $M_1 \subseteq$ A and A $\cap M_2 \ll_G M_2$ and hence A $\cap M_2 \ll_G M$ by [7]. Now A = A ∩ M = A ∩ ($M_1 \oplus M_2$). So by modular law, A = $M_1 \oplus$(A $\cap M_2$), let S = A ∩ M_2. Thus A = $M_1 \oplus$ S where $M_1 \subseteq_\oplus$M and $S \ll_G M$.

3→1) Let A be a cofinite submodule of M. By (3), A can be written as A = B ⊕ S, where B is a direct summand of M and $S \ll_G M$. To show that $\frac{A}{B} \ll_G \frac{M}{B}$. Let $\frac{M}{B} = \frac{A}{B} + \frac{X}{B}$ where $\frac{X}{B} \subseteq_e \frac{M}{B}$, then M = A + X ,$X \subseteq_e M$, But $S \ll_G M$, so that X=M and hence $\frac{X}{B} = \frac{M}{B}$ and hence $\frac{A}{B} \ll_G \frac{M}{B}$.

Remark 3.3: Let M be an R-module then M is C-G-lifting if and only if for each cofinite submodule A of M there is a decomposition M = $M_1 \oplus M_2$ such that $M_1 \subseteq$A and (A $\cap M_2) \ll_G$M.

Remarks and Examples 3.4:

1- Every lifting is C-G-lifting , for example: Z_4 as Z-module is C-G-lifting.
2- The converse of (1) in general is not true , consider Q as Z-module , since the only cofinite submodule of Q is Q , hence $\exists\{0\} \subseteq Q$, $\{0\} \subseteq_\oplus Q$, $Q = \{0\} \oplus Q$, $Q \cap \{0\} = 0 \ll_G Q$, thus Q is C-G-lifting but not lifting .
3- Consider Z_{24} as Z-module, each of the following submodule $(\bar{2}), (\bar{4}), (\bar{6}), (\bar{8})$, $(\overline{12}), (\bar{0})$ is G-small .Tak N=$(\bar{2})$, $N = (\bar{0}) \oplus N$,$(\bar{0}) \subseteq_\oplus Z_{24}$,$N \ll_G Z_{24}$ Similarly the other submodules satisfy condition (3) of Theorem (3.2). Now take N=$(\bar{3})$, $N = (\bar{3}) \oplus (\bar{0})$, $(\bar{3}) \subseteq_\oplus Z_{24}$,and $(\bar{0}) \ll_G$. Also $Z_{24} = (\bar{3}) \oplus (\bar{8})$, $(\bar{3}) \ll_G Z_{24}$,and $(\bar{8}) \ll_G Z_{24}$, hence Z_{24} as Z-module is C-G-lifting .

Proposition 3.5: Let M = $M_1 \oplus M_2$ be R-module with $M_1, M_2 \subseteq$M and M is distributive , provided $N \cap M_i \neq M_i$ for all i = 1, 2 and N⊆M . if M_1, M_2 are C-G-lifting then M is C-G-lifting .

Proof: Let N be cofinite submodule of M , then N=$(N \cap M_1) \oplus (N \cap M_2)$,$N \cap M_1 \subset M_1$ and $N \cap M_2 \subset M_2$.

Now $\frac{M}{N} = \frac{M_1 \oplus M_2}{N} = \frac{M_1+N}{N} \oplus \frac{M_2+N}{N}$
$= \frac{M_1+(N\cap M_1)+(N\cap M_2)}{(N\cap M_1)\oplus(N\cap M_2)} \oplus \frac{M_2+(N\cap M_1)+(N\cap M_2)}{(N\cap M_1)\oplus(N\cap M_2)}$

$$\cong \frac{M_1}{(N \cap M_1)} \oplus \frac{M_2}{((N \cap M_2)}$$

Now $\frac{\frac{M}{N}}{\frac{M_1+N}{N}} \cong \frac{M}{M_1+N} = \frac{M_1 \oplus M_2}{M_1+N} = \frac{(M_1+N)+M_2}{(M_1+N)} \cong \frac{M_2}{(M_1+N)\cap M_2} = \frac{M_2}{N\cap M_2}$, therefore

$\frac{M_2}{N \cap M_2}$ *is finitely generated.*

Similarly $\frac{M_1}{N\cap M_1}$ *is finitely generated* , hence $N \cap M_1$ and $N \cap M_2$ are cofinite submodules of M_1 and M_2 respectively . Since M_1 and M_2 are C-G-lifting , then $\exists K_1$ *a direct summand of* M_1 , $M_1 = K_1 \oplus K_1'$, $K_1' \subseteq M$ *such that* $\frac{N\cap M_1}{K_1} \ll_G \frac{M_1}{K_1}$

And $\exists K_2$ *a direct summand of* M_2 , $M_2 = K_2 \oplus K_2'$, $K_2' \subseteq M$ *such that*

$\frac{N\cap M_2}{K_2} \ll_G \frac{M_2}{K_2}$,Thus M=$K_1 \oplus K_2 \oplus K_1' \oplus K_2'$, then $K_1 \oplus K_2$ is a direct summand of M.

N=$(N \cap M_1) \oplus (N \cap M_2)$ and $\frac{(N\cap M_1)\oplus(N\cap M_2)}{K_1 \oplus K_2} \ll_G \frac{M}{K_1 \oplus K_2}$.

Then M is C-G-lifting.

Proposition 3.6: Let N be proper submodule of M , if M is C-G-lifting then M/N is C-G-lifting .

Proof: Let $\frac{K}{N} \subset \frac{M}{N}$ and $K \neq M$. Such that $\frac{K}{N}$ is cofinite submodule of $\frac{M}{N}$, then $\frac{M/N}{K/N} \cong \frac{M}{K}$ is finitely generated , then K is a cofinite submodule of M , since M is C-G-lifting then $\exists K_1$ *a direct summand of M* such that $\frac{K}{K_1} \ll_G \frac{M}{K_1}$ then $\frac{K/N}{K_1/N} \ll_G \frac{M/N}{K_1/N}$, [7] .

Corollary 3.7: The nonzero homomorphic image of C-G-lifting module is C-G-lifting.

Corollary 3.8: The direct summand of C-G-lifting is again C-G-lifting .

Proposition 3.9: Every C-G-hollow module is C-G-lifting .

Proof: Let N ⊂M such that M/N is finitely generated. Then M = $(\bar{0})$⊕M, $(\bar{0})$ ⊆N, N ∩M = N$\ll_G M$. Thus M is C-G-lifting.

.**Remark 3.10:**The converse of Proposition (3.9) is not true in general for example Z_{12} as Z-module is C-G-lifting but it is not C-G-hollow , since Take N= $(\bar{2})$ is cofinite submodule of Z_{12} which is not G-small .

But under certain condition we have :-

Proposition 3.11: Let M be a non zero indecomposable R-module then the following are equivalent :-

1- M is C-G-hollow .
2- M is C-G-lifting .

Proof: 1 → 2)by proposition 3.9 .

2 → 1) Let N be a proper cofinite submodule of M , by (2), ∃ K⊆N such that M=K⊕K' , K'∩N $\ll_G$ M . but M is indecomposable then either K=0 or K=M , if K=M , then N=M which is a contradiction , thus K=0 , hence K'=M and K'∩N=M∩N$\ll_G$ M =N .

Notice that Z as Z-module (by proposition 3.11) is not C-G-lifting module since it is not C-G-hollow .

Remark 3.12: If M is C-G-hollow , then M needn't be indecomposable , for example Z_6 as Z-module is C-G-hollow , but not indecomposable .

4-Cofinite Generalized Hollow $lifting_g$Module

In this section we introduce cofinite generalized hollow $lifting_g$ module as a generalization of generalized hollow $lifting$ module .

Definition 4.1:-An R-module M is called cofinite generalized hollow $lifting_g$ module (for short C-G-hollow $ifting_g$ module) , if for every cofinite submodule N of M with $\frac{M}{N}$ is G-hollow , there exist a direct summand K of M , $k \subseteq N$, such that $M = K \oplus K'$, $K' \subseteq M$, and $N \cap K' \ll_G M$.

Examples and Remarks 4.2:-

1- Every semi-simple module is C-G-hollow $lifting_g$ module . in particular it is clear that Z_6 as Z-module is C-G-hollow $lifting_g$ module.
2- Every lifting module is C-G-hollow $lifting_g$ module.
3- Every hollow module is C-G-hollow $lifting_g$ module.
4- The converse of (2) and (3) is not true in general , consider Q as Z-module , since the only cofinite submodule of Q is Q , hence Q is C-G-hollow $lifting_g$, but not lifting and hence not hollow.
5- If M is C-G-hollow , then M is C-G-hollow $lifting_g$ module , to see that let N be a cofinite submodule of M such that M/N is G-hollow, then $N \ll_G M$, hence $\exists 0 \subset N$, $0 \subseteq_\oplus M$, $M = 0 \oplus M$, $M \cap N = N \ll_G M$, thus N is C-G- $lifting_g$ module .
6- It is clear that Z as Z-module is not C-G-hollow $lifting_g$ module. To see that , assume that Z is C-G-hollow $lifting_g$ module , consider $4Z \subseteq Z$,$\frac{Z}{4Z}$ is hollow , hence C-G-hollow , then $\exists K \subseteq_\oplus Z$, $K \subseteq 4Z$. But Z is indecomposable , then K=0 or K=Z if K=Z then 4Z=Z which is a contradiction . then K=0 hence $4Z \ll_G Z$ which is a contradiction . then Z is not C-G-hollow $lifting_g$.

Proposition 4.3: Let M be a non zero indecomposable module , then the following are equivalent:

1- M is C-G-hollow $lifting_g$ module .
2- M is C-G-hollow or else M has no G-hollow factor module for every cofinite submodule of M .

Proof:1→2) suppose that M has a G-hollow factor module for every cofinite submodule of M , and let N be a proper cofinite submodule of M , then by assumption , M/N is G-hollow , by (1) $\exists K \subseteq N , K \subseteq_\oplus M$, i.e $M = K \oplus K'$, for $K' \subseteq M , N \cap K' \ll_G M$, but M is indecomposable thus either K=M or K=0 , if K=M , then N=M which is a contradiction hence K=0 , i.e $K' = M$ and $N \cap K' = N \cap M = N \ll_G M$. therefore M is C-G-hollow module

2→1 Clear.

Proposition 4.4: Let M be any R-module, then the following are equivalent :

1. M is a C-G-hollow $lifting_g$ module.

2. Every cofinite submodule N of M , with $\frac{M}{N}$ is G-hollow, has a G-supplement K in M such that $N \cap K \subseteq_\oplus N$.

Proof:- 1 → 2) Let N *be a cofinite submodule of M with* $\frac{M}{N}$ is G-hollow since M is a C-G-hollow $lifting_g$ module , then $\exists K \subseteq_\oplus M$, K⊆ Nsuch that $M = K \oplus K'$ and $N \cap K' \ll_G M$. thus M=N+K' , $N \cap$

$K' \ll_G M$, hence $N \cap K' \ll_G M$. To prove $\cap K' \subseteq_{\oplus} N$. since $= K \oplus K'$, then $N \cap M = N = (N \cap K) \oplus (N \cap K') = K \oplus (N \cap K')$, hence $N \cap K' \subseteq_{\oplus} N$ .2 → 1) Let N be a submodule of M , with $\frac{M}{N}$ is G-hollow by (2) , $\exists K \subseteq M$, M=N+K , $N \cap K \ll_G M$ and $N \cap K \subseteq_{\oplus} N$, N=($N \cap K) \oplus L$, $L \subseteq N$. M=($N \cap K) \oplus L + K$ =L+K . let $x \in L \cap K$ then $x \in L$ and $x \in K$; since $L \subseteq N$ then $x \in N$, then $x \in L$ and $x \in N \cap K$. but $L \oplus (N \cap K)$=0 , then $x = 0$, then $M = L \oplus K$ i.e $K \subseteq_{\oplus} M$ and $N \cap K \ll_G M$.

Proposition 4.5:- let M be an R-module , then M is C-G-hollow $lifting_g$ module if and only if for every cofinite submodule N of M with $\frac{M}{N}$ is G-hollow , then there exists an idempotent $f \in End(M)$ $with$ $f(M) \subset N$ and $(1-f)(N) \ll_G (1-f)(M)$.

Proof:- →)Assume that M is C-G-hollow $lifting_g$ module , let $N \subseteq M$ with $\frac{M}{N}$ is G-hollow , then (by proposition 4.4) N has a G-supplement K in M such that $N \cap K \subset_{\oplus} N$, then $M = N + K$, $N \cap K \ll_G M$, $N = (N \cap K) \oplus L$ for $L \subseteq N$.

Note That : $M = N + K = (N \cap K) + L + K = L + K$, and $N \cap L \cap K = 0$ $then$ $L \cap K = 0$ then $M = K \oplus L$.

Let $f: M \to L$ be the projection map $(M) \subset L \subset N$.

It is enough to show that $(1-f)(N) \ll_G (1-f)(M)$, one can easily to show that $(1-f)(N) = N \cap (1-f)(M) = N \cap K \ll_G (M)$.

←) Let N be a cofinite submodule of M with $\frac{M}{N}$ is G-hollow . by assumption ∃ an idempotent $f \in End(M)$ such that $f(M) \subseteq N$ and $(1-f)(N) \ll_G (1-f)(M)$ and clearly $M = f(M) \oplus (1-f)(M)$ and $N \cap (1-f)(M) = (1-f)(N) \ll_G (1-f)(M)$, thus M is C-G-hollow $lifting_g$.

Proposition 4.6: Let M be a G-hollow module , Then the following are equivalent:

1. M is a C-G-hollow $lifting_g$module.

2. M is a C-G-lifting module.

Proof : 1→2 let N be a cofinite submodule of M , then by [3],$\frac{M}{N}$ is C-G-hollow and by (1) M is C-G-lifting.

2→1 Clear.

Proposition 4.7: Let M be an R-module , M is a C-G-hollow $lifting_g$ module if and only if every cofinite submodule N of M such that $\frac{M}{N}$ G-hollow, can be written as N = K ⊕ L , where K is a direct summand of M and L$\ll_G (M)$.

Proof : →)Let N⊆ M , with $\frac{M}{N}$ is G- hollow , since M be a C-G-hollow $lifting_g$ module , then there exist a direct summand K of M , K ⊆ N $M = K \oplus K'$, $K' \subseteq M$ and N∩ $K' \ll_G M$, then N =$N \cap M = N \cap (K \oplus K')$ = K⊕(N ∩ K').

←) Let N⊆ M , with $\frac{M}{N}$ is G- hollow , then by (2) $N = K \oplus L$ where K is a direct summand of M and L$\ll_G$M , then M= $K \oplus K'$ and $K' \cap N = K' \cap (K \oplus L) = K' \cap L \subseteq L \ll_G M$. Hence M is C-G-hollow $lifting_g$ module .

Proposition 4.8 : Let M be any R-module and let $N \subseteq M$, if M is a C-G-hollow $lifting_g$ module , then $\frac{M}{N}$ is a C-G-hollow $lifting_g$module.

Proof: let $\frac{K}{N} \subseteq \frac{M}{N}$ such that $\frac{\frac{M}{N}}{\frac{K}{N}} \cong \frac{M}{K}$ is G-hollow . Since M is C-G-hollow $lifting_g$, then ∃L$\subseteq_{\oplus}$ M, $M = L \oplus L'$, L⊆K, $L' \cap K \ll_G M$. Now $\frac{M}{N} = \frac{L \oplus L'}{N} = \frac{L+N}{N} \oplus \frac{L'+N}{N}$ and $\frac{L'+N}{N} \cap \frac{K}{N} = \frac{(L'+N) \cap K}{N} = \frac{(L' \cap K)+N}{N} \ll_G \frac{M}{N}$.

Corollary 4.9: The homomorphic image of C-G-hollow $lifting_g$ module is again C-G-hollow $lifting_g$.

Corollary 4.10:

The direct summand of C-G-hollow $lifting_g$ module is again C-G-hollow $lifting_g$ module.

Proposition 4.11 :- Let $M = M_1 \oplus M_2$ be duo module , if M_1 , M_2 are C-G-hollow $lifting_g$, then M is C-G-hollow $lifting_g$.

Proof :- Let N be a $cofinite$ $submodule$ of M such that $\frac{M}{N}$ is G-hollow , then $N = (N \cap M_1) \oplus (N \cap M_2)$.

$$\frac{M}{N} = \frac{M_1 \oplus M_2}{N} = \frac{M_1 + N}{N} \oplus \frac{M_2 + N}{N} \cong \frac{M_1}{M_1 \cap N} + \frac{M_2}{M_2 \cap N}.$$

Thus $\frac{\frac{M}{N}}{\frac{M_2}{M_2 \cap N}} \cong \frac{M_2}{M_2 \cap N}$. since $\frac{M}{N}$ is G-hollow , then $\frac{M_2}{M_2 \cap N}$ is G-hollow , and similarly $\frac{M_1}{M_1 \cap N}$ is G-hollow , since M_1 , M_2 are C-G-hollow $lifting_g$ module , since M_1 , M_2 are C-G-hollow $lifting_g$ module , then $\exists K_1 \subseteq_{\oplus} M_1$, $K_1 \subseteq M_1 \cap N$ such that M_1=$K_1 \oplus L_1$, $L_1 \subseteq M_1$ and $L_1 \cap (M_1 \cap N) \ll_G M$.

$\exists K_2 \subseteq_{\oplus} M_2$, $K_2 \subseteq M_2 \cap N$ such that $M_2 = K_2 \oplus L_2$, $L_2 \subseteq M_2$ and $L_2 \cap (M_2 \cap N) \ll_G M$.

M= $M = M_1 \oplus M_2 = K_1 \oplus L_1 \oplus K_2 \oplus L_2 = K_1 \oplus K_2 \oplus L_1 \oplus L_2$ then $K_1 \oplus K_2 \subseteq_{\oplus} M$ and N=(N ∩ $M_1) \oplus (N \cap M_2)$ and $\frac{(N \cap M_1) \oplus (N \cap M_2)}{K_1 \oplus K_2} \ll_G \frac{M}{K_1 \oplus K_2}$, then M is C-G-hollow $lifting_g$

Corollary 3.12 :- let $M = M_1 \oplus M_2 \oplus \dots \dots \oplus M_n$ be a duo module if $\forall i = 1,2, \dots, n$, M_i is G-hollow $lifting_g$ then M is G-hollow $lifting_g$.

Proposition 4.13 : Let M be an R-module with $Rad_g(M) = 0$, then M is C-G-hollow $lifting_g$ module if and only if every submodule N of M with $\frac{M}{N}$ is G-hollow is a direct summand of M .

Proof:- →) Let $N be\ a\ cofinite\ submodule\ of\ M$ with $\frac{M}{N}$ is G-hollow , since M is C-G-hollow $lifting_g$, then $\exists K \subseteq_{\oplus} M, K \subseteq N$ and $M = K \oplus K'$, $N \cap K' \ll_G M$, then , $hence N \cap K' \subseteq Rad_g(M) = 0$ thus $M = N \oplus K'$ hence $N \subseteq_{\oplus} M$.

←) Let $N\ be\ a\ cofinite\ submodule\ of\ M$, N is cofinite in M and $\frac{M}{N}$ is G-hollow , hence $N \subseteq_{\oplus} M$, then $M = N \oplus K$, $K \subseteq M$ and $N \cap K = 0 \ll_G M$, hence M is C-G-hollow $lifting_g$ module.

Proposition 4.14:- Let R be a non zero indecomposable and M is projective R-module , if M is C-G-hollow $lifting_g$ module , then $\forall a \in M\ with\ \frac{M}{Ra}$ is G-hollow and Ra is cofinite in M either Ra is projective summand of M or Ra $\ll_G M$.

Proof:- Let $a \in M$ with $\frac{M}{Ra}$ is G-hollow then by proposition (4.7) $Ra = K \oplus L$ for $L \subseteq Ra$ and K is a direct summand of M . $L \ll_G M$. Let $\emptyset: R \to Ra$ defined by : $\emptyset(r) = ra$, $\forall r \in R$, $\emptyset$ is epimorphisim . let $\rho: Ra \to K$ be the projective map . $\rho \circ \emptyset :\ R \to K$ is an epimorphism . consider the following :-

$$\mathbf{0} \to \mathbf{ker}(\boldsymbol{\rho} \circ \emptyset) \xrightarrow{i} \boldsymbol{R} \xrightarrow{\rho \circ \emptyset} \boldsymbol{K} \to \mathbf{0}$$

Where i is the inclusion map . since K is a direct summand of M and M is projective then K is projective , the sequence is splites thus ker$(\rho \circ \emptyset)$ is a direct summand of R .

But ker$(\rho \circ \emptyset)$=$\{r \in R\ ; (\rho \circ \emptyset) = 0\}$

= $\{r \in R\ ; \emptyset(r) \in L\}$=$\emptyset^{-1}(L)$.

Thus $\emptyset^{-1}(L)$ is a direct summand of R , but R is indecomposable thus $\emptyset^{-1}(L)$=0 or $\emptyset^{-1}(L)$=R. Hence either L=0 thus Ra=K , hence Ra is projective direct summand or $\emptyset^{-1}(L) = R$, thus $\emptyset\emptyset^{-1}(L)$= $\emptyset(R)$

then L = Ra

But $\cap\ Ra \ll_G M$, hence. Hence $Ra \ll_G M$.

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