GLOBAL STABILITY OF PREY-PREDATOR WITH HARVESTING AND HOLLING TYPE IV FUNCTIONAL RESPONSE

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ABSTRACT: *This study has proposed to discuss the mathematical model of phytoplankton and herbivorous phytoplankton with a functional response of the fourth type of Holling. The local stability of the mathematical system has also been discussed. Furthermore, the global dynamics of the system were analyzed with the help of the Lyapunov function. At the end, theoretical results enhanced using numerical simulations.*

Keywords: Holling type IV functional response, harmful phytoplankton, herbivorous zooplankton, global stability.

1. INTRODUCTION.

Economic prosperity and environmental balance are always incompatible with interests. The provision of food is one of the necessities of life and amenities provided by humans and adversely affect the ecological structure of nature. This, often, does not lead to the extinction of a kind of life. If we plan properly, we can prevent this extinction. Such planning must be either by force or by offense. For example, if individuals in an area are engaged in a particular activity, this causes severe damage to the ecosystem of that area. If activity is indispensable, the governing body of the region should plan an organizational study that will keep the ecosystem a little harmless. These activities harvest, which has a strong impact on the dynamic development of the population of that area affected by it. The phenomenon of harvesting is one of the major and interesting problems from an ecological and economic point of view. The exploitation of biological resources and the harvesting of habitats are usually practiced in the management of fisheries, forestry and wildlife. The management of multi-species fisheries needed to maintain ecological balance is hampered by the over-exploitation of many traditional fish stocks and the growing interest in harvesting new types of food from the sea. Some researchers have studied and analyzed the problem of predator-prey interactions under the fixed rate of harvesting or fixed quotas of harvest of either species or both at a time. For example, Brauer and Soudak [5-8].

2. Mathematical model formulation

Consider the simple prey-predator system with Holling type IV functional response which can be written as:

$$\begin{aligned} \frac{dx}{dt} &= (a-bx)x - \frac{\alpha\gamma xy}{x^2+\gamma x+\gamma\beta} \\ \frac{dy}{dt} &= \frac{e\alpha\alpha\gamma x}{x^2+\gamma x+\gamma\beta} - hy - q_0 Ey \end{aligned} \qquad (1)$$

Here $x(t)$ and $y(t)$ represent the densities of prey and predator at time t respectively. While the parameters $a>0$ is the intrinsic growth rate of the prey population; $b>0$ is the strength of the specific intracellular competition between the prey ; we can explain the parameter $\beta>0$ that it is a semi-saturation constant in the absence of any inhibitory effect; the parameters $\gamma>0$ is a direct measure of the predator immunity from the prey; the predator consumer consume their food according to Holling type IV of functional response, where $\alpha>0$ is the predation rate on the predator; $e>0$ is the conversion rate of predation into higher level species; here $E>0$ is harvesting effort and $q_0>0$ is the catch ability coefficient. The catch-rate function $q_0 E$ is based on the catch-per-unit-effort (CPUE). Finally $h>0$ represent the natural death rate for the predator. The initial condition for system (1) may be taken as any point in the region $R^2_+ = \{(x,y): x\ge 0, y\ge 0\}$. Obviously, the interaction functions in the right hand side of system (1) are continuously differentiable functions on R^2_+, hence they are Lipschitizian. Therefore the solution of system (1) exists and is unique. Further, all the solutions of system (1) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1) which initiate in R^2_+ are uniformly bounded.

Proof. Let $((x(t), y(t))$ be any solution of the system (1) with non-negative initial condition (x_0, y_0). According to the first equation of system (1) we have

$$\frac{dx}{dt} \le (a-bx)x$$

Then by solving this differential inequality we obtain that

$$x(t) \le \frac{ax_0}{ae^{-at}+(1-e^{-at})bx_0}$$

Thus $\lim\limits_{t\to\infty} \text{Sup}\, x(t) \le M$ where $M = \max\left\{\frac{a}{b}, x_0\right\}$.

Define the function: $W(x,y) = x + \frac{1}{e}y$

So the time derivative of $W(t)$ along the solution of the system (1)

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{e}\frac{dy}{dt}$$

$$\frac{dW}{dt} \le (a+h)x - h(x + \tfrac{1}{e}y)$$

$$\frac{dW}{dt} + hW \le (a+h)M$$

Again by solving the above linear differential inequality we get

$$\frac{dW}{dt} \le \frac{(a+h)M}{h} + W(0)e^{-ht} - \frac{(a+h)}{h}Me^{-ht}$$

Consequently, for $t\to\infty$ we have

$$0 \le W(t) \le \frac{(a+h)M}{h}$$

Hence all solution of system (1) enter the region $\Omega = \{((x(t), y(t) \in R^2_+ : x(t) + \frac{1}{e} y(t) \le \frac{(a+h)M}{h} + \varepsilon \text{ for any } \varepsilon > 0\}$

■

2. Existence of fixed points

The system (1) have at most three non-negative equilibrium points, two of them namely $F_0 = (0,0)$, $F_x = (\frac{a}{b}, 0)$ always exist. While the existence of other equilibrium points is shown in the following:

The positive equilibrium point $F_{xy} = (x^*, y^*)$ exists in the interior of the first quadrant if and only if there is a positive solution to the following set of algebraic nonlinear equations:

$$a - bx - \frac{\alpha\gamma y}{x^2 + \gamma x + \gamma\beta} = 0 \quad (2a)$$

$$\frac{e\alpha\alpha\gamma}{x^2 + \gamma x + \gamma\beta} - h - q_0 E = 0 \quad (2b)$$

From (2.2a) we have

$$y^* = \left(a - bx^*\right) \frac{x^{*2} + \gamma x^* + \gamma\beta}{\alpha\gamma}$$

Clearly, $y^* > 0$ if the following condition holds

$$a > bx^*$$

While x^*, represents the positive root to the following equation

$$f(x) = A_2 x^2 + A_1 x + A_0 \quad (3)$$

Where

$$A_2 = -(h + q_0 E), \qquad A_1 = \gamma(e\alpha - h - q_0 E),$$

$$A_0 = -\gamma\beta(h + q_0 E)$$

So by using Descartes rule of signs, Eq. (3) has either no positive root and hence there is no equilibrium point or two positive roots depending on the following condition holds:

$$e\alpha > h + q_0 E$$

3. The stability analysis

In this section the stability (locally as well as globally) analysis of the above mentioned fixed points of system (1) are investigated analytically.

The **vibrational** matrix of system (1) at the equilibrium point $F_0 = (0,0)$ can be written as

$$J_0 = J(F_0) = \begin{bmatrix} a & 0 \\ 0 & -(h + q_0 E) \end{bmatrix}$$

$$\lambda_{01} = a > 0, \ \lambda_{02} = -(h + q_0 E) < 0$$

Therefore, the equilibrium point F_0 is a saddle point.

The **variational** matrix of system (1) at the equilibrium point $F_x = (\frac{a}{b}, 0)$ can be written as

$$J_x = J(F_x) = \begin{bmatrix} -a & \frac{-\alpha\gamma ab}{a^2 + \gamma b(a + \beta b)} \\ 0 & \frac{e\alpha\alpha\gamma a}{a^2 + \gamma b(a + \beta b)} - h - q_0 E \end{bmatrix}$$

Hence, the eigenvalues of J_x are:

$$\hat{\lambda}_1 = -a < 0, \ \hat{\lambda}_2 = \frac{e\alpha\alpha\gamma a}{a^2 + \gamma b(a + \beta b)} - h - q_0 E$$

Therefore, F_x is locally asymptotically stable if and only if

$$\frac{e\alpha\alpha\gamma a}{a^2 + \gamma b(a + \beta b)} < h + q_0 E \quad (4a)$$

While F_x is saddle point provided that

$$\frac{e\alpha\alpha\gamma a}{a^2 + \gamma b(a + \beta b)} > h + q_0 E \quad (4b)$$

Finally, the **variational** matrix of system (1) at the positive equilibrium point $F_{xy} = (x^*, y^*)$ in the $\text{Int.}R^2_+$ can be written as:

$$J_{xy} = J(F_{xy}) = \begin{bmatrix} \left(-b + \frac{\alpha\gamma y^*(2x^* + \gamma)}{R^{*2}}\right) x^* & \frac{-\alpha\gamma x^*}{R} \\ \left(\frac{e\gamma\gamma\alpha(\gamma - x^{*2})}{R^{*2}} - \theta D\right) y^* & 0 \end{bmatrix}$$

Note that according to the stability theorem for the two dimensional dynamical system, $F_{xy} = (x^*, y^*)$ is locally asymptotically stable provided that

$$\text{Trace } (J_{xy}) = T = a_{11} < 0, \ |J_{xy}| = D = -a_{12} a_{21} > 0$$

Now since

$$T = \left(-b + \frac{\alpha\gamma y^*(2x^* + \gamma)}{R^{*2}}\right) x^* \quad (5a)$$

$$D = \frac{\alpha\gamma x^* y^*}{R^{*3}} \left(e\gamma\gamma\alpha(\gamma - x^{*2}) - \theta D R^{*2}\right) \quad (5b)$$

Therefore the positive equilibrium point $F_{xy} = (x^*, y^*)$ of system (1) is locally asymptotically stable in $\text{Int.}R^2_+$ under the following necessary and sufficient conditions

$$b > \frac{\alpha\gamma y^*(2x^* + \gamma)}{R^{*2}} \quad (6a)$$

$$e\gamma\gamma\alpha(\gamma - x^{*2}) > \theta D R^{*2} \quad (6b)$$

Now we will study the continuity of the system (1). It is common knowledge that the system remains only if each type continues. Mathematically, if the system (1) continues if the system is not resolved with the initial positive state of the Omega boundary sets at the border levels of its own domain. However, biology means that all forms are survivors. In the following theory the condition of constancy of the system (1) is established using the Gard and Hallam technique [12].

Theorem (2). The system (1) uniformly persists provided that condition (4b) holds.

Proof. Consider the following function, $\sigma(x, y) = x^{p_1} y^{p_2}$ where $p_i, i = 1,2$ undetermined positive constants.

Obviously, $\sigma(x, y)$ is C^1 positive function defined on R^2_+, and $\sigma(x, y) \to 0$, if $x \to 0$ or $y \to 0$. Now since

$$\Psi(x, y) = \frac{\sigma'(x,y)}{\sigma(x,y)} = p_1 \frac{x'}{x} + p_2 \frac{y'}{y}$$

Therefore

$$\Psi(x, y) = p_1\left[a - bx - \frac{\alpha\gamma y}{x^2+\gamma x+\gamma\beta}\right] + p_2\left[\frac{e\alpha\alpha\gamma}{x^2+\gamma x+\gamma\beta} - h - q_0 E\right]$$

Note that, since $F_0 = (0,0)$ and $F_x = (\frac{a}{b},0)$ are the only possible omega limit sets of the solution of system (1) on the boundary of $Int.R^2_+$, in addition

$$\Psi(F_0) = ap_1 - dp_2$$
$$\Psi(F_x) = \left(\frac{e\alpha\alpha\gamma a}{a^2+\gamma b(a+\beta b)} - h - q_0 E\right)p_2 > 0$$

Clearly $\Psi(E_0) > 0$ for all sufficiently large positive value of p_1 with respect to p_2, while $\Psi(E_x) > 0$, for all values of p_2 under condition (4b).Hence σ represents persistence function and system (1) is uniformly persistent. ■

Since system (1) may have either two fixed points or no equilibrium points in the $Int.R^2_+$ of the F_{xy}. The global stability of the equilibrium point F_x in R^2_+ is investigated as shown in the following

4. Global stability of the system

In this section the global stability of the equilibrium points F_x in R^2_+ is investigated as shown in the following theorem.

Theorem (3). Assume that the equilibrium point F_x is locally asymptotically stable in the R^2_+, and let the following conditions:

$$h \geq \frac{e\alpha\alpha\gamma a}{a^2+\gamma b(a+\beta b)} \qquad (7)$$

Hold, then F_x is globally asymptotically stable in the R^2_+.

Proof. Consider the following positive definite function:

Where $\bar{x} = \frac{a_1}{b}$. Clearly $U_1 : R^2_+ \to R,$ and is a C^1 positive definite function, where $c_i, (i = 1,2)$ are positive constants to be determined. Now, since the derivative of U_1 along the trajectory of system (2.1) can be written as:

$$\frac{dU_1}{dt} = -c_1 b(x - \bar{x})^2 - (c_1 - c_2 e)\frac{\alpha\gamma xy}{x^2+\gamma x+\gamma\beta} - \left(c_2 h - \frac{c_1\alpha\gamma\bar{x}}{x^2+\gamma x+\gamma\beta}\right)y - c_2 q_0 Ey$$

Since, we have $x \leq \frac{a}{b}$ then by choosing the positive constants as $c_1 = 1$ and $c_2 = \frac{1}{e}$ gives:

$$\frac{dU_1}{dt} \leq -b\left(x - \frac{a}{b}\right)^2 - \left(\frac{h}{e} - \frac{\alpha\gamma ab}{a^2+\gamma b(a+\beta b)}\right)y$$

Therefore, $\frac{dU_1}{dt} < 0$ under conditions (7) hence U_1 is strictly Lyapunov function. Therefore, F_x is globally asymptotically stable in the R^2_+. ■

5. Numerical analysis

In this section the global dynamics of system (1) is studied numerically. System (1) is solved numerically for different sets of parameters and for different sets of initial conditions, and then the attracting sets and their trajectory as function of time are drown as shown below. Now, for the following set of hypothetical parameters

$$a = 0.25, b = 0.2, \alpha = 1, \gamma = 0.75, \beta = 2, e = 0.35, h = 0.01, \quad q_0 = 0.1, E = 0.02 \qquad (8)$$

The attracting sets along with their time series of system (1) are drown in Fig (1). Note that from now onward, in the trajectory as function of time figures, we will use the following representation: blue color represents the trajectory of phytoplankton, green color represents the trajectory of zooplankton.

Clearly, as shown in Fig. (1), system (1) has a globally stable positive equilibrium point $F_{xy} = (0.08,0.48)$ in the $Int.R^2_+$ □, hence all the species coexists and the system persists. However, for the parameters values given by Eq. (8) with the intrinsic growth rate $a = 0.5$, system (1) approaches to the periodic dynamics in $Int.R^2_+$, see the following figure.

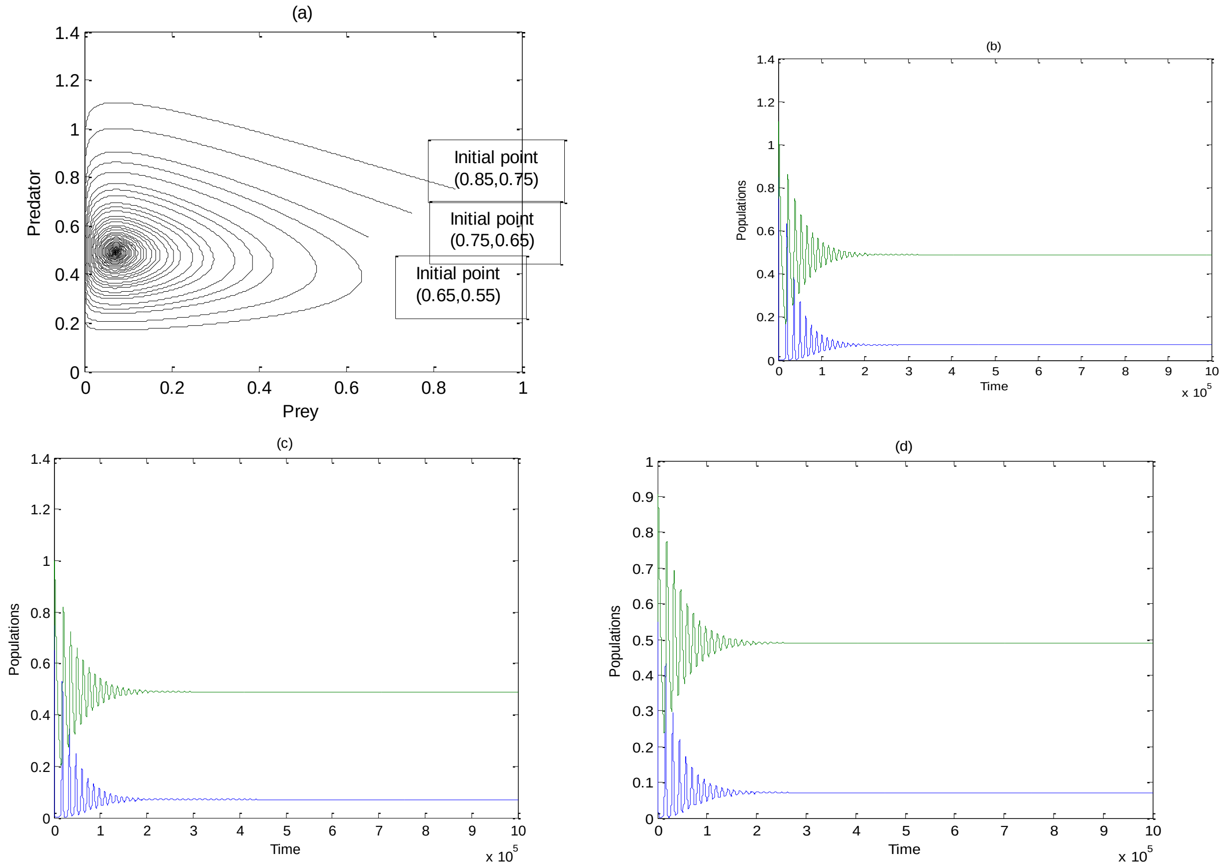
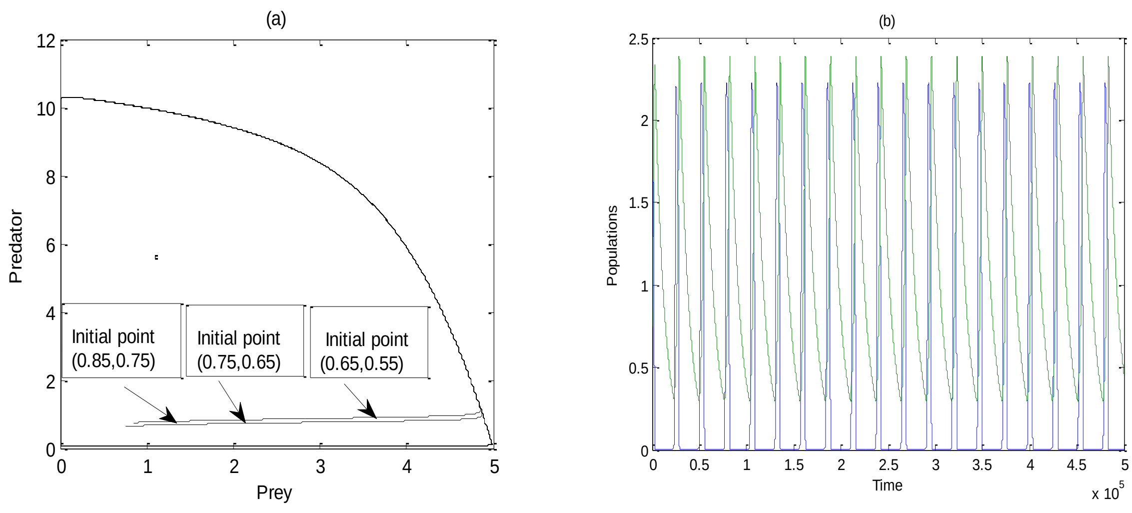


Fig. (1): (a) the solution of system (1) approaches asymptotically to the positive fixed point starting from different initial values for the data given by Eq. (8). (b) Trajectory as function of time in (a) starting at (0.85, 0.75). (c) Trajectory as function of time in (a) starting at (0.75, 0.65) (d) Trajectory as function of time in (a) starting at (0.65, 0.55).



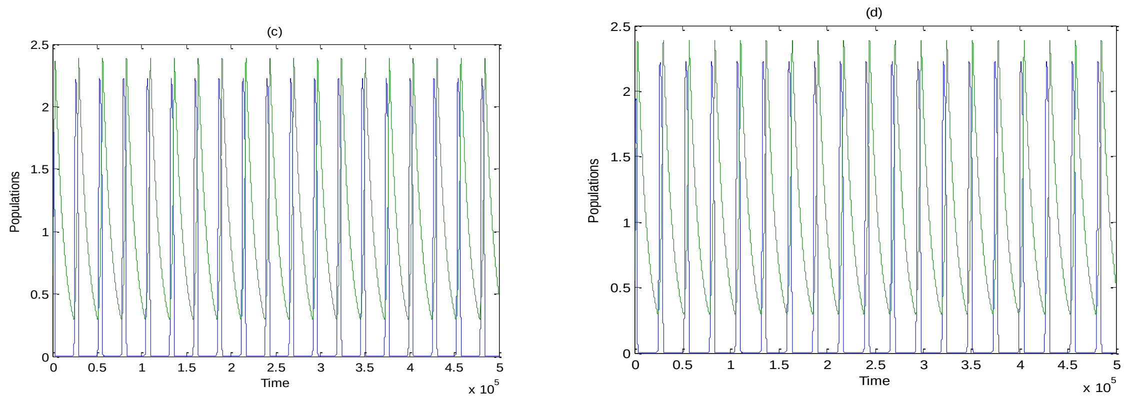


Fig. (2): (a) globally asymptotically stable limit cycle of system (1) starting from different initial values for the data given by Eq. (8) with $a = 0.5$. (b) Trajectory as function of time in (a) starting at (0.85, 0.75). (c) Trajectory as function of time in (a) starting at (0.75, 0.65) (d) Trajectory as function of time in (a) starting at (0.65, 0.55).

6. DISCUSSION AND CONCLUSION

In this paper, a mathematical model consisting of a Holling type IV phytoplankton- zooplankton model with intra specific competition has been studied analytically as well as numerically. The condition for the system (1) to be uniformly bounded and persistence have been derived. The local as well as global stability of the proposed system has been studied. The effect of intrinsic growth rate of the phytoplankton species on the dynamical behavior of system (1) is studied numerically and the trajectories of the system are drowned. According to these formats the following conclusions are obtained:

1. For the set of hypothetical parameters values given in Eq. (8), system (1) approaches asymptotically to a globally asymptotically stable point
2. $F_{xy} = (x^*, y^*)|''$.|
3. As the intrinsic growth rate of the phytoplankton decreasing then the system (1) approaches to an asymptotically stable positive equilibrium point, otherwise the system has periodic dynamics. So this parameter has a stabilizing effect on the system.

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