# ON $(n,m)$-NORMAL POWERS WEIGHTED COMPOSITION OPERATORS ON HARDY SPACE $\mathbb{H}^2$.

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***ABSTRACT:*** *An operator $T \in B(H)$ is called $(n,m)$-normal powers operator if $T^n(T^m)^* = (T^m)^*T^n$ for some nonnegative integers $n$ and $m$. In this paper we characterized $(n,m)$-normal powers weighted composition on Hardy space $\mathbb{H}^2$.*

**Keywords**: Composition operators, weighted composition operators, $(n,m)$-normal powers operators

## .1. INTRODUCTION.

Let $\mathbb{U}$ denote the open unite disc in the complex plan, $\mathbb{H}^\infty$ denotes the collection of all bounded holomorphic functions on $\mathbb{U}$ and let $\mathbb{H}^2$ is consisting of all holomorphic functions on $\mathbb{U}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e) $f(z) = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. More precisely $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{H}^2$ if and only if $\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty$.The inner product inducing the $\mathbb{H}^2$ norm is given by $\langle f,g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$
where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Given any holomorphic self-map $\varphi$ of $\mathbb{U}$, recall that [9] the composition operator is defined as follows $C_\varphi(h) = ho\varphi \quad (h \in \mathbb{H}^2)$. It is called the composition operator with symbol $\varphi$, is necessarily bounded. Moreover, let $f \in \mathbb{H}^\infty$, the operator $T_f$ defined by $T_f(h(z)) = f(z)h(z), \quad (z \in \mathbb{U}, h \in \mathbb{H}^2)$
is called the Toeplitz operator on $\mathbb{H}^2$ with symbol $f$ . Since $f \in \mathbb{H}^\infty$, then $T_f$ is called a holomorphic Toeplitz operator. If $T_f$ is a holomorphic Toeplitz operator, then the operator $T_f C_\varphi$ is bounded and has the form

$$T_f C_\varphi g = f(g \circ \varphi) \qquad (g \in \mathbb{H}^2).$$

It is called the weighted composition operator with symbols $f$ and $\varphi$ [7]. The weighted composition operator is denoted by

$$\mathcal{W}_{f,\varphi}\, g = f(g \circ \varphi) \qquad (g \in \mathbb{H}^2).$$

For given holomorphic self-maps $f$ and $\varphi$ of $\mathbb{U}$, $\mathcal{W}_{f,\varphi}$ is bounded operator even if $f \notin \mathbb{H}^\infty$ .To see a trivial example, consider $\varphi(z) = p$ where $p \in \mathbb{U}$ and $f \in \mathbb{H}^2$, then for all $g \in \mathbb{H}^2$, we have

$$\|\mathcal{W}_{f,\varphi}\, g\| = \|g(p)\|\|f\| = \|f\| \,|\langle g, K_p\rangle| \le \|f\| \,\|g\| \,\|K_p\| .$$

In fact, if $f \in \mathbb{H}^\infty$ , then $\mathcal{W}_{f,\varphi}$ is bounded operator on $\mathbb{H}^2$ with norm

$$\|\mathcal{W}_{f,\varphi}\| = \|T_f C_\varphi\| \le \|f\|_\infty \|C_\varphi\| = \|f\|_\infty \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

We collect some properties of Toeplitz and composition operators in the following known results.

***Lemma (1.1)***: Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$, then

(a) $C_\varphi T_f = T_{fo\varphi}\, C_\varphi$.

(b) $T_g T_f = T_{gf}$.

(c) $T_{f+\gamma g} = T_f + \gamma T_g$.

(d) $T_f^* = T_{\bar{f}}$.

**Proposition (1.2):[1]** Let $\varphi$ and $\psi$ be two holomorphic self-map of $\mathbb{U}$, then

**1.** $C_\varphi^n = C_{\varphi_n}$ for all positive integer n, where $\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n-times}$.

**2.** $C_\varphi$ is the identity operator if and only if $\varphi$ is the identity map.

**3.** $C_\varphi = C_\psi$ if and only if $\varphi = \psi$.

**4.** The composition operator cannot be zero operator.

For each $\alpha \in \mathbb{U}$, the reproducing kernel at $\alpha$ ,defined by $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$.
It is easily seen that the family $\{K_\alpha\}_{\alpha\in\mathbb{U}}$ forms a dense subset of $\mathbb{H}^2$. In [4], the adjoint of weighted composition operator on the reproducing kernel at $\alpha$ is as follows
$\mathcal{W}_{f,\varphi}^* K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}$.

If $\varphi(z) = (az+b)/cz+d)$ is linear-fractional self-map of $\mathbb{U}$, Cowen in [5] establishes $C_\varphi^* = T_g C_\sigma T_h^*$, where the Cowen auxiliary functions $g$, $\sigma$ and $h$ are defined as follows:
$g(z) = 1/(-\bar{b}z + \bar{d})$, $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ and $h(z) = cz + d$ .
$Therefore \;\; W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^* = T_g C_\sigma T_h^* T_f^*$.

Recall that an operator $T \in B(H)$ is called normal if $\|Tx\| = \|T^*x\|$ for all $x \in H$. In [2] the author introduced the $(n,m)$-normal powers operators as follows: an operator $T \in B(H)$ is called $(n,m)$-*normal powers* operator if $T^n(T^m)^* = (T^m)^*T^n$ for some nonnegative integers $n$ and m. Moreover, $T$ is called $(n,m)$-*unitary powers* operator if and only if $T^n(T^m)^* = (T^m)^*T^n = I$ for some nonnegative integers $n$ and m. In the following theorem the author gives a necessary condition for $T$ to be $(n,m)$-normal powers operators.

**Proposition (1.3):** Let $T \in B(H)$. If $T$ is $(n,m)$-normal powers operator, then $T^{nm}$ is normal operator.

In [4] Bourdon and Narayan characterized normal weighted composition operator on $\mathbb{H}^2$. In this paper, we give a characterization of $(n,m)$-*normal powers* weighted composition operator on $\mathbb{H}^2$ when $\varphi$ has interior fixed point of $\mathbb{U}$.

## 2. $(n,m)$-*normal powers* weighted composition operator on $\mathbb{H}^2$.

First, Cowen [6] described the normal composition operator as follows.

**Theorem (2.1):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$. Then $C_\varphi$ is normal if and only if $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$.

The following consequence describes the $(n,m)$-*normal powers* composition operator on $\mathbb{H}^2$.

**Theorem (2.2):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$. Then $C_\varphi$ is $(n,m)$-*normal powers* if and only if $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$.

**Proof:** If $C_\varphi$ is $(n,m)$-*normal powers*, then by proposition(1.3) $C_\varphi^{nm} = C_{\varphi_{nm}}$ is normal operator. Thus by theorem(2.1) we have $\varphi_{nm}(z) = \lambda z$ for some $|\lambda| \le 1$. Hence it is easily seen that $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$.

The converse is straightforward by the fact that every normal operator is $(n,m)$-*normal powers* ■

**Corollary(2.3):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$. Then $C_\varphi$ is $(n,m)$-*normal powers* if and only if $C_\varphi$ is normal.

**Corollary(2.4):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$. Then $C_\varphi$ is $(n,m)$-*unitary powers* if and only if $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$ such that $\overline{\lambda^m}\lambda^n = 1$ for some nonnegative integers $n$ and $m$.

**Proof:** If $C_\varphi$ is $(n,m)$-*unitary powers*, then it is $(n,m)$-*normal powers.* Thus by corollary(2.3) and theorem(2.1) $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$. Moreover, for each $\alpha\epsilon\mathbb{U}$ we have

$C_\varphi^n (C_\varphi^m)^* K_\alpha(z) = C_{\varphi_n}(C_{\varphi_m})^* K_\alpha(z) = (C_{\varphi_m})^* C_{\varphi_n} K_\alpha(z) = K_\alpha(z)$. Hence for each $\alpha\epsilon\mathbb{U}$ we have

$$\begin{aligned} C_\varphi^n (C_\varphi^m)^* K_\alpha(z) &= C_{\varphi_n}(C_{\varphi_m})^* K_\alpha(z) \\ &= C_{\varphi_n} K_{\varphi_m(\alpha)}(z) = K_{\varphi_m(\alpha)}(\varphi_n(z)) \\ &= K_\alpha(z). \end{aligned}$$

It follows that for each $\alpha\epsilon\mathbb{U}$,

$\frac{1}{1-\overline{\varphi_m(\alpha)}\varphi_n(z)} = \frac{1}{1-\bar{\alpha}z}$. Then,

$$\frac{1}{1-\overline{\lambda^m}\lambda^n\bar{\alpha}z} = \frac{1}{1-\bar{\alpha}z}.$$

Thus $\overline{\lambda^m}\lambda^n = 1$. The converse is clear ■

**Proposition (2.5):** Let $\varphi$ be a non-constant holomorphic self-map of $\mathbb{U}$ and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator, then either $f = 0$ or $f$ never vanishes on $\mathbb{U}$.

**Proof:** Assume that $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator such that $f(\beta) = 0$ for some $\beta\epsilon\mathbb{U}$. Thus $\mathcal{W}_{f,\varphi}^* K_\beta = \overline{f(\beta)} K_{\varphi(\beta)} = 0$. But by proposition(1.3) $\mathcal{W}_{f,\varphi}^{nm}$ is normal. Hence

$\|\mathcal{W}_{f,\varphi}^{nm} K_\beta\| = \|(\mathcal{W}_{f,\varphi}^{nm})^* K_\beta\| = \|(\overline{f(\beta)})^{nm} K_{\varphi(\beta)}\|$=0.

Therefore for each $\beta\epsilon\mathbb{U}$ $\mathcal{W}_{f,\varphi}^{nm} K_\beta = 0$. This implies that $\mathcal{W}_{f,\varphi}^{nm} = 0$. Hence by [8] we have

$\mathcal{W}_{f,\varphi}^{nm} = T_{f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1})} C_{\varphi_{nm}} = 0$. But by proposition(1.2) (4) we have $T_{f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1})} = 0$. It follows that $f(\varphi_i(\mathbb{U})) = 0$ for some $1 \le i \le nm-1$. But $\varphi$ is non-constant, then by open mapping theorem $f = 0$ on $\mathbb{U}$.

**Proposition (2.6):** Let $\varphi$ be a non-constant holomorphic self-map of $\mathbb{U}$, $f \in \mathbb{H}^\infty/\{0\}$. If $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator, then $\varphi$ is univalent ■

**Proof:** If $\varphi$ is not univalent on $\mathbb{U}$, then there exists $a, b \in \mathbb{U}$ such that $a \ne b, \varphi(a) = \varphi(b)$. Since $f \ne 0$, then by proposition(2.5) we get that $f(a) \ne 0, f(b) \ne 0$. Put $g = \frac{K_a}{f(a)} - \frac{K_b}{f(b)}$. Since $a \ne b$, then $g \ne 0$. Therefore, it is easily seen that $\mathcal{W}_{f,\varphi}^* g = K_{\varphi(a)} - K_{\varphi(b)} = 0$. But by proposition(1.3) $\mathcal{W}_{f,\varphi}^{nm}$ is normal, then $\|\mathcal{W}_{f,\varphi}^{nm} g\| = \|(\mathcal{W}_{f,\varphi}^{nm})^* g\|$ =0. This implies that $\mathcal{W}_{f,\varphi}^{nm} g = 0$. Therefore $\mathcal{W}_{f,\varphi}^{nm} g = T_{f(f\circ\varphi)(f o\varphi_2)\dots(f o\varphi_{nm-1})} C_{\varphi_{nm}} g = 0$. It implies that $(f\circ\varphi(z))(f\circ\varphi_2(z))\dots(f\circ\varphi_{nm-1}(z))g(\varphi_{nm}(z)) = 0$. Since $f \ne 0$, then by proposition(2.5) $f(\varphi_i(\mathbb{U})) \ne 0$ for each $1 \le i \le nm-1$. It follows that $g(\varphi_{nm}(\mathbb{U})) = 0$ . But $\varphi$ is non-constant, then by open mapping theorem $g = 0$ on $\mathbb{U}$, which a contradiction. Therefore, $\varphi$ is univalent ■

Now we are ready to discuss the sufficient condition for $(n,m)$-*normal powers* operator when $\varphi$ has an interior fixed point of $\mathbb{U}$.

**Proposition (2.7):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$, $f \in \mathbb{H}^\infty$ such that $\varphi(p) = p$ for some $p \in \mathbb{U}$. If $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator, then

$$f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1}) = \frac{(f(p))^{nm}K_p}{K_p\circ\varphi_{nm}}.$$

**Proof:** Since $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers*, then by proposition(1.3) $\mathcal{W}_{f,\varphi}^{nm}$ is normal. But $(\mathcal{W}_{f,\varphi}^{nm})^* K_p = (\mathcal{W}_{f,\varphi}^*)^{nm} K_p = (\overline{f(p)})^{nm} K_p$. Hence $K_p$ is an eigenvector for $(\mathcal{W}_{f,\varphi}^{nm})^*$ corresponding to eigenvalue $(\overline{f(p)})^{nm}$. But $\mathcal{W}_{f,\varphi}^{nm}$ is normal, then $K_p$ is an eigenvector for $\mathcal{W}_{f,\varphi}^{nm}$ corresponding to eigenvalue $f(p)^{nm}$ (see [3]). Therefore $\mathcal{W}_{f,\varphi}^{nm} K_p = f(p)^{nm} K_p$.Thus

$T_{f(f\circ\varphi)(f o\varphi_2)\dots(f o\varphi_{nm-1})} C_{\varphi_{nm}} K_p = f(p)^{nm} K_p$. It follows that $f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1})(K_p\circ\varphi_{nm}) = f(p)^{nm}K_p$. This implies that

$f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1}) = \frac{(f(p))^{nm}K_p}{K_p\circ\varphi_{nm}}$ ■

Now, since $K_0 \equiv 1$, then by Proposition(2.7) we get an immediate result.

**Corollary (2.8):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$, $f \in \mathbb{H}^\infty$ such that $\varphi(0) = 0$. If $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator, then $f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1})$ is constant and $C_{\varphi_{nm}}$is $(n,m)$-*normal powers*.

From corollary(2.8) and theorem(2.2) we conclude the following consequence.

**Corollary (2.9):** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$ with $\varphi(0) = 0$ and $f \in \mathbb{H}^\infty$ such that $|f(z)| = 1$ on $\mathbb{U}$. Then $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator if and only if $f(f\circ\varphi)(f\circ\varphi_2)\dots(f\circ\varphi_{nm-1})$ is constant and $\varphi(z) = \lambda z$ for some $|\lambda| \le 1$.

**Proposition(2.10):** Let $\varphi$ be a linear fractional self-map of $\mathbb{U}$ and $f(f\circ\varphi)(f o\varphi_2)\dots(f o\varphi_{i-1}) = K_{\sigma_{i(0)}}(z)$, $i = n, m$. Then $\mathcal{W}_{f,\varphi}$ is $(n,m)$-*normal powers* operator if and only if

$$\frac{\overline{d_m}d_n}{(\overline{d_m}d_n - \overline{c_m}c_n) - (\overline{b_m}d_n - \overline{a_m}c_n)z} C_{\varphi_n\circ\sigma_m} = \frac{\overline{d_m}d_n}{(\overline{d_m}d_n - \overline{b_m}b_n) - (\overline{b_m}a_n - \overline{d_m}c_n)z} C_{\sigma_m\circ\varphi_n}$$

where $\sigma_i$ is the Cowen auxiliary function of $\varphi_i$.

**Proof:** Recall that if $\varphi$ is a linear fractional self-map of $\mathbb{U}$, then $C_\varphi^* = T_g C_\sigma T_h^*$, where the Cowen auxiliary functions $g, \sigma$ and $h$ are defined as follows:

$g(z) = 1/(-\bar{b}z + \bar{d})$, $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ *and* $h(z) = cz + d$ .

Since $\varphi$ is a linear fractional self-map of $\mathbb{U}$, then it is clear that $C_{\varphi_i}$ is also a linear fractional self-map of $\mathbb{U}$. Therefore, $C_{\varphi_i}^* = T_{g_i} C_{\sigma_i} T_{h_i}^*$, where the Cowen auxiliary functions $g_i$, $\sigma_i$ and $h_i$ are defined as follows:

$$g_i(z) = \frac{1}{-\bar{b}_i z + \bar{d}_i},$$

$$\sigma_i(z) = \frac{\bar{a}_i z - \bar{c}_i}{-\bar{b}_i z + \bar{d}_i},$$

and $h_i(z) = c_i z + d_i$.

Note that, $K_{\sigma_{i(0)}}(z) = \frac{d_i}{c_i z + d_i}$, then $K_{\sigma_{i(0)}}(z) h_i = d_i$, $i = n, m$. Thus for each $v \in \mathbb{H}^2$ we get

$\left(W_{f,\varphi}^{m}\right)^{*}W_{f,\varphi}^{n}v$
$=\left(T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}C_{\varphi_m}\right)^{*}T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}v$
$=C_{\varphi_m}^{*}T_{\overline{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}}T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}v$

$=T_{g_m}C_{\sigma_m}T_{h_m}^{*}T_{\overline{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}}T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}v$
$=T_{g_m}C_{\sigma_m}T_{\overline{h_m K_{\sigma_{m(0)}}}}\ T_{K_{\sigma_{n(0)}}}C_{\varphi_n}v$
$=\overline{d_m}T_{g_m}C_{\sigma_m}T_{K_{\sigma_{n(0)}}}C_{\varphi_n}v$
$=\overline{d_m}T_{g_m}T_{K_{\sigma_{n(0)}}\circ\sigma_m}C_{\sigma_m}C_{\varphi_n}v$
$=\overline{d_m}T_{g_m\left(K_{\sigma_{n(0)}}\circ\sigma_m\right)}C_{\varphi_n\circ\sigma_m}v$
$=\overline{d_m}\,.\,g_m.K_{\sigma_{n(0)}}\circ\sigma_m\,.\,v\circ\varphi_n\circ\sigma_m.$

Similarly,

$W_{f,\varphi}^{n}\left(W_{f,\varphi}^{m}\right)^{*}v$
$=T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}\left(T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}C_{\varphi_m}\right)^{*}v$
$T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}C_{\varphi_m}^{*}T_{\overline{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}}v$

$=T_{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{n-1})}C_{\varphi_n}T_{g_m}C_{\sigma_m}T_{h_m}^{*}T_{\overline{f(f\circ\varphi)(fo\varphi_2)\ldots(fo\varphi_{m-1})}}\ v$
$=T_{K_{\sigma_{n(0)}}}C_{\varphi_n}T_{g_m}C_{\sigma_m}T_{\overline{h_m K_{\sigma_{m(0)}}}}\ v$
$=\overline{d_m}\,T_{K_{\sigma_{n(0)}}}T_{g_m\circ\varphi_n}C_{\varphi_n}C_{\sigma_m}v$
$=\overline{d_m}T_{K_{\sigma_{n(0)}}(g_m\circ\varphi_n)}C_{\sigma_m\circ\varphi_n}v$
$=\overline{d_m}\,.\,K_{\sigma_{n(0)}}.\ g_m\circ\varphi_n\,.\,v\circ\sigma_m\circ\varphi_n.$

as desired.

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