CHARACTERISTICS OF KUMARASWAMY DISTRIBUTION PARAMETERESTIMATIONWITH PROBABILITY WEIGHTED MOMENT (PWM) AND MAXIMUM LIKELIHOOD ESTIMATION (MLE) METHODS

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ABSTRACT*: The distribution of Kumaraswamy has two parameters namely (a, b) with parameters a and b are shape parameters which show the shape of the curve. This Kumaraswamy distribution parameter estimation is obtained by using Probability Weighted Moment (PWM) and Maximum Likelihood Estimation (MLE) methods. This study examined the characteristics of Kumaraswamy distribution parameters estimator (a, b) using the Probability Weighted Moment (PWM) method and Maximum Likelihood Estimation (MLE) method which includes unbiased, minimum variance, consistency, and sufficient statistics characteristics. The results indicated that parameter estimators (a, b) have good estimator characteristics which are unbiased, consistent, minimum variance, and sufficient statistics to attain the Cramer-Rao lower bound.*

Keywords: Kumaraswamy Distribution, Probability Weighted Moment (PWM), Maximum Likelihood Estimation (MLE), Unbiased, minimum variance.

1. INTRODUCTION

Kumaraswamy distribution is a continuous distribution of probabilities defined on intervals [0, 1]. Kumaraswamy distribution is very similar to Beta distribution but it is simpler for use in simulation studies because of its simple and closed shape of both the probability distribution function and the cumulative distribution function. This distribution was first proposed by [1]. The probability distribution function of the Kumaraswamy distribution is:

$$f(x) = abx^{a-1}(1-x^a)^{b-1},\ a > 0, b > 0, 0 < x < 1 \quad (1)$$

where the cumulative distribution function of the Kumaraswamy distribution is as follows:

$$f(x) = 1 - (1 - x^a)^b \quad (2)$$

For a detailed discussion of the Kumaraswamy distribution it can be seen in [2]. The parameter estimation is a process that uses a statistical sample to predict unknown population parameters. To estimate a parameter there are several commonly used estimation methods: Maximum Likelihood, Method of Moments (MM), Ordinary Least Square (OLS), Generalized Methods of Moment (GMM), and Probability Weighted Moment (PWM) [3 – 9]. A good estimator should satisfy some of the estimator characteristics of an expected probability such as unbiased, minimum variance, consistency, efficiency, sufficient statistics, and completeness (complete statistics). There are many forms of Kumaraswamy distribution discussed in the literatures. Nadarajah and Eljabri in 2013 proposed, simple generalization of GP distribution, Kumaraswamy GP (KumGP) distribution [10], and in [11] exp-Kumaraswamy distribution with its properties and its applications was discussed.Mostafa et al. In [12] the Estimation for parameters of the Kumaraswamy distribution based on general progressive type II censoring was studied. These studied are derived by using the maximum likelihood and Bayesian approaches.

El-Sayed and Mohamed in 2014 discussed a new distribution so called the Kumaraswamy – Kumaraswamy (KW-KW) distribution, as a special model from the class of Kumaraswamy Generalized (KW-G) distributions [13]. In their paper, they discussed the pdf, the cdf, the quantiles, the median, the mode, the mean deviation, the entropy, order statistics, L-moments and parameters estimation based on maximum likelihood. The order statistics of Kumaraswamy distribution was discussed in 2009 [14]. In [15] a new class of generalized distributionscalled the Exponentiated Kumaraswamy Lindley (EKL) distribution was disccused. This class of distributions contains the Kumaraswamy Lindley (KL), generalized Lindley (GL), and Lindley(L) distributions as special cases.In [16] Adepoju et al. discussed about the Kumaraswamy Fisher Snedecor Distribution, they discussed some statistical properties of the proposed distribution such as moments, moment generating function, and the asymptotic behavior.

This study discussed further the Kumaraswamy distribution using Probability Weighted Moment (PWM) method and Maximum Likelihood Estimation (MLE) method by looking for characteristic of estimation which includes unbiased, minimum variance, consistency and sufficient statistics.

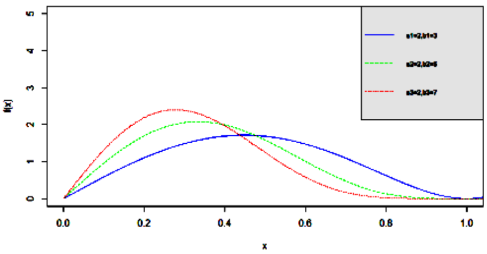
2. METHOD

The steps of the method used in this study are as follows:

1. Creating the curve of probability distribution function of Kumaraswamy distribution with parameters a and b using software R.
2. Finding the value of E (X) and the value of Var (X) from Kumaraswamy distribution.
3. Estimating Kumaraswamy parameter (a, b) using Maximum Likelihood Estimation (MLE) method.
4. Estimating Kumaraswamy parameter (a, b) using Probability Weighted Moment (PWM) method.
5. Examining the characteristic of unbiased from estimator of each parameter a and b of the Probability Weighted Moment (PWM) method.
6. Examining the characteristic of consistency from estimator parameters a and b from the Probability Weighted Moment (PWM) method.
7. Examining the minimum variance of Kumaraswamy distribution.
8. Examining the characteristic of sufficient statistics of the Kumaraswamy distribution.
9. Simulating using Software R for Probability Weighted Moment (PWM) method and Maximum Likelihood Estimation (MLE) method.

2. RESULTS AND DISCUSSION

3.1 Curve of Probability Distribution Function of Kumaraswamy Distribution (a, b)



Fit. 1. The curve of the pdf ofKumaraswamy distribution when the value of a fixed and b increases

The curves of probability distribution function of Kumaraswamy distribution when the value of b increases and a fixed. Blue curve shows the value of a = 2 and b = 3, for the green curve a = 2 and b = 5, while for the red curve the value shows a = 2 and b = 7. From the figure, it can be seen that the greater the parameter value of b, the more pointed shape of the resulting curve.

2.1. The Value of $E(X)$ and $Var(X)$ Kumaraswamy Distribution

From the Kumaraswamy distribution we get the value for the expectation of E(X) that is:

$$E(X) = \frac{b\,\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}{\Gamma\left(\frac{1}{a}+1+b\right)} \qquad (3)$$

Then the $Var(X)$ of Kumaraswamy distribution is obtained as follows:

$$Var(X) = b\,\Gamma(b)x\left[\frac{\Gamma\left(\frac{2}{a}+1\right)}{\Gamma\left(\frac{2}{a}+1+b\right)} - b\,\Gamma(b)\left(\frac{\Gamma\left(\frac{1}{a}+1\right)}{\Gamma\left(\frac{1}{a}+1+b\right)}\right)^2\right] \qquad (4)$$

2.2. Maximum Likelihood Estimation (MLE) Kumaraswamy Distribution (a,b)

From the joint probability distribution function of equation (1) will be formed likelihood function $L(a,b)$that is:

$$L(a,b) = (ab)^n \prod_{i=1}^{n} x_i^{a-1}(1-x_i^a)^{b-1} \qquad (5)$$

So from the equation (5), the maximum function of likelihood is obtained as follows:

$$\ln L(a,b) = n\ln a + n\ln b + (a-1)\sum_{i=1}^{n}\ln(x_i) + (b-1)\sum_{i=1}^{n}\ln(1-x_i^a) \qquad (6)$$

Next, to estimate the Kumaraswamy distribution parameter it can be obtained by finding the first derivative of equation (6) with respect to the parameters a and b and equating to zero.

Parameter a ($\hat{a}$)

$$a = \frac{n}{-\sum_{i=1}^{n}\ln(x_i) + (b-1)\sum_{i=1}^{n}\frac{-x_i^a\ln(x_i)}{1-x_i^a}} \qquad (7)$$

Parameter b ($\hat{b}$)

$$b = \frac{n}{-\sum_{i=1}^{n}\ln(1-x_i^a)} \qquad (8)$$

In determining the estimation in equations (7) and (8) of Kumaraswamy distribution by the Maximum Likelihood Estimation (MLE) method which cannot be solved analytically, this can be solved by numerical iteration method that is Newton Raphson's method.

Newton Raphson Method

The steps in Newton Raphson's method are:

1. Determining the starting value $a_0 = \hat{a}$ and $b_0 = \hat{b}$
2. Determining the first derivative and the second derivative of $z = f(a,b)$:i.e.:

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\ln(x_i) + (b-1)\sum_{i=1}^{n}\ln(1-x_i^a)$$

$$\frac{\partial \ln L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln(1-x_i^a)$$

$$\frac{\partial^2 \ln L}{\partial^2 a} = -\frac{n}{a^2} + (b-1)\sum_{i=1}^{n}\frac{-xi^a\ln^2(x_i)}{(1-x_i^a)^2}$$

$$\frac{\partial^2 \ln L}{\partial^2 b} = -\frac{n}{b^2}$$

3. Defining g_θ as the first gradient vector and derivative vector of its parameters:

$$g_\theta = \begin{bmatrix} \frac{n}{a} + \sum_{i=1}^{n}\ln(x_i) + (b-1)\sum_{i=1}^{n}\ln(1-x_i^a) \\ \frac{n}{b} + \sum_{i=1}^{n}\ln(1-x_i^a) \end{bmatrix}$$

4. Next defining the Hessian matrix H_θ where the Hessian Matrix or second derivative matrix to its parameters, denoted by H_θ are:

$$H_\theta = \begin{bmatrix} -\frac{n}{a^2} + (b-1)\sum_{i=1}^{n}\frac{-x_i^a\ln^2(x_i)}{(1-x_i^a)^2} & \frac{n}{b} + \sum_{i=1}^{n}\frac{-ax_i^{a-1}}{1-xi^a} \\ b-1)\sum_{i=1}^{n}\frac{-x_i^a\ln^2(x_i)}{(1-x_i^a)^2} & -\frac{n}{b^2} \end{bmatrix}$$

5. Iteration will stop when $\left\|\begin{matrix} a_{\theta+1} - a_\theta \\ b_{\theta+1} - b_\theta \end{matrix}\right\| < \varepsilon$, where ε is the specified error limit.

2.3. Probability Weighted Moment (PWM)for Kumaraswamy Distribution

To estimate the parameters of Kumaraswamy distribution (a, b) we have to find the inverse function of the cumulative distribution function, while the inverse functionas in [3] is obtained as follows:

$$x = \left[1-(1-F)^{\frac{1}{b}}\right]^{\frac{1}{a}} \qquad (9)$$

Next to search for Probability Weighted Moment (PWM) of Kumaraswamy distribution by searching for the-r moment is as follows:

$$M_{1,s,0} = \int_0^1 x(F)[F(x)]^s\,df$$

$$= \int_0^1 \left[1-(1-F)^{\frac{1}{b}}\right]^{\frac{1}{a}}[F(x)]^s\,df$$

So the obtained PWM form to estimate parameter a and b from Kumaraswamy distribution is:

$$M_{1,s,0} = b\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{\Gamma\left(\frac{1}{a}+1\right)\Gamma(bk+b)}{\Gamma\left(\frac{1}{a}+1+bk+b\right)} \qquad (10)$$

Parameter Estimator a and b

After obtaining the-r momentM_s)then the next step is to determine the estimators for parameters a and b:

Parameter estimator b ($\hat{b}$)

$$\hat{b} = \frac{\hat{M}_0\,\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)} \qquad (11)$$

Parameter estimator a ($\hat{a}$)

$$\hat{a} = \frac{\left(\frac{\hat{M}_0}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}\right)\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b)\Gamma\left(\frac{1}{a}+1+2b\right)-\Gamma(2b)\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1+2b\right)}\right)}{\hat{M}_1} \qquad (12)$$

Examining the Unbiased Characteristic

After obtaining the respective estimators from the Kumaraswamy distribution, the next step is to examine the unbiased characteristic for estimator a and b with the definition of unbiased are $E(\hat{a}) = a$, and $E(\hat{b}) = b$.

Parameter estimator b ($\hat{b}$)

$$E(\hat{b}) = E\left[\frac{\hat{M}_0\,\Gamma(\frac{1}{a}+1+b)}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}\right]$$

$$E(\hat{b}) = b$$

Parameter estimator a ($\hat{a}$)

$$E(\hat{a}) = E\left[\frac{\left(\frac{\hat{M}_0}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}\right)\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b)\Gamma\left(\frac{1}{a}+1+2b\right)-\Gamma(2b)\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1+2b\right)}\right)}{\hat{M}_1}\right]$$

$$E(\hat{a}) = a$$

Thus, $\hat{a}$ and $\hat{b}$ are unbiased estimator for a and b.

Examining the Consistency Characteristic

$$P(|\hat{a}-a| < \varepsilon) \geq 1 - \frac{\mathrm{Var}\left(\frac{\left(\frac{\hat{M}_0}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}\right)\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b)\Gamma\left(\frac{1}{a}+1+2b\right)-\Gamma(2b)\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1+2b\right)}\right)}{\hat{M}_1}\right)}{\varepsilon^2}$$

Since $\lim_{n\to\infty} P(|\hat{a}-a| < \varepsilon) \geq 1$ then $\hat{a} \xrightarrow{P} a$ ($\hat{a}$ converges in probability). Thus $\hat{a}$ is a consistent predictor of a.

2.4. Examining the Minimum Variance ofKumaraswamyDistributionParameterEstimator (a,b)

To examine the minimum variance of Kumaraswamy distribution, the steps required are finding the Fisher information matrix and Cramer-Rao inequality.

Fisher Information Matrix of Kumaraswamy Distribution Parameter Estimator (a, b)

The Fisher information matrix form of Kumaraswamy distribution (a, b) is

$$L(a,b) = -\begin{bmatrix} E\left(\frac{\partial^2 \ln L}{\partial a^2}\right) & E\left(\frac{\partial^2 \ln L}{\partial a\,\partial b}\right) \\ E\left(\frac{\partial^2 \ln L}{\partial b\,\partial a}\right) & E\left(\frac{\partial^2 \ln L}{\partial b^2}\right) \end{bmatrix}$$

Next, we will look for elements of the Fisher information matrix by determining the function of L from the probability distribution function of Kumaraswamy distribution as follows:

Ln L(a,b)

$$= n\ln a + n\ln b + (a-1)\sum_{i=1}^{n}\ln(xi) + (b-1)\sum_{i=1}^{n}\ln(1-xi^a)$$

Then theelements of Fisher information matrix are:

$$E\left(\frac{\partial^2 \ln L}{\partial^2 b}\right) = -\frac{n}{b^2}\,E\left(\frac{\partial}{\partial a}\left(\frac{\partial \ln L}{\partial b}\right)\right)$$

$$= \frac{n}{b} - nab\left(\frac{\Gamma\left(\frac{1}{a}\right)\Gamma(b-1)}{\Gamma\left(\frac{1}{a}+b-1\right)}\right)$$

$$E\left(\frac{\partial}{\partial b}\left(\frac{\partial \ln L}{\partial a}\right)\right) = (bn)\left(-ab\sum_{k=0}^{\infty}(-1)^k\binom{b-3}{k}\frac{2}{(ak+2a)^3}\right)$$

$$E\left(\frac{\partial^2 \ln L}{\partial^2 a}\right)$$

$$= -\frac{n}{a^2} + (bn-n)\left(-ab\sum_{k=0}^{\infty}(-1)^k\binom{b-3}{k}\frac{2}{(ak+2a)^3}\right)$$

So Fisher information matrix is obtained as follows:

To examine the consistency characteristic in estimating a parameter of the Kumaraswamy distribution by using Chebysev's Theorem[6,9] is as follows:

Parameter estimator b ($\hat{b}$)

$$P(|x-\delta| < \varepsilon) \geq 1 - \frac{\mathrm{Var}(\delta)}{\varepsilon^2}$$

So: $P(|\hat{b}-b| < \varepsilon) \geq 1 - \frac{\mathrm{Var}(\hat{b})}{\varepsilon^2}$

$$P(|\hat{b}-b| < \varepsilon) \geq 1 - \frac{\mathrm{Var}\left(\frac{\hat{M}_0\,\Gamma(\frac{1}{a}+1+b)}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}\right)}{\varepsilon^2}$$

$$\lim_{n\to\infty} P(|\hat{b}-b| < \varepsilon) \geq 1$$

Since $\lim_{n\to\infty} P(|\hat{b}-b| < \varepsilon) \geq 1$ then $\hat{b} \xrightarrow{P} b$ ($\hat{b}$converges in probability). Thus $\hat{b}$ is a consistent predictor of b.

Parameter estimator a ($\hat{a}$)

$$P(|x-\delta| < \varepsilon) \geq 1 - \frac{\mathrm{Var}(\delta)}{\varepsilon^2}$$

So: $P(|\hat{a}-a| < \varepsilon) \geq 1 - \frac{\mathrm{Var}(\hat{a})}{\varepsilon^2}$

$$L^{-1}(a,b) = \frac{1}{W}\begin{bmatrix} \frac{n}{b^2} & \frac{n}{b} - nab\left(\frac{\Gamma\left(\frac{1}{a}\right)\Gamma(b-1)}{\Gamma\left(\frac{1}{a}+b-1\right)}\right) \\ -(bn-n)\,M & \frac{n}{a^2} + (bn-n)\,M \end{bmatrix}$$

Where W is the determinant of L(a,b), or W=|L(a,b)| and $M = \left(ab\sum_{k=0}^{\infty}(-1)^k\binom{b-3}{k}\frac{2}{(ak+2a)^3}\right)$.

Cramer-Rao Inequality for Variance of Kumaraswamy Distribution Parameter Estimator (a, b)

To determine the Cramer-Rao inequality, Fisher information matrix is used as follows:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{E\left[\left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2\right]} = \frac{1}{I(\theta)} = I^{-1}(\theta)$$

Cramer-Rao inequality is obtained from the inverse of the Fisher information matrix to obtain the following:

$$\mathrm{Var}(\hat{a},\hat{b}) \geq L^{-1}(a,b) = \frac{1}{W}\begin{bmatrix} \frac{n}{b^2} & \frac{n}{b} - nab\left(\frac{\Gamma\left(\frac{1}{a}\right)\Gamma(b-1)}{\Gamma\left(\frac{1}{a}+b-1\right)}\right) \\ -(bn-n)\,M & \frac{n}{a^2} + (bn-n)\,M \end{bmatrix}$$

the parameter estimator (a, b) is an efficient predictor because the range of parameter estimators (a, b) attain the Cramer-Rao lower bound.

2.5. Examining the Sufficient Statistics of Kumaraswamy Distribution(a,b)

To find sufficient statistics of Kumaraswamy distribution, the Neyman-Fisher Theorem is used as follows:

$$f(x) = abx^a(1-x^a)^{b-1}$$

$$f(x;a,b) = \prod_{i=1}^{n} abx^a(1-x^a)^{b-1}$$

$$f(x;a,b) = (ab)^n\prod_{i=1}^{n} x_i{}^a(1-x_i{}^a)^{b-1}$$

$$f(x;a,b) = \underbrace{(ab)^n\prod_{i=1}^{n} x_{ii}^a(1-x_i{}^a)^{b-1}}_{g(x_1,x_2,\ldots,x_n;a,b)} \cdot \underbrace{1}_{h(x_1,x_2,\ldots,x_n)}$$

Thus the Kumaraswamy distribution is a sufficient statistic because in $g(x_1, x_2, \ldots, x_n; a, b)$ functions containing parameters of the Kumaraswamy distribution where $h(x_1, x_2, \dots, x_n)$ contain no more parameters.

2.6. Simulation of Parameter Estimator a and b with Probability Weighted Moment (PWM) Method and Maximum Likelihood Estimation (MLE)

Simulation of Kumaraswamy distribution with parameters a and b are performed using software R version 3.3.2. The sample data used are n = 20, 50, 80, 100, and 150. In this simulation the mean value, standard deviation, variance, bias, and Mean Square Error (MSE) using Probability Weighted Moment (PWM) method and Maximum Likelihood Estimation (MLE) Method can also be seen. The explanation is as follows:

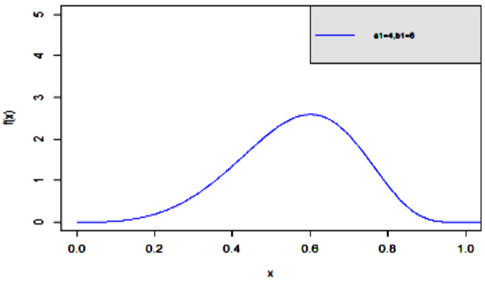


Fig.2.Curve Graphic a = 4 and b = 6

Table 1. Value of Parameter estimation a = 4 and b = 6 by Probability Weighted Moment (PWM) Method and Maximum Likelihood Estimation (MLE) Method.

Sample Size	Estimated Value	Parameter			
		a=4		b=6	
		Method PWM ($\hat{a}$)	Method MLE($\hat{a}$)	Method PWM ($\hat{b}$)	Method MLE ($\hat{b}$)
20	Mean	0.617011641	4,077201141	6,356606	6,232711786
	Standard Deviation	0.003158366	0,786530665	0,133652	0,720150928
	Variance	9.98E-06	0,618630487	0,017862909	0,518617359
	Bias	3.382988359	-0,077201141	-0,356605795	-0,232711786
	MSE	11.44462021	0,624590503	0,145031	0,572772134
50	Mean	1,590695435	4,15026258	6,346388607	6,108271048
	Standard Deviation	0,005933877	0,720843914	0,096532674	0,679051432
	Variance	3,52109E-05	0,519615949	0,009318557	0,461110847
	Bias	2,409304565	-0,15026258	-0,346388607	-0,108271048
	MSE	5,804783699	0,542194792	0,129303624	0,472833467
80	Mean	2,56752802	4,157698741	6,375465	6,185160021
	Standard Deviation	0,00649345	0,695205909	0,066432	0,603808772
	Variance	4,21649E-05	0,483311256	0,004411977	0,364585033
	Bias	1,432471983	-0,157698741	-0,375464696	-0,185160021
	MSE	2,05201815	0,508180149	0,145386	0,398869266
100	Mean	3,217035	4,141447374	6,37123	6,253934112
	Standard Deviation	0,007105	0,601272448	0,057858	0,514199127
	Variance	5,0488E-05	0,361528557	0,003347597	0,264400742
	Bias	0,782964847	-0,141447374	-0,371230085	-0,253934112
	MSE	0,613084	0,381535917	0,141159	0,328883275
150	Mean	4,841048	4,163279072	6,367069	6,230612204
	Standard Deviation	0,009661	0,542250819	0,052126	0,50556352
	Variance	9,33342E-05	0,294035951	0,002717082	0,255594473
	Bias	-0,84104824	-0,163279072	-0,367069417	-0,230612204
	MSE	0,707455	0,320696006	0,137457	0,308776462

4. CONCLUSIONS

Based on the curve of probability distribution function of Kumaraswamy, it is found that (a) and (b) are shaped parameters in which if the values of a and b are increased, the curves are sharp and the curve width is smaller. When the parameters values of a and b are decreased, the curves are slope and the curve width is greater. Kumaraswamy distribution parameter estimator with Probability Weighted Moment (PWM) method is

$$\hat{b} = \frac{M_0\,\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1\right)\Gamma(b)}$$

$$\hat{a} = \frac{b\,\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b)\Gamma\left(\frac{1}{a}+1+2b\right)-\Gamma(2b)\Gamma\left(\frac{1}{a}+1+b\right)}{\Gamma\left(\frac{1}{a}+1+b\right)\Gamma\left(\frac{1}{a}+1+2b\right)}\right)}{M_1}$$

While Kumaraswamy distribution parameter estimator with Maximum Likelihood Estimation (MLE) method cannot be solved analytically, it can use Newton Raphson iteration method. Kumaraswamy distribution parameter estimator with Probability Weighted Moment (PWM) method and Maximum Likelihood Estimation (MLE) method performs a good estimator. The simulation results show of small and consistent variance and small bias values. MLE method is more efficient than PWM method because MSE value for MLE method is smaller than MSE value for PWM method.

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