A QUALITATIVE STUDY OF AN ECO-EPIDEMIOLOGICAL MODEL WITH (SI) EPIDEMIC DISEASE IN PREY AND (SIS) EPIDEMIC DISEASE IN PREDATOR INVOLVING A HARVESTING

Haidar Ghanem Sufaeh[1] and Azhar Abbas Majeed[2]
[1,2]Department of mathematics, Collage of Science
University of Baghdad, Baghdad, Iraq
[1]E-mail: haidarghanem@yahoo.com
[2]*E-mail: azhar_abbas_m@scbaghdad.edu.iq*

ABSTRACT :*In this paper, an eco-epidemiological prey – predator model is proposed for study . The model includes SI infectious disease in prey which is transmitted by external source and the contact between the susceptible and infected species involving the harvesting on the infected prey, and SIS disease in predator species which is spread by contact between susceptible individuals and infected individuals. The epidemics cannot transmitted from prey to predator by predation or conversely. Two types of functional response for describing the predation as well as linear incidence for describing the transition of disease are used, the model is proposed and analyzed. Based on the assumption above, all possible equilibrium points are analyzed quantitatively by mathematical: methods. The locally and globally dynamics of the model are presented, also the effect of the disease and harvest on the dynamics of the system is discussed using numerical simulation.*

Keywords: Eco-epidemiological model, SI epidemic disease, Prey-predator model, Harvest, Lyapunov function.

1 INTRODUCTION

Mathematicians and biologists, including medical scientists, have achieved many successes through out history in their work together. In the life sciences, advanced mathematical results were used. Examples are given by the development of stochastic processes and statistical methods to solve a variety of population problems in demography, epidemics and genetics, ecology, and most joint work between biologists, physicists, chemists and engineers involves synthesis and analysis of mathematical structures. Pythagoras, Aristotle, Fibonacci, Cardano, Bernoulli, Euler, Fourier, Laplace, Gauss, von Helmholtz, Riemann, Einstein, Thompson, Turing, Wiener, von Neumann, Thom, and Keller are names associated with both significant applications of mathematics to life science problems and significant developments in mathematics motivated by the life sciences [1].

Prey-predator models are of great interest to researchers in mathematics and ecology because they deal with environmental problems such as community's morbidity and how to control it, optimal harvest policy to sustain a community, and other [2].

Mathematical models predation are amongst the oldest in ecology. The Italian mathematician Volterra is said to have developed his ideas about predation from watching the rise and fall of Adriatic fishing fleets. When fishing was good, the number of fishermen increased, drawn by the success of others. After a time, the fish declined, perhaps due to over-harvest, and then the number of fishermen also declined. After some time, the cycle repeated [3].

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance.

On other hand, the term epidemiology deals with the study of the spread of diseases in species. It is well known that when the infected individual still infective and the susceptible individual still susceptible for all the time, the disease is called SI disease. However, when the infection does not leads to immunity, so that infective becomes susceptible again after recovery, the disease is called SIS disease. Finally, when infective has permanent immunity after recover, the disease is called SIR disease. Further, after the pioneering work of Kermack _ Mckendrick [4] which is based on classical susceptible, infected, recovered model, the field of epidemiology has come into sight and recovered a lot of attention from the researches [5,6].

Moreover, whereas many diseases are transmitted in the species not only through contact, but also directly from environment, Majeed and Shawka [7] studied prey -predator model involving SI and SIS infectious disease in prey population and the disease transmitted within the same species by contact and external source. In addition to Khalaf and et.al [8] , considered and studied prey -predator model involving SIS infectious disease in prey population this disease passed from a prey to predator through attacking of predator to prey and the disease transmitted within the same species by contact and external source, while Naji and Mustafa [9] , proposed and analyzed a prey -predator model involving SI infectious disease in prey and the disease transmitted within the same species by contact.

Bairagi et.al [10] studied prey -predator model with harvest and disease , and they assumed hat the harvest can remove a parasite .

A functional response in ecology is the intake rate of a consumer as a function of food density (the amount of food available in a given ecotype). It is associated with the numerical response, which is the reproduction rate of a consumer as a function of food density. Following Holling, functional responses are generally classified into three types, which are called Holling's type I, II, and III [11].

Ali [12]. studied prey -predator model involving SI infectious disease in prey and predator population with harvesting and the disease transmitted within the same species by contact .

In this article, an eco-epidemiological mathematical system consisting of prey -predator model involving SI disease in prey with harvesting in infectious prey population and SIS disease in predator population is proposed. Moreover, in this system, the susceptible predator consumes the susceptible prey according to Holling type II response function and consumes the infected prey according to linear type of functional response , while the infected predator attack the infected prey only linearly as well as linear incidence rate for describing the transition of diseases are used .

2 Mathematical Model:

In this section, an eco -epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by $N(T)$, interacting with predator whose total population at time T is denoted by $P(T)$. It is assumed that both the prey and the predator populations are infected by different infectious diseases. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1-There is an $SI-$type of epidemic disease in prey and SIS- type of epidemic disease the predator population's, SI epidemic disease divides the prey population into two classes namely $S_1(T)$ that represents the density of susceptible prey at time T and $I_1(T)$ which represents the density of infected prey at time T. Therefore at any time T, we have $N(T)=S_1(T)+I_1(T)$. on other hand SIS epidemic disease also divides the predator population into two classes namely $S_2(T)$ that represents the density of susceptible predator at time T and $I_2(T)$ which represents the density of infected predator at time T. Therefore at any time T, we have $P(T)=S_2(T)+I_2(T)$.

2-It is assumed that only susceptible prey S_1 is capable of reproducing in logistically with carrying capacity $k > 0$ and intrinsic growth rate constant $r > 0$, the infected prey I_1 is removed before having the possibility of reproducing. However, the infected prey population I_1 still contribute with S_1 to population growth toward the carrying capacity.

3-The disease is transmitted within the same species by contact with an infected individual at infection rates $\gamma_1 > 0$ and $\gamma_2 > 0$ for the prey and predator respectively. In addition , there is an external source of disease causes incidence with the disease within the specific population at an external infection rate α_1.

4-The disease disappear and infected individuals become susceptible again in the predator at recover rate α_2.

5-The susceptible predator consumes the susceptible and infected prey according to Holling type-II and Lotka - Volterra of functional response with maximum attack rate $a_1 > 0$ and half saturation rate $b > 0$ for susceptible prey , and maximum attack rate $a_2 > 0$ for infected prey , while the infected predator consume the infected prey only according to Lotka -Volterra of functional response with maximum attack rate $a_3 > 0$. However the constants e_i; i=1,2,3 represent the conversion rates.

6-In the absence of the prey the susceptible and infected predator decay exponentially with natural death rate $d_2 > 0$.

7-The disease may causes death with a constant death rate β for the prey while $d_1 > 0$ represent the natural death for prey.

8-Finally, the infected prey is harvest with constant rate $h > 0$.

According to the above assumptions, the proposed mathematical model can be represented mathematically by the following set of first order non-linear differential equations, while the block diagram of this model can be illustrated in figure (2.1).

$$
\begin{aligned}
\frac{dS_1}{dT} &= rS_1\left(1-\frac{S_1+I_1}{k}\right)-\gamma_1 S_1 I_1 - d_1 S_1 - \alpha_1 S_1 - \frac{a_1 S_1 S_2}{b+S_1} \\
\frac{dI_1}{dT} &= \gamma_1 S_1 I_1 + \alpha_1 S_1 - \beta I_1 - a_2 I_1 S_2 - a_3 I_1 I_2 - d_1 I_1 - h I_1 \qquad (2.1) \\
\frac{dS_2}{dT} &= e_1\frac{a_1 S_1 S_2}{b+S_1} + e_2 a_2 I_1 S_2 - \gamma_2 I_2 S_2 - d_2 S_2 + \alpha_2 I_2 \\
\frac{dI_2}{dT} &= e_3 a_3 I_1 I_2 - (d_2+\alpha_2) I_2 + \gamma_2 I_2 S_2
\end{aligned}
$$

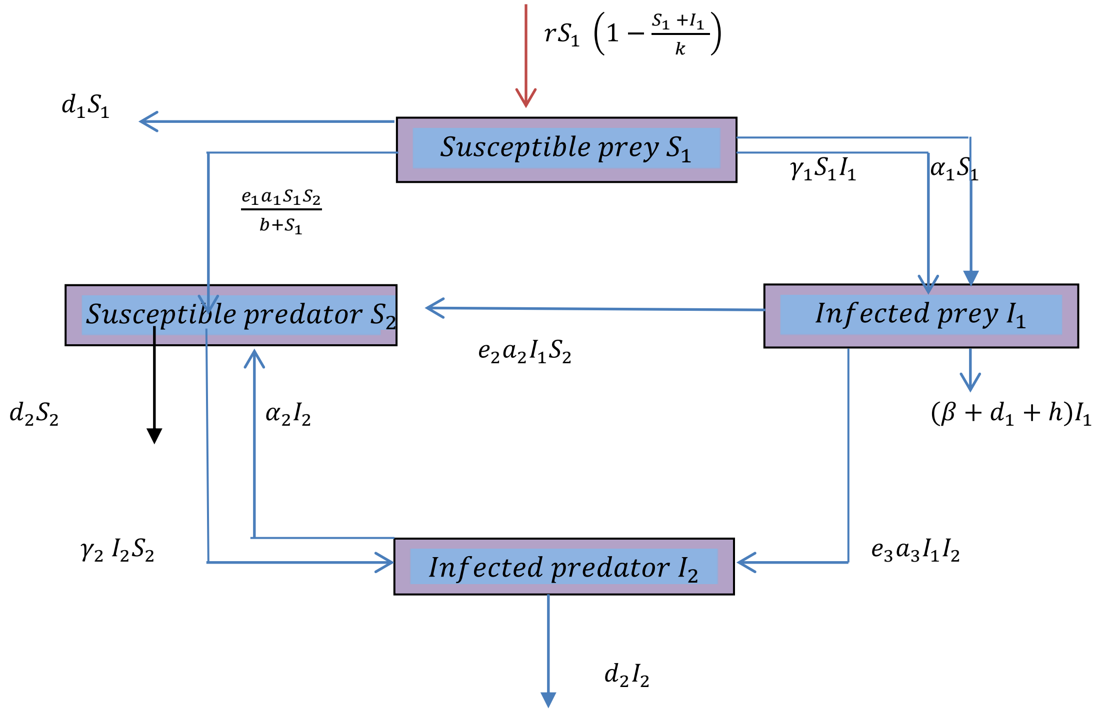


Fig. (2.1): Block diagram for prey -predator model given by system (2.1)

Note that the above proposed model has seventeen parameters which makes the mathematical analysis of the system difficult. So in order to reduce the number of parameters and determine which parameter represents the control parameter, the following dimensionless variables are used:

$$t = r\,T\,, x = \frac{S_1}{k}\ \ , y = \frac{I_1}{k}\ \ , z = \frac{S_2}{k}\,, w = \frac{I_2}{k}.$$

Then system (2.1) can be written in the following dimensionless form:

$$\frac{dx}{dt} = x\left(1 - x - y - u_1 y - (u_2 + u_3) - \frac{u_4 z}{u_5 + x}\right)$$
$$= f_1(x, y, z, w)$$
$$\frac{dy}{dt} = y\big(u_1 x - u_6 z - u_7 w - (u_3 + u_8\ + u_9)\big) \quad (2.2)$$
$$+u_2 x = f_2(x, y, z, w)$$
$$\frac{dz}{dt} = z\left(\frac{u_{10}\, x}{u_5 + x} + u_{11} y - u_{12} w - u_{13}\right) + u_{14} w$$
$$= f_3(x, y, z, w)$$
$$\frac{dw}{dt} = w\big(u_{15}\, y + u_{12} z - (u_{13} + u_{14})\big)$$
$$= f_4(x, y, z, w)\ ,$$

where :

$$u_1 = \frac{\gamma_1 k}{r}\,, u_2 = \frac{\alpha_1}{r}\,, u_3 = \frac{d_1}{r}\,, u_4 = \frac{a_1}{r}\,, u_5 = \frac{b}{k}\,,$$
$$u_6 = \frac{a_2 k}{r}\ \ , u_7 = \frac{a_3 k}{r}\,, u_8 = \frac{\beta}{r}\,, u_9 = \frac{h}{r}\,,$$
$$u_{10} = \frac{e_1\, a_1}{r}\,, u_{11} = \frac{e_2\, a_2\, k}{r}, u_{12} = \frac{\gamma_2 k}{r}\ ,$$
$$u_{13} = \frac{d_2}{r}, u_{14} = \frac{\alpha_2}{r}\,, u_{15} = \frac{e_3 a_3 k}{r}\ .$$

Represent the dimensionless parameter of system (2.2). It is observed that the number of parameters have been reduced from seventeen in the system (2.1) to fifteen in the system (2.2).

Since the density of any species cannot be negative, therefore we will solve system (2.2) with the following initial condition $x(0) \geq 0, y(0) \geq 0,\ z(0) \geq 0$ and $w(0) \geq 0$.

It is easy to verify that all the interaction functions f_1, f_2, f_3 and f_4 on the right hand side of system (2.2) are continuous and have continuous partial derivatives on R_+^4 with respect to dependent variables x , y , z and w. Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem (2.1): All the solutions of system (2.2) which initiate in R_+^4 are uniformly bounded.

Proof.

Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2.2) with non -negative initial condition $(x(0), y(0), z(0), w(0))$. According to the first equation of system (2.2) we have:

$$\frac{dx}{dt} \leq x(1 - x).$$

Clearly according to the theory of differential inequality, we get:

$\lim_{t\to\infty} \sup x(t) \leq 1$. Define the function $M_1(t) = x(t) + y(t) + z(t) + w(t)$.

Therefore,

$\frac{dM_1}{dt} < 2x - x^2 - (1+u_3)x - (u_6 - u_{11})zy - (u_7 - u_{15})wy - (u_3 + u_8 + u_9)y - u_{13}z - (u_{13} + u_{14})w - \frac{(u_4 - u_{10})xz}{u_5 + x}$.

Now, since the conversion rate constant from prey population to predator population can't be exceeding the maximum predation rate constant of predator population to prey population, hence from the biological point of view, always $u_{10} < u_4$, $u_{11} < u_6$ and $u_{15} < u_7$, hence it is obtained that:

$\frac{dM_1}{dt} \leq 2 - DM_1$, where

D= min $\{ (1+u_3), (u_3 + u_8 + u_9), u_{13}, u_{13} + u_{14} \}$.

Now, by using the comparison theorem [13] on the above differential inequality, we get that:

$M_1(t) \leq \frac{2}{D} + \left(M_1(0) - \frac{2}{D}\right) e^{-Dt}$.

Thus $0 \leq M_1(t) \leq \frac{2}{D}$ as $t \to \infty$. Hence all the solutions of system (2.2) are uniformly Bounded and the proof is complete ■

Existence of equilibrium points

In the following, the conditions for the existence of all possible equilibrium points of the system (2.2) are discussed. System (2.2) results in the following seven equilibrium points.

1) The vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ always exist.

2) The disease -predator free equilibrium point $E_1 = (\hat{x}, 0, 0, 0)$, where

$\hat{x} = 1 - (u_2 + u_3)$. $(2.2a)$

Note that equation $(2.2a)$ is a positive, provided that:

$(u_2 + u_3) < 1$. $(2.2b)$

3) The predator - free equilibrium point $E_2 = (\bar{x}, \bar{y}, 0, 0)$ exists if and only if there is a positive solution to the following set of equations:

$1 - x - (1+u_1)y - (u_2+u_3) = 0$ $(2.3a)$

$u_1xy + u_2x - (u_3 + u_8 + u_9)y = 0$ $(2.3b)$

From equation (2.3a) we have,

$x = 1 - (1+u_1)y - (u_2+u_3)$. $(2.3c)$

Now, by substituting equation (2.3c) in equation: $(2.3b)$ we get:

$A_2y^2 + A_2y + A_3 = 0$ $(2.3d)$

where :

$A_1 = -(1+u_1)u_1$.

$A_2 = u_1(1-2u_2) - u_2 - (1+u_1)u_3 - (u_8+u_9)$.

$A_3 = (1 - (u_2+u_3))\, u_2$.

Note that by using Descartes rule of sign equation $(2.3d)$ has a unique positive root namely $\bar{y}$ provided that $A_3 > 0$ which is holds by condition (2.2b).

Substituting the value of y in $(2.3c)$ yield that $x(\bar{y}) = \bar{x}$ which is positive if the following condition hold:

$(1+u_1)\bar{y} + (u_2+u_3) < 1$ $(2.3e)$

4) The disease - free equil ibrium point $E_3 = (\dot{x}, 0, \dot{z}, 0)$ exists if and only if there is a positive solution to the following set of equations:

$1 - x - (u_2+u_3) - \frac{u_4z}{u_5+x} = 0$ $(2.4a)$

$\frac{u_{10}x}{u_5+x} - u_{13} = 0$ $(2.4b)$

From equation $(2.4b)$ we have,

$\dot{x} = \frac{u_5u_{13}}{u_{10} - u_{13}}$ $(2.4c)$

Note that $\dot{x}$ is positive provided that the following condition holds:

$u_{10} > u_{13}$ $(2.4d)$

Now, by substituting equation (2.4c) in equation $(2.4a)$ we get:

$\dot{z} = \frac{u_5+\dot{x}}{u_4}(1 - \dot{x} - u_2 - u_3)$ $(2.4e)$

Note that $\dot{z}$ is a positive if in addition to condition $(2.4d)$, the following condition holds

$(\dot{x} + u_2 + u_3) < 1$ $(2.4f)$

5) The infected -predator free equilibrium point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$ exists if and only if there is a positive solution to the following set of equations:

$1 - x - (1+u_1)y - (u_2+u_3) - \frac{u_4z}{u_5+x} = 0$ $(2.5a)$

$u_1xy + u_2x - u_6zy - (u_3+u_8+u_9)y = 0$ $(2.5b)$

$\frac{u_{10}x}{u_5+x} + u_{11}y - u_{13} = 0$ $(2.5c)$

From equation $(2.5b)$ and $(2.5c)$ we have,

$y = \frac{u_{13}(u_5+x) - u_{10}x}{u_{11}(u_5+x)}$ $(2.5d)$

By substituting $(2.5d)$ in $(2.5b)$ we get ,

$z = -\frac{1}{u_6}\Big((u_3+u_8+u_9 - u_1x) - \frac{u_2u_{11}(u_5+x)x}{u_{13}(u_5+x) - u_{10}x} \Big)$ $(2.5e)$

Now, put equation $(2.5d)$ and $(2.5e)$ in $(2.5a)$ we get,

$B_1x^3 + B_2x^2 + B_3x + B_4 = 0$, $(2.5f)$

where:

$B_1 = u_6u_{11}(u_{10} - u_{13})$,

$B_2 = u_6u_{10}u_{11}(u_2+u_3) + u_6u_{10}u_{13} - u_4u_{11}(u_1(u_{13} - u_{10})) + u_6(1+u_1)[(2u_{10}u_{13} - (u_{13}{}^2 + u_{10}{}^2)] + u_6u_{11}[u_{10} - u_{13}(2u_5+1)] + u_{11}(u_5u_6u_{10} - u_2u_4u_{11})$,

$B_3 = -2u_5u_6u_{13}[(u_2+u_3)u_{11} + u_{13}(1+u_1)] + u_4u_{11}(u_{13} - u_{10})(u_3+u_8+u_9) - u_2u_{11} - u_5u_{11}(u_6u_{10} + u_1u_4u_{13}) - 2u_5u_6u_{13}[u_{11} - u_{10}(2+u_1)]$,

$B_4 = u_4u_{11}(u_3+u_8+u_9) + u_5u_6u_{10} - u_5u_{10}u_{13}(1+u_1) - u_5u_6u_{11}(u_2+u_3)$.

Note that by using Descartes rule of sign equation $(2.5f)$ has a unique positive root, namely $\bar{\bar{x}}$ provided that one of the following conditions hold:

$B_1 > 0$, $B_2 > 0$, $B_4 < 0$,
$B_1 > 0$, $B_3 < 0$, $B_4 < 0$, (2.5g)
$B_1 < 0$, $B_2 < 0$, $B_4 > 0$,
$B_1 < 0$, $B_3 > 0$, $B_4 > 0$.

Substituting the value of $\bar{\bar{x}}$ in $(2.5d)$ and $(2.5e)$ yield that $z(\bar{\bar{x}}) = \bar{\bar{z}}$ and $y(\bar{\bar{x}}) = \bar{\bar{y}}$ which are positive if the following conditions hold:

$$\frac{u_2\ u_{11}\ (u_5\ +\bar{\bar{x}})\bar{\bar{x}}}{u_{13}\ (u_5\ +\bar{\bar{x}})-u_{10}\ \bar{\bar{x}}} + u_1\ \bar{\bar{x}}\ >\ (u_3 + u_8 + u_9) \quad (2.5h)$$

$$u_{13}\ (u_5\ + \bar{\bar{x}}) > u_{10}\ \bar{\bar{x}}. \quad (2.5i)$$

Therefore, the infected predator free equilibrium point: $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$ of system (2.2) exists uniquely in the $Int.R^3_+$ of $xyz-$space under conditions (2.5g)-(2.5i)

6) The infected prey free equilibrium point $E_5 = (\tilde{x}, 0, \tilde{z}, \widetilde{w})$ exists if and only if there is a positive solution to the following set of equations:

$$1 - x - (u_2 + u_3) - \frac{u_4 z}{u_5 + x} = 0 \quad (2.6a)$$

$$\frac{u_{10}\ zx}{u_5 + x} - u_{12}wz - u_{13}\ z + u_{14}\ w = 0 \quad (2.6b)$$

$$u_{12}\ z - (u_{13} + u_{14}) = 0 \quad (2.6c)$$

From equation $(2.6c)$ we have,

$$\tilde{z} = \frac{(u_{13}+u_{14})}{u_{12}}. \quad (2.6d)$$

Now, by substituting equation (2.6d) in equation $(2.6a)$ we get:

$$\gamma_1 x^2 + \gamma_2 x + \gamma_3 = 0, \quad (2.6e)$$

where:

$\gamma_1 = -u_{12}$,

$\gamma_2 = u_{12}\ (1 - u_5\ - u_2\ - u_3\ ),$.

$\gamma_3 = u_5 u_{12}\ - u_5 u_{12}\ (u_2\ + u_3\ )\ -\ u_4\ (u_{13}\ +\ u_{14}\ )$.

Note that by using Descartes rule of sign equation $(2.6e)$ has a unique positive root namely $\tilde{x}$ if $\gamma_3 > 0$.

Now $\gamma_3 > 0$, if in addition to the condition $(2.2b)$, the following condition holds:

$$u_5 u_{12}\ (1 - (u_2\ + u_3\ )) > u_4\ (u_{13}\ +\ u_{14}\ ) \quad (2.6f)$$

Substituting $\tilde{x}$ and $\tilde{z}$ in $(2.6b)$ yield that

$$w(\tilde{x}) = \widetilde{w} = (\frac{u_{10}\ \tilde{x}}{u_{13}\ (u_5+\tilde{x})} - 1)\tilde{z},$$

which is positive if the following condition holds :

$$1 < \frac{u_{10}\ \tilde{x}}{u_{13}(u_5+\tilde{x})}. \quad (2.6g)$$

7) The positive (coexistence) equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$ exists if and only if there is a positive solution to the following set of equations:

$$1 - x - (1 + u_1)y - (u_2 + u_3) - \frac{u_4\ z}{u_5+x} = 0, \quad (2.7a)$$

$$u_1 xy + u_2 x - u_6 zy - (u_3 + u_8 + u_9)y - u_7 wy\ = 0, \quad (2.7b)$$

$$\frac{u_{10}\ z\ x}{u_5+x} + u_{11} zy - u_{12} wz - u_{13} z + u_{14} w = 0, \quad (2.7c)$$

$$u_{12} z + u_{15} y - (\ u_{13} + u_{14}) = 0. \quad (2.7d)$$

From equation (2.7d) we have,

$$z = \frac{1}{u_{12}}\big((u_{13} + u_{14}) - u_{15} y\big). \quad (2.7e)$$

Also, from equation $(2.7c)$ we have,

$$w = (\frac{u_{10}\ x}{u_5+x} - u_{13} + u_{11} y)\frac{z}{(u_{12} z - u_{14})}. \quad (2.7f)$$

Now, by substituting equation $(2.7e)$ in $(2.7f)$ yield the following equation:

$$w = \left(\frac{u_{10}\ x}{u_5+x} - u_{13} + u_{11} y\right)\frac{(u_{13}+u_{14}) - u_{15} y}{u_{12}\ (u_{13} - u_{15} y\ )}. \quad (2.7g)$$

Then by substituting equation $(2.7e)$ in $(2.7a)$ and $(2.7g)$ in $(2.7b)$ yield the following two isoclines:

$$g_1(x,y) = 1 - x - (1 + u_1)y - (u_2 + u_3) - \frac{u_4\big((u_{13} + u_{14}) - u_{15} y\big)}{u_{12}(u_5+x)} = 0 \quad (2.7h)$$

$$g_2(x,y) = u_1 xy + u_2 x - (u_3 + u_8 + \ u_9)y - \frac{u_6}{u_{12}}\big((u_{13} + u_{14}) - u_{15} y\big)y\ - \left(\frac{u_{10}}{u_5+x} + u_{13} - u_{11} y\right)\left(\frac{u_7 y (u_{13}+u_{14}) - u_{15} y)}{u_{12}\ (u_{13} - u_{15} y)}\right) = 0. \quad (2.7i)$$

Now from equation $(2.7h)$ we notice that, when $y \to 0$, then $\to x_1$, where x_1 represents a positive root of the following second order polynomial equation:

$$N_1\ x^2 +\ N_2\ x +\ N_3 =\ 0\ , \quad (2.7j)$$

where:

$N_1 = u_{12}$,

$N_2 =\ u_{12}(u_2\ + u_3\ + u_5\ - 1)$,

$N_3 = u_4\ (u_2\ + u_3\ +\ u_{14}) - u_5\ u_{12}(\ u_2\ + u_3\ -\ 1)$.

Straightforward computation shows that equation $(2.7j)$ has a unique positive root namely x_1 if $N_3 < 0$.

Therefore, $(2.7j)$ has unique positive root if the following condition holds:

$$u_4\ (u_2\ + u_3\ +\ u_{14}) + u_5 u_{12} < u_5\ u_{12}(\ u_2\ + u_3\ ) \quad (2.7r)$$

Further, from equation $(2.7i)$ we notice that, when $y \to 0$, then $x = 0$,

Now, from equation $(2.7h)$ we have:

$\frac{dx}{dy} = -\left(\frac{\partial g_1}{\partial y}\right) / \left(\frac{\partial g_1}{\partial x}\right)$. So, $\frac{dx}{dy} < 0$ if one of the following of conditions hold:

$$\left(\frac{\partial g_1}{\partial y}\right) > 0, \left(\frac{\partial g_1}{\partial x}\right) > 0\ \ OR\ \ \left(\frac{\partial g_1}{\partial y}\right) < 0, \left(\frac{\partial g_1}{\partial x}\right) < 0\ . \quad (2.7k)$$

Further, from $(2.7i)$ we notice that:

$\frac{dx}{dy} = -\left(\frac{\partial g_2}{\partial y}\right) / \left(\frac{\partial g_2}{\partial x}\right)$. So, $\frac{dx}{dy} > 0$ if one set of the following sets of conditions hold:

$$\left(\frac{\partial g_2}{\partial y}\right) > 0, \left(\frac{\partial g_2}{\partial x}\right) < 0\ \ OR\ \ \left(\frac{\partial g_2}{\partial y}\right) < 0, \left(\frac{\partial g_2}{\partial x}\right) > 0. \quad (2.7l)$$

Then the two isoclines (2.7h) and $(2.7i)$ intersect at a unique positive point (x^*, y^*) if we substituting the value of x^* and y^* in $(2.7e)$ and $(2.7g)$ yield that $z(y^*) = z^*$ and $w(x^*, y^*) = w^*$ which are positive if and only if the following conditions hold:

$$\frac{u_{13}}{u_{11}} - \frac{u_{10}}{u_{11}}\left(\frac{x^*}{u_5+x^*}\right) < y^* < \min\{\left(\frac{u_{13}+u_{14}}{u_{15}}\right), \frac{u_{13}}{u_{15}}\}\ , \quad (2.7m)$$

$$\frac{u_{13}}{u_{11}} > \frac{u_{10}}{u_{11}}\left(\frac{x^*}{u_5+x^*}\right). \quad (2.7n)$$

These are present the conditions of existence of $E_6 = (x^*, y^*, z^*, w^*)$.

3 Local stability analysis.

In this section, we analyzed the local stability of the model (2.2) around each equilibrium point and discussed through computing the Jacobian matrix $J(x\ ,y,z\ ,w)$ and their eigenvalues. The Jacobian matrix $J(x,y,z\ ,\mathrm{w})$ of the system (2.2) at each of them can be written:

$$J=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial \mathrm{w}}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \mathrm{w}}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial \mathrm{w}}\\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial \mathrm{w}}\end{bmatrix}, \qquad (3.1)$$

where f_i ; 1,2,3,4 are given in system (2.2) and,

$$\frac{\partial f_1}{\partial x}=1-2x-(1+u_1)\,\mathrm{y}-(u_2+u_3)-\frac{u_4 z}{u_5+x}+\frac{u_4 z x}{(u_5+x)^2}\ ,\frac{\partial f_1}{\partial \mathrm{y}}-(1+u_1)x,\frac{\partial f_1}{\partial z}=-\frac{u_4 x}{u_5+x},$$

$$\frac{\partial f_1}{\partial w}=0\ ,\frac{\partial f_2}{\partial x}=u_1 y+u_2\ ,$$

$$\frac{\partial f_2}{\partial y}=u_1 x-u_6 z-u_7 w-(u_3+u_8+u_9)\ ,$$

$$\frac{\partial f_2}{\partial z}=-u_6 y\ \ ,\frac{\partial f_2}{\partial w}=-u_7 y\ ,$$

$$\frac{\partial f_3}{\partial x}=\frac{u_{10}u_5 z}{(u_5+x)^2}\quad ,\frac{\partial f_3}{\partial y}=u_{11}z,$$

$$\frac{\partial f_3}{\partial z}=\frac{u_{10}x}{u_5+x}+u_{11}y-u_{12}w-u_{13}\ ,$$

$$\frac{\partial f_3}{\partial w}=u_{14}-u_{12}\,z\quad ,\frac{\partial f_4}{\partial x}=0\quad ,\frac{\partial f_4}{\partial y}=u_{15}w\quad ,$$

$$\frac{\partial f_4}{\partial z}=u_{12}w\ ,$$

$$\frac{\partial f_4}{\partial w}=u_{15}y+u_{12}z-(u_{13}+u_{14})\ .$$

3. 1 Stability of equilibrium point $\boldsymbol{E_0=(0,0,0,0)}$

At E_0 the Jacobian matrix becomes:

$$J_0{=}J(E_0)=\left[a_{ij}\right]_{4\times 4}\ , \qquad (3.1a)$$

where,

$a_{11}=1-(u_2+u_3), a_{12}=a_{13}=a_{14}=0,$
$a_{21}=u_2\ , a_{22}=-(u_3+u_8+u_9), a_{23}=a_{24}=0,$
$a_{31}=a_{32}=0, a_{33}=-u_{13}, a_{34}=u_{14\ ,}$
$a_{41}=a_{42}=a_{43}=0, a_{44}=-(u_{13}+u_{14}).$

Then the characteristic equation of $J(E_0)$ is given by:

$$(1-(u_2+u_3)-\lambda)\,(-(\,u_3+u_8+u_9)-\lambda\,)(-u_{13}-\lambda\,)(-(\,u_{13}+u_{14})-\lambda)\ =0.$$

The eigenvalues are $\lambda_{0\,x}=1-(u_2+u_3)\ ,\lambda_{0\,y}=-(u_3+u_8+u_9),\lambda_{0\,z}=-u_{13}$ and $\lambda_{0\,w}=-(u_{13}+u_{14}).$

Thus, the equilibrium point E_0 , is locally asymptotically stable in the. R_+^4 provided that,

$u_2{+}\,u_3>1.$ (3.1b)

Otherwise, E_0 is unstable.

3.2 Stability of equilibrium point $\boldsymbol{E_1=(\hat{x},0,0,0)}$

At E_1 the Jacobian matrix becomes:

$$J_1{=}J(E_1)=\left[b_{ij}\right]_{4\times 4}\ , \qquad (3.2)$$

where,

$b_{11}=-\hat{x}, b_{12}=-(1+u_1)\hat{x}\ , b_{13}=-\frac{u_4\,\hat{x}}{u_5+\hat{x}},$
$b_{14}=0, b_{21}=u_2\ , b_{22}=u_1\hat{x}-(u_3+u_8{+}u_9),$
$b_{23}=b_{24}=0, b_{31}=b_{32}=0, b_{33}=\frac{u_{10}\hat{x}}{u_5+\hat{x}}-u_{13},$
$b_{34}=u_{14\ ,}b_{41}=b_{42}=b_{43}=0,$
$b_{44}=-(u_{13}+u_{14}).$

Then the characteristic equation of $J(E_1)$ is given by:

$$[\ \lambda^2+B\lambda+\mathrm{C}\ ]\ \left(\frac{u_{10}\hat{x}}{u_5+\hat{x}}-\ u_{13}-\lambda\right)(-(\,u_{13}+u_{14})-\lambda)=0\ ,$$

where: $B=\ (1-u_1)\hat{x}+\ (u_3+u_8+u_9)$.

$\mathrm{C}=(\,u_3+u_8+u_9+\ u_1\,(u_2-\hat{x})+u_2\,)\hat{x}$.

So, either

$$[\ \lambda^2+B\lambda+C\ ]=0, \qquad (3.2a)$$

which gives two eigenvalues of $J(E_1)$ with negative real part provided that

the following conditions: hold:

$u_1<1$ (3.2b)
$\hat{x}<u_2$ (3.2c)

Or

$$(\ \tfrac{u_{10}\hat{x}}{u_5+\hat{x}}-u_{13}-\lambda)\,(-\,(u_{13}{+}u_{14})\,-\lambda\,)=0, \qquad (3.2d)$$

which gives the other two eigenvalues of $J(E_1)$ by:

$\lambda_{1z}=\ \frac{u_{10}\hat{x}}{u_5+\hat{x}}-u_{13}$
<0 under the condition

$\frac{u_{10}\hat{x}}{u_5+\hat{x}}<u_{13},$ (3.2e)

$\lambda_{1w}=-(u_{13}+u_{14})<0$.

So, the equilibrium point E_1 is local asymptotically stable in the . R_+^4 . However, it is unstable otherwise.

3.3 Stability of equilibrium point $\boldsymbol{E_2=(\bar{x},\bar{y},0,0)}$

At E_2 the Jacobain matrix becomes:

$$J_2=J(E_2)=\ \left[n_{ij}\right]_{4\times 4}. \qquad (3.3)$$

where,

$n_{11}=-\hat{x}, n_{12}=-(1+u_1)\bar{x}\ , n_{13}=-\frac{u_4\,\bar{x}}{u_5+\bar{x}}$
$n_{14}=0, n_{21}=\boldsymbol{u_1}\bar{y}+u_2\ , n_{22}=-u_2\ \frac{\bar{x}}{\bar{y}}\ ,$
$n_{23}=-u_6\bar{y}\ , n_{24}=0, n_{31}=n_{32}=0,$
$n_{33}=\frac{u_{10}\ \bar{x}}{u_5+\bar{x}}+u_{11}\bar{y}-u_{13},$
$n_{34}=u_{14\ ,}n_{41}=n_{42}=n_{43}=0,$
$n_{44}=\ \ u_{15}\bar{y}\ -(\ u_{13}+u_{14}).$

Then the characteristic equation of $J(E_2)$ is given by:

$$[\lambda^2 + Q_1\lambda + Q_2][(u_{15}\bar{y} - (u_{13}+u_{14}) - \lambda)\left(\frac{u_{10}\bar{x}}{u_5+\bar{x}} + u_{11}\bar{y} - u_{13} - \lambda\right) = 0,$$

where:

$Q_1 = \left(1 + \frac{u_2}{\bar{y}}\right)\bar{x}$ and,

$Q_2 = \bar{x}\left[u_2\frac{\bar{x}}{\bar{y}} + (1+u_1)(u_1\bar{y} + u_2)\right]$

So, either

$[\lambda^2 + Q_1\lambda + Q_2] = 0$,

which gives two eigenvalues of $J(E_2)$ with negative real part.

Or

$$(u_{15}\bar{y} - (u_{13}+u_{14}) - \lambda)\left(\frac{u_{10}\bar{x}}{u_5+\bar{x}} + u_{11}\bar{y} - u_{13} - \lambda\right) = 0, \quad (3.3a)$$

which gives the other two eigenvalues of $J(E_2)$ by:

$\lambda_{2z} = (u_{15}\bar{y} - (u_{13}+u_{14})) < 0$, provided that

$(u_{13}+u_{14}) > u_{15}\bar{y}$ (3.3b)

$\lambda_{2w} = \left(\frac{u_{10}\bar{x}}{u_5+\bar{x}} + u_{11}\bar{y} - u_{13}\right) < 0$, provided that

$\left(\frac{u_{10}\bar{x}}{u_5+\bar{x}} + u_{11}\bar{y}\right) < u_{13}$. (3.3$c$)

Then E_2 is locally asymptotically stable in the R_+^4. However, it is unstable otherwise.

3.4 Stability of equilibrium point $E_3 = (\dot{x}, 0, \dot{z}, 0)$

At E_3 the Jacobian matrix becomes:

$J(E_3) = [k_{ij}]_{4\times4}$, (3.4)

where:

$k_{11} = \dot{x}\left(-1 + \frac{u_4\dot{z}}{(u_5+\dot{x})^2}\right), k_{12} = -(1+u_1)\dot{x},$

$k_{13} = -\frac{u_4\dot{x}}{u_5+\dot{x}}, k_{14} = 0, k_{21} = u_2,$

$k_{22} = u_1\dot{x} - u_6\dot{z} - (u_3 + u_8 + u_9), k_{23} = 0,$

$k_{24} = 0, k_{31} = \frac{u_5u_{10}\dot{z}}{(u_5+\dot{x})^2}, k_{32} = u_{11}\dot{z},$

$k_{33} = 0, k_{34} = u_{14} - u_{12}\dot{z}, k_{41} = 0, k_{42} = 0,$

$k_{43} = 0, k_{44} = u_{12}\dot{z} - (u_{13}+u_{14}).$

Then the characteristic equation of $J(E_3)$ is given by:

$[\lambda^3 + V_1\lambda^2 + V_2\lambda + V_3](k_{44} - \lambda) = 0,$ (3.4a)

where:

$V_1 = -(k_{11} + k_{22})$

$V_2 = k_{11}k_{22} - k_{12}k_{21} - k_{13}k_{31}$

$V_3 = k_{13}k_{31}k_{22} - k_{13}k_{21}k_{32}$.

So, either

$(k_{44} - \lambda) = 0$, which gives (3.4b)

$\lambda_{3w} = k_{44} < 0$, provided that

$u_{12}\dot{z} < (u_{13}+u_{14})$. (3.4$c$)

Or

$[\lambda^3 + V_1\lambda^2 + V_2\lambda + V_3] = 0$. (3.4$d$)

Using Routh Hurwitz criterion implies equation (3.4d) has roots with negative real part if and only if $V_1 > 0$, $V_3 > 0$ and $V_1V_2 - V_3 > 0$.

Now $V_1 > 0$, provided that:

$1 > \frac{u_4\dot{z}}{(u_5+\dot{x})^2}$ (3.4e)

$u_1\dot{x} < u_6\dot{z} + (u_3 + u_8 + u_9)$ (3.4f)

$V_3 > 0$ due to condition (3.4f).

Further, it is easy to check that:

$V_1V_2 - V_3 = k_{11}(-k_{22}(k_{11}+k_{22}) + k_{12}k_{21} + k_{13}k_{31}) + k_{21}(k_{12}k_{22} + k_{13}k_{32}),$

Now, the first terms is positive by conditions (3.4e)and (3.4f) while the second term is positive if in addition to (3.4f), the following condition holds:

$\dot{x}(1+u_1)(u_6\dot{z} + (u_3+u_8+u_9) - u_1\dot{x}) > u_{11}\dot{z}\left(\frac{u_4\dot{x}}{u_5+\dot{x}}\right).$ (3.4g)

So, all the eigenvalues of $J(E_3)$ have negative real part under the above given conditions and hence E_3 is locally asymptotically stable. However, it is unstable otherwise.

3.5 Stability of equilibrium point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$

At E_4 the Jacobian matrix becomes:

$J(E_4) = [c_{ij}]_{4\times4}$, (3.5)

where:

$c_{11} = -\bar{\bar{x}} + \frac{u_4\bar{\bar{x}}\bar{\bar{z}}}{(u_5+\bar{\bar{x}})^2}, c_{12} = -(1+u_1)\bar{\bar{x}},$

$c_{13} = -\frac{u_4\bar{\bar{x}}}{u_5+\bar{\bar{x}}}, c_{14} = 0, c_{21} = u_1\bar{\bar{y}} + u_2,$

$c_{22} = -u_2\frac{\bar{\bar{x}}}{\bar{\bar{y}}}, c_{23} = -u_6\bar{\bar{y}}, c_{24} = -u_7\bar{\bar{y}},$

$c_{31} = \frac{u_5u_{10}\bar{\bar{z}}}{(u_5+\bar{\bar{x}})^2}, c_{32} = u_{11}\bar{\bar{z}}, c_{33} = 0,$

$c_{34} = u_{14} - u_{12}\bar{\bar{z}}, c_{41} = 0, c_{42} = 0, c_{43} = 0,$

$c_{44} = u_{12}\bar{\bar{z}} + u_{15}\bar{\bar{y}} - (u_{13}+u_{14})$

Then the characteristic equation of $J(E_4)$ is given by:

$[\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3](c_{44} - \lambda) = 0$, (3.5$a$)

where:

$L_1 = -(c_{11} + c_{22})$.

$L_2 = c_{11}c_{22} - c_{12}c_{21} - c_{13}c_{31} - c_{23}c_{32}$.

$L_3 = c_{23}c_{32}c_{11} - c_{32}c_{21}c_{13} + c_{13}(c_{31}c_{22} - c_{12}c_{23})$

So, either

$(c_{44} - \lambda) = 0$ (3.5b)

$\lambda_{4w} = c_{44} < 0$, provided that

$u_{12}\bar{\bar{z}} + u_{15}\bar{\bar{y}} < (u_{13}+u_{14})$, (3.5$c$)

or

$\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 = 0$. (3.5$d$)

Using Routh Hurwitz criterion implies equation (3.5d) has roots with negative real part if and only if $L_1 > 0$, $L_3 > 0$ and $L_1L_2 - L_3 > 0$.

Now $L_1 > 0$, provided that:

$\frac{u_4\bar{\bar{x}}\bar{\bar{z}}}{(u_5+\bar{\bar{x}})^2} < 1$, (3.5e)

while $L_3 > 0$, under condition (3.5e). Further, it is easy to check that:

$$L_1L_2 - L_3 = L_1(c_{11}c_{22} - c_{21}c_{12}) + c_{31}(c_{11}c_{13} + c_{23}c_{12}) + c_{32}(c_{22}c_{23} + c_{21}c_{13}).$$

Now, the first and second terms are positive by condition (3.5e), while the third term is positive if the following condition holds:

$\left(\frac{u_1\bar{\bar{y}}+u_2}{u_5+\bar{\bar{x}}} < \frac{u_2u_6}{u_4}\right)$. (3.5f)

So, all the eigenvalues of $J(E_4)$ have negative real part under the given conditions and hence E_4 is locally asymptotically stable. However, it is unstable otherwise.

3.6 Stability of equilibrium point $E_5 = (\tilde{x}, 0, \tilde{z}, \tilde{w})$

At E_5 the Jacobian matrix becomes:

$J(E_5) = [d_{ij}]_{4\times4}$, (3.6)

where:

$$d_{11} = -\tilde{x} + \frac{u_4\tilde{x}\tilde{z}}{(u_5+\tilde{x})^2}, d_{12} = -(1+u_1)\tilde{x},$$

$$d_{13} = -\frac{u_4\tilde{x}}{u_5+\tilde{x}}, d_{14} = 0, d_{21} = u_2,$$

$$d_{22} = u_1\tilde{x} - u_6\tilde{z} - u_7\tilde{w} - (u_3+u_8+u_9),$$

$$d_{23} = 0, d_{24} = 0, d_{31} = \frac{u_5u_{10}\tilde{z}}{(u_5+\tilde{x})^2},$$

$$d_{32} = u_{11}\tilde{z}, d_{33} = -u_{14}\frac{\tilde{w}}{\tilde{z}},$$

$$d_{34} = u_{14} - u_{12}\tilde{z}, d_{41} = 0,$$

$$d_{42} = u_{15}\tilde{w}, \quad d_{43} = u_{12}\tilde{w}, \quad d_{44} = 0.$$

Then the characteristic equation of $J(E_5)$ is given by:

$[\lambda^4 + \widetilde{Q_1}\lambda^3 + \widetilde{Q_2}\lambda^2 + \widetilde{Q_3}\lambda + \widetilde{Q_4}] = 0$, (3.6a)

where:

$$\widetilde{Q_1} = -(d_{11}+d_{22}+d_{33}),$$

$$\widetilde{Q_2} = d_{11}d_{22} + d_{33}(d_{11}+d_{22}) - (d_{12}d_{21} + d_{13}d_{31} + d_{34}d_{43}),$$

$$\widetilde{Q_3} = d_{34}d_{43}(d_{11}+d_{22}) - d_{33}(d_{11}d_{22} - d_{12}d_{21}) - d_{13}(d_{21}d_{32} - d_{31}d_{22}),$$

$$\widetilde{Q_4} = d_{34}[d_{43}(d_{12}d_{21} - d_{11}d_{22}) - d_{13}d_{21}d_{42}].$$

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of eq. (3.6a), have negative real parts if and only if $\widetilde{Q_1} > 0$, $\widetilde{Q_3} > 0$, $\widetilde{Q_4} > 0$ and

$$\Delta_1 = (\widetilde{Q_1}\widetilde{Q_2} - \widetilde{Q_3})\widetilde{Q_3} - \widetilde{Q_1}^2\widetilde{Q_4} > 0.$$

Now $\widetilde{Q_1} > 0$, provided that the following conditions hold:

$1 > \frac{u_4\tilde{z}}{(u_5+\tilde{x})^2}$ (3.6b)

$u_1\tilde{x} < u_6\tilde{z} + u_7\tilde{w} + (u_3+u_8+u_9)$. (3.6c)

And, $Q_3 > 0$, if in addition to conditions (3.6b) and (3.6c), the following condition holds:

$u_{14} < u_{12}\tilde{z}$. (3.6d)

Also, $Q_4 > 0$, if in addition to conditions $(3.6b - 3.6d)$, the following condition holds:

$$\frac{u_2u_3}{u_5+\tilde{x}} < u_{12}(u_2(1+u_1) + (1 - \frac{u_4\tilde{z}}{(u_5+\tilde{x})^2})(u_6\tilde{z} + u_7\tilde{w} + u_3 + u_8 + u_9 - u_1\tilde{x}d$$

Further, it is easy to check that:

$\Delta_1 = U_1 - U_2$, where:

$$U_1 = (d_{11}+d_{22})(d_{11}d_{22} + d_{34}d_{43} + d_{33}^2)[-d_{34}d_{43}(d_{11}+d_{22}) + d_{33}(d_{11}d_{22} - d_{12}d_{21}) + d_{13}(d_{21}d_{32} - d_{31}d_{22}] + (d_{11}+d_{22}+d_{33})[(d_{12}d_{21} + d_{13}d_{31} + d_{34}d_{43}) \ \{d_{34}d_{43}(d_{11}+d_{22})d_{33}(d_{11}d_{22} - d_{12}d_{21} - d_{13}(d_{21}d_{32} - d_{31}d_{22})\}],$$

$$U_2 = (d_{11}+d_{22}+d_{33})^2 d_{34}[d_{43}(d_{12}d_{21} - d_{11}d_{22}) - d_{13}d_{21}d_{43}] - (d_{21}d_{32} - d_{31}d_{22})[d_{13}d_{34}d_{43}(d_{11}+d_{22}) - d_{13}d_{33}(d_{11}d_{22} - d_{12}d_{21}) + d_{12}d_{21}d_{13}d_{33} - d_{13}^2(d_{21}d_{32} - d_{31}d_{22})] + d_{12}d_{21}d_{33}[d_{34}d_{43}(d_{11}+d_{22}) - d_{33}(d_{11}d_{22} - d_{12}d_{21})].$$

Hence $\Delta_1 > 0$ if in addition to conditions (3.6b)-(3.6d), the following conditions hold:

$u_{12}(u_{12}\tilde{z} - u_{14})\tilde{w} < \tilde{x}\left(1 - \frac{u_4\tilde{x}}{(u_5+\tilde{x})^2}\right)(u_6\tilde{x} + u_7\tilde{w} + u_3 + u_8 + u_9 - u_1\tilde{x}) + u_{14}^2\frac{\tilde{w}^2}{\tilde{z}^2}$ (3.6f)

$U_1 > U_2$. (3.6g)

So, all the eigenvalues of $J(E_5)$ have negative real part under the given conditions and hence E_5 is locally asymptotically stable. However, it is unstable otherwise.

3.7 Stability of equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$

At E_6 the Jacobian matrix becomes:

$J(E_6) = [r_{ij}]_{4\times4}$, (3.7)

where:

$$r_{11} = -x^* + \frac{u_4x^*z^*}{(u_5+x^*)^2}, r_{12} = -(1+u_1)x^*,$$

$$r_{13} = -\frac{u_4x^*}{u_5+x^*}, r_{14} = 0, r_{21} = u_2 + u_1y^*,$$

$$r_{22} = -u_2\frac{x^*}{y^*}, r_{23} = -u_6y^*, r_{24} = -u_7y^*,$$

$$r_{31} = \frac{u_5u_{10}z^*}{(u_5+x^*)^2}, r_{32} = u_{11}z^*, r_{33} = -u_{14}\frac{w^*}{z^*},$$

$$r_{34} = u_{14} - u_{12}z^*, r_{41} = 0, r_{42} = u_{15}w^*,$$

$$r_{43} = u_{12}w^*, r_{44} = 0.$$

Then the characteristic equation of $J(E_6)$ is given by:

$\lambda^4 + B_1^*\lambda^3 + B_2^*\lambda^2 + B_3^*\lambda + B_4^* = 0$, (3.7a)

where:

$$B_1^* = -(N_1 + N_2 + N_3),$$

$$B_2^* = N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_{10} + N_{11},$$

$$B_3^* = -N_1(N_{11} + N_8) - N_3N_8 + N_{12} - N_1N_7 + N_3(N_4 - N_9) - N_2(N_{11} + N_{10}) + N_{13},$$

$$B_4^* = N_1(N_3N_8 + N_{15}) + N_1N_2N_{11} + N_{14} + N_{16}.$$

And,

$N_1 = r_{11}$, $N_2 = r_{22}$, $N_3 = r_{33}$, $N_4 = N_1N_2$,

$N_5 = N_2N_3$, $N_6 = N_1N_3$,

$N_7 = -r_{23}r_{32}$, $N_8 = -r_{42}r_{24}$, $N_9 = -r_{21}r_{12}$. $N_{10} = -r_{31}r_{13}$, $N_{11} = -r_{43}r_{34}$

$N_{12} = -(r_{34}r_{32}r_{24} + r_{32}r_{13}r_{21})$,

$N_{13} = -(r_{31}r_{23}r_{12} + r_{42}r_{34}r_{23})$.
$N_{14} = N_{10}N_8 + N_9 N_{11}$,
$N_{15} = r_{32}r_{24}r_{43} + r_{42}r_{34}r_{23}$.
$N_{16} = -(r_{31}r_{12}r_{24}r_{43} + r_{34}r_{21}r_{13}r_{42})$.

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of eq. $(3.7a)$, have negative real parts if and only if $B_1^* > 0$, $B_3^* > 0$, $B_4^* > 0$ and
$\Delta_2 = (B_1^* B_2^* - B_3^*)B_3^* - B_1^{*2}B_4^* > 0$.
Clearly $B_1^* > 0$, provided that:

$$\frac{u_4 z^*}{(u_5+x^*)^2} < 1 \qquad (3.7b)$$

$Now\ in B_3^*$, the first term is positive if in addition to conditions $(3.7b)$, the following conditions hold :

$$(u_{14} > u_{12}z^*), \qquad (3.7c)$$

$$(u_7 u_{15} y^*) > u_{12}(u_{14} - u_{12}z^*). \qquad (3.7d)$$

The second to fourth terms $of\ B_3^*$ are positive under conditions $(3.7b)$ and $(3.7c)$, while the fifth term $of\ B_3^*$ is positive if the following condition holds :

$$(1 + u_1)(u_2 + u_1 y^*) > \frac{u_2 x^*}{y^*}\left(1 \frac{u_4 z^*}{(u_5 + x^*)^2}\right), \qquad (3.7e)$$

while the last two terms of B_3^* are positive if in addition to condition $(3.7c)$ the following condition holds :

$$\frac{1+u_1}{u_{15}} < \frac{w^*(u_5+x^*)^2(u_{14}-u_{12}z^*)}{u_5 u_{10} x^* y^*} < \frac{u_4}{u_{12}} \qquad (3.7f)$$

Now $B_4^* > 0$, if in addition to conditions $(3.7b)$ and $(3.7c)$, the following condition holds:

$$\frac{u_{12}(1+u_1)}{u_4 u_{15}(u_2+u_1 y^*)} < \frac{(u_{14}-u_{12}z^*)(u_5+x^*)}{u_5 u_7 u_{10} x^* y^*} < \frac{u_4 u_{15}}{u_{12}\,[(1+u_1)(u_2+u_1 y^*)x^* + N_1 N_2](u_5+x^*)^2} \qquad (3.7g)$$

Further, it is easy to check that :
$\Delta_2 = \rho_1 - \rho_2$, where :

$$\rho_1 = (N_2 + N_2 + N_3)[(N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_{10})\{N_1 + N_3(N_8 - (N_4 - N_9)) + N_2(N_{11} - N_{10}) - N_{13}\} - (N_1 + N_2 + N_3)\{N_1(N_2 N_{11} + N_{15}) + N_9 N_{11}\}] - (N_8 + N_{11})[N_1(N_8 + N_{11}) + 2N_1 N_3(N_4 - N_9)] ,$$

$$\rho_2 = -(N_1 + N_2 + N_3)[N_1 N_{11}(N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_{10}) + N_3 N_{11}(N_8 - (N_4 - N_9)) - N_{11}(N_{12} - N_1 N_7 - N_2(N_{11} - N_{10}) + N_{13}) + N_1 N_{11}(N_8 + N_{11}) - (N_2 + N_2 + N_3)\{N_8 (N_1 N_3 + N_{10}) + N_{16}\}] + (N_4 - N_9)[N_2^2(N_{11} - N_{10}) + 2N_2(N_3 N_8 + N_1 N_7 - N_2(N_4 - N_9) - N_{13})] + N_3^2(1 + N_8^2) - N_1 N_7[2(N_{12} - N_3 N_8 + N_{13}) - N_1 N_7] + 2N_1(N_8 + N_{11})[N_3 N_8 - N_{12} + N_2(N_{11} - N_{10}) - N_{13} + N_1 N_7] + N_{12}[N_{12} - 2N_3 N_8 + 2N_{13}] - 2N_3 N_8 N_{13},$$

and, hence $\Delta_2 > 0$ if in addition to conditions (3.7a)-(3.7g) the following condition holds:

$$\rho_1 > \rho_2. \qquad (3.7h)$$

So, all the eigenvalues of $J(E_6)$ have negative real part under the given conditions and hence E_6 is locally asymptotically stable. However, it is unstable otherwise.

4 Global stability analysis:

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable of system (2.2) is studied analytically by use the suitable of Lyapunov functions as shown in the following theorems.

Theorem (4.1):
Assume that the vanishing equilibrium point $E_0 = (0,0,0,0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_0 is globally asymptotically stable provided that the following condition holds:

$$u_3 > 1 . \qquad (4.1a)$$

Proof: Consider the following function:
$F_0(x, y, z, w) = x + y + z + w.$
It is easy to see that $F_0(x, y, z, w) \epsilon\, C^1(R_+^4, R)$, and $F_0(E_0) = 0$, and $F_0(x, y, z, w) > 0$; $\forall (x, y, z, w) \neq E_0$. Now by differentiating F_0 with respect to time t and going some algebraic handling, given that:

$$\frac{dF_0}{dt} = x(1 - u_3) - x(x + y) - zy(u_6 - u_{11}) - (u_4 - u_{10})\frac{xz}{u_5+x} - wy(u_7 - u_{15}) - u_{13}(z + w) - (u_3 + u_8 + u_9)y$$

Now, according to the biological facts mentioned in theorem (2.1) , always $u_{10} < u_4$, $u_{11} < u_6$ and $u_{15} < u_7$, we obtain that:

$$\frac{dF_0}{dt} < x(1 - u_3) - u_{13}(z + w) - (u_3 + u_8 + u_9)y.$$

Thus, $\frac{dF_0}{dt}$ is negative definite and hence F_0 is Lyapunov function under the condition $(4.1a)$, and the proof is complete ■

Theorem (4.2):
Assume that the disease – predator free equilibrium point $E_1 = (\hat{x}, 0, 0, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_1 is globally asymptotically stable on the sub region $\acute{\omega}_1 \subseteq R_+^4$ provided that the following conditions hold:

$$\hat{\theta}_1 > \hat{\theta}_2 , \qquad (4.2a)$$

where:

$$\hat{\theta}_1 = (x - \hat{x})^2 + u_{13}(w + z) + xy .$$

$$\hat{\theta}_2 = [(1 + u_1)y + \frac{u_4}{u_5} z]\,\hat{x} + u_2 x .$$

Proof: Consider the following function

$$F_1(x, y, z, w) = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + y + z + w$$

It is easy to see that $F_1(x, y, z, w) \epsilon\, C^1(R_+^4, R)$, and $F_1(E_1) = 0$, and $F_1(x, y, z, w) > 0$; $\forall (x, y, z, w) \neq E_1$. Now by differentiating F_1 with respect to time t and going some algebraic handling, given that:

$$\frac{dF_1}{dt} = -(x - \hat{x})^2 - xy + (1 + u_1)\hat{x}y$$

$$-zy(u_6-u_{11})-\ wy(u_7-u_{15}) -(u_4-u_{10})\frac{x\,z}{u_5+x}-u_{13}(w+z) -(u_3+u_8\ +u_9\ )y+u_2x\ +\frac{u_4\hat{x}\,z}{u_5+x}.$$

Now, according to the biological facts mentioned in theorem (2.1), always , $u_{10}<u_4$, $u_{11}<u_6$ and $u_{15}<u_7$, we obtain that:

$$\frac{dF_1}{dt}<\ -(\ x-\ \hat{x})^2\ -u_{13}(w+z)\ +[(1+u_1)y\ +\ \frac{u_4}{u_5}z)]\ \hat{x}+u_2x$$

Then, $\frac{dF_1}{dt}<-\hat{\theta}_{\ 1}+\hat{\theta}_{\ 2}$.

Thus $\frac{dF_1}{dt}<0$ under the condition $(4.2a)$. hence E_1 is a globally asymptotically stable on the sub region $\acute{\omega}_1\subseteq R_+^4$ and then the proof is complete ■

Theorem (4.3):

Assume that the predator free equilibrium point $E_2=(\bar{x}\,,\bar{y}\,,0\,,0\,)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_2 is globally asymptotically stable on the sub region $\acute{\omega}_2\subseteq R_+^4$ provided that the following conditions hold:

$$\left(\frac{u_2}{\bar{y}}-1\right)\le 2\sqrt{\frac{u_2x}{y\bar{y}}}\ , \qquad (4.3a)$$

$$\bar{\theta}_1>\bar{\theta}_2\ , \qquad (4.3b)$$

where:

$$\bar{\theta}_1=\left[\ (x-\bar{x})-\sqrt{\frac{u_2x}{y\bar{y}}}(y-\bar{y})\right]^2+u_{13}(z+w)\ ,$$

$$\bar{\theta}_2=\ \frac{u_4\bar{x}\ z}{u_5+x}+\bar{y}(u_6z+u_7w)\ .$$

Proof: Consider the following function:

$$F_2(x,y,z,w)=\left(\ x-\bar{x}-\bar{x}ln\frac{x}{\bar{x}}\right)+\left(y-\bar{y}-\ \ \bar{y}ln\frac{y}{\bar{y}}\right)+z+w$$

It is easy to see that $F_2(x,y,z,w)\epsilon\ C^1(R_+^4,R)$ and $F_2(E_2)=0$, and $F_2(x,y,z,w)>0,\ \forall(x,y,z,w)\neq E_2$. Now by differentiating F_2 with respect to time t and going some algebraic handling, given that:

$$\frac{dF_2}{dt}=-(x-\bar{x})^2+\left(\frac{u_2}{\bar{y}}-1\right)(x-\bar{x})(y-\ \bar{y}\ )-\frac{u_2x}{y\bar{y}}\ (y-\bar{y})^2-(u_4-u_{10})\ \frac{x\,z}{u_5+x}\ +\frac{u_4\,\bar{x}z}{u_5+x}\ -(u_6-u_{11})yz-(u_7-u_{15})yw+\bar{y}(u_6z+u_7w)-u_{13}(z+w).$$

Now, according to condition (4.3a) and the biological facts mentioned in theorem (2.1) , always $u_{10}<u_4$, $u_{11}<u_6$ and $u_{15}<u_7$, we obtain that:

$$\frac{dF_2}{dt}<-\left[\ (x-\bar{x})-\sqrt{\frac{u_2x}{y\bar{y}}}(y-\bar{y})\right]^2-u_{13}(z+w)+\frac{u_4\bar{x}\ z}{u_5+x}+\bar{y}(u_6z+u_7w)=-\bar{\theta}_1+\bar{\theta}_2$$

Thus $\frac{dF_2}{dt}<0$ under the condition $(4.3b)$. Hence E_2 is a globally asymptotically stable on the sub region $\acute{\omega}_2\subseteq R_+^4$ and then the proof is complete ■

Theorem (4.4):

Assume that the disease free equilibrium point $E_3=(\dot{x},0,\dot{z},0)$ of system (2.2) is locally asymptotically stable in R_+^4.Then E_3 is globally asymptotically stable on the sub region $\acute{\omega}_3\subseteq R_+^4$ provided that the following conditions hold:

$$\frac{u_5u_{10}}{(u_5+x)(u_5+\dot{x})}\le 2\ , \qquad (4.4a)$$

$$\dot{\theta}_1>\ \dot{\theta}_2, \qquad (4.4b)$$

where:

$$\dot{\theta}_1=\left[(x-\ \dot{x})-(z-\dot{z})\right]^2+xy,$$

$$\dot{\theta}_2=(1+u_1)\dot{x}y+\frac{u_4}{u_5}(\dot{x}z+\dot{x}z)+u_2x+u_{12}\dot{z}w+(z-\dot{z})^2.$$

Proof: Consider the following function

$$F_3(x,y,z,w)=\left(\ x-\dot{x}-\dot{x}\ ln\frac{x}{\dot{x}}\right)+y+\left(z-\dot{z}-\dot{z}ln\frac{z}{\dot{z}}\right)+w$$

It is easy to see that $F_3(x,y,z,w)\epsilon\ C^1(R_+^4,R)$,and $F_3(E_3)=0$, and $F_3(x,y,z,w)>0\,;\forall(x,y,z,w)\neq E_3$. Now by differentiating F_3 with respect to time t and doing some algebraic handling, given that:

$$\frac{dF_3}{dt}=-\left[(x-\dot{x})^2-\frac{u_5u_{10}}{(u_5+x)(u_5+\dot{x})}(x-\dot{x})(z-\dot{z})+(z-\dot{z})^2\right]-xy+(1+u_1)\dot{x}y-\frac{u_4xz}{u_5+x}+\frac{u_4\dot{x}z}{u_5+\dot{x}}+\frac{u_4x\dot{z}}{u_5+\dot{x}}-\frac{u_4\dot{x}\dot{z}}{u_5+\dot{x}}-(u_6-u_{11})yz-(u_7-\ u_{15})yw-(u_3+u_8+u_9)y+u_2x-\dot{z}\left(u_{11}y+u_{14}\frac{w}{z}\right)-\ u_{13}\,w+u_{12}\dot{z}w+(z-\dot{z})^2.$$

Now, according to condition (4.4a) and the biological facts mentioned in theorem (2.1), $u_{11}<u_6$ and $u_{15}<u_7$, we obtain that:

$$\frac{dF_3}{dt}<-[(x-\dot{x})-(z-\dot{z})]^2-xy+(1+u_1)\dot{x}y+\frac{u_4}{u_5}(\dot{x}z+\dot{x}z)+\ u_2x+u_{12}\dot{z}w+\ (z-\dot{z})^2 =-\dot{\theta}_1+\dot{\theta}_2\ .$$

Thus, $\frac{dF_3}{dt}\ <0$ under the condition $(4.4b)$, and hence E_3 is a globally asymptotically stable on the sub region $\acute{\omega}_3\subseteq R_+^4$ and then the proof is complete ■

Theorem (4.5):

Assume that the infected prey free equilibrium point $E_4=(\bar{\bar{x}}\,,\bar{\bar{y}},\bar{\bar{z}},0\,)$ of system (2.2) is locally asymptotically stable. Then E_5 is globally asymptotically stable in the sub region $\acute{\omega}_4\subseteq R_+^4$ provided that the following condition holds:

$$\bar{\bar{\theta}}_1>\bar{\bar{\theta}}_2\ , \qquad (4.5a)$$

where:

$$\bar{\bar{\theta}}_1\ =\ \left[\ (x-\bar{\bar{x}})-\frac{1}{2}(y-\bar{\bar{y}})\right]^2+u_{13}w\ .$$

$$\bar{\bar{\theta}}_2=\ \frac{1}{4}(y-\bar{\bar{y}}\ )^2+u_6(y\bar{\bar{z}}+\bar{\bar{y}}z)+(u_7\bar{\bar{y}}+\ u_{12}\bar{\bar{z}})w+\left(\frac{u_4-u_{10}}{u_5}\right)(\bar{\bar{x}}z+x\bar{\bar{z}})\ .$$

Proof: Consider the following function:

$$F_4(x,y,z,w)=\left(\ x-\bar{\bar{x}}-\bar{\bar{x}}\ ln\frac{x}{\bar{\bar{x}}}\right)+\ \ (y-\bar{\bar{y}}-\bar{\bar{y}}ln\frac{y}{\bar{\bar{y}}})+(z-\bar{\bar{z}}-\bar{\bar{z}}ln\frac{z}{\bar{\bar{z}}})+w$$

It is easy to see that $F_4(x,y,z,w)\,\epsilon\, C^1(R_+^4,R)$, $and\ F_4(E_4)=0$,and $F_4(x,y,z,w)>0$; $\forall(x,y,z,w)\neq E_4$. Now by differentiating F_4 with respect to time t and going some algebraic handling, given that:

$$\frac{dF_4}{dt}=-(x-\bar{\bar{x}})^2-(x-\bar{\bar{x}})-\frac{1}{4}(y-\bar{\bar{y}})^2-(u_4-u_{10})\frac{xz}{(u_5+x)}-(u_4-u_{10})\frac{\bar{\bar{x}}\bar{\bar{z}}}{u_5+\bar{\bar{x}}}-(u_6-u_{11})zy-(u_7-u_{15})yw-\frac{u_{10}\,x\,\bar{\bar{z}}}{(u_5+x)}-\frac{u_{10}\,z\,\bar{\bar{x}}}{(u_5+\bar{\bar{x}})}+\frac{u_4\,z\,\bar{\bar{x}}}{(u_5+x)}+\frac{u_4\,x\,\bar{\bar{z}}}{(u_5+\bar{\bar{x}})}+u_6(y\bar{\bar{z}}+\bar{\bar{y}}z)-(u_6-u_{11})\bar{\bar{y}}\bar{\bar{z}}+u_2(x+\bar{\bar{x}})-u_2\left(\frac{x\bar{\bar{y}}}{y}+\frac{\bar{\bar{x}}y}{\bar{\bar{y}}}\right)+u_7\bar{\bar{y}}w-u_{11}(y\bar{\bar{z}}+\bar{\bar{y}}z)+u_{12}\bar{\bar{z}}w-u_{14}\frac{\bar{\bar{z}}w}{z}-u_{13}w+\frac{1}{4}(y-\bar{\bar{y}})^2+u_7\bar{\bar{y}}w.$$

Now, according to the biological facts mentioned in theorem (2.1) $u_{10}<u_4$, $u_{11}<u_6$ and $u_{15}<u_7$, we obtain that:

$$\frac{dF_4}{dt}<-\left[(x-\bar{\bar{x}})-\frac{1}{2}(y-\bar{\bar{y}})\right]^2-u_{13}w+\frac{1}{4}(y-\bar{\bar{y}})^2+u_6(y\bar{\bar{z}}+\bar{\bar{y}}z)+(u_7\bar{\bar{y}}+u_{12}\bar{\bar{z}})w+\left(\frac{u_4-u_{10}}{u_5}\right)(\bar{\bar{x}}z+x\bar{\bar{z}})$$
$$=-\bar{\bar{\theta}}_1+\bar{\bar{\theta}}_2$$

Thus, $\frac{dF_4}{dt}<0$ under the conditions (4.5a) with the biological fact $(u_{10}<u_4)$, and . Hence E_5 is a globally asymptotically stable on the sub region $\acute{\omega}_4\subseteq R_+^4$ and then the proof is complete ■

Theorem (4.6) :

Assume that the positive equilibrium point $E_5=(\tilde{x},0,\tilde{z},\widetilde{w})$ of system (2.2) is locally asymptotically stable. Then E_5 is globally asymptotically stable on the sub region $\acute{\omega}_5\subseteq R_+^4$ provided that the following condition holds:

$$\frac{u_{14}}{z}\leq 2\sqrt{\frac{u_{14}w}{z\tilde{z}}}, \qquad (4.6a)$$
$$\tilde{\theta}_1>\tilde{\theta}_2 \qquad (4.6b)$$

where:

$$\tilde{\theta}_1=\left[\sqrt{\frac{u_{14}w}{z\tilde{z}}}(z-\tilde{z})-(w-\widetilde{w})\right]^2+(x-\tilde{x})^2+xy+(u_3+u_8+u_9)y$$
$$\tilde{\theta}_2=(1+u_1)\tilde{x}y+\frac{(u_4-u_{10})}{u_5}(x\tilde{z}+\tilde{x}z)+u_2x+(w-\widetilde{w})^2 .$$

Proof: Consider the: following function:

$$F_5(x,y,z,w)=\left(x-\tilde{x}-\tilde{x}\,ln\frac{x}{\tilde{x}}\right)+y+\left(z-\tilde{z}-\tilde{z}\,ln\frac{z}{\tilde{z}}\right)+\left(w-\widetilde{w}-\widetilde{w}ln\frac{w}{\widetilde{w}}\right)$$

It is easy to see that $F_5(x,y,z,w)\epsilon\, C^1(R_+^4,R)$, and $F_5(E_5)=0$, and $F_5(x,y,z,w)>0$; $\forall(x,y,z,w)\neq E_5$ Now by differentiating F_5 with respect to time t and going some algebraic handling, given that:

$$\frac{dF_5}{dt}=-(x-\tilde{x})^2-xy+(1+u_1)\tilde{x}y-(u_4-u_{10})\frac{x\,z}{(u_5+x)}+u_4\left(\frac{\tilde{x}\,z}{(u_5+x)}+\frac{x\,\tilde{z}}{(u_5+\tilde{x})}\right)-u_{10}\left(\frac{\tilde{x}\,z}{(u_5+x)}+\frac{x\,\tilde{z}}{(u_5+\tilde{x})}\right)-(u_6-u_{11})yz-(u_3+u_8+u_9)y+u_2x-(u_4-u_{10})\frac{\tilde{x}\,\tilde{z}}{(u_5+\tilde{x})}-y(u_{11}\tilde{z}+u_{15}\widetilde{w})-\frac{u_{14}w}{z\tilde{z}}(z-\tilde{z})^2+\frac{u_{14}}{z}(z-\tilde{z})(w-\widetilde{w})-(w-\widetilde{w})^2+(w-\widetilde{w})^2.$$

Now, according to condition (4.6a) and the biological facts mentioned in theorem (2.1) $always,$ $u_{10}<u_4$, $u_{11}<u_6$ and $u_{15}<u_7$, we obtain that:

$$\frac{dF_5}{dt}<-\left[\sqrt{\frac{u_{14}w}{z\tilde{z}}}(z-\tilde{z})-(w-\widetilde{w})\right]^2-(x-\tilde{x})^2-xy-(u_3+u_8+u_9)y+(1+u_1)\tilde{x}y+\frac{(u_4-u_{10})}{u_5}(x\tilde{z}+\tilde{x}z)+u_2x+(w-\widetilde{w})^2$$
$$=-\tilde{\theta}_1+\tilde{\theta}_2.$$

Then $\frac{dF_5}{dt}<0$ under the condition:(4.6b) with the biological fact $(u_{10}<u_4)$, and Hence E_5 is a globally asymptotically stable on the sub region $\acute{\omega}_5\subseteq R_+^4$ and then the proof is complete ■

Theorem (4.7):

Assume that the positive equilibrium point $E_6=(x^*,y^*,z^*,w^*)$ of system (2.2) is locally asymptotically stable. Then E_6 is globally asymptotically stable on the sub region $\acute{\omega}_6\subseteq R_+^4$ provided that the following conditions hold:

$$\frac{u_2}{y^*}-1\leq 2\sqrt{\frac{u_2x}{yy^*}} \qquad (4.7a)$$
$$\frac{u_{14}}{z^*}\leq 2\sqrt{\frac{u_{14}w}{zz^*}} \qquad (4.7b)$$
$$\theta_1^*>\theta_2^*, \qquad (4.7c)$$

where:

$$\theta_1^*=\left[(x-x^*)-\sqrt{\frac{u_2x}{yy^*}}(y-y^*)\right]^2+\left[\sqrt{\frac{u_{14}w}{zz^*}}(z-z^*)-(w-w^*)\right]^2.$$
$$\theta_2^*=\frac{1}{u_5}\left((u_4-u_{10})xz^*+u_4x^*z\right)+(u_6-u_{11})(yz^*+y^*z)+(u_7-u_{15})(y^*w+yw^*)+(w-w^*)^2.$$

Proof: Consider the following function:

$$F_6(x,y,z,w)=\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\left(z-z^*-z^*\ln\frac{z}{z^*}\right)+\left(w-w^*-w^*\ln\frac{w}{w^*}\right)$$

It is easy to see that $F_6(x,y,z,w)\epsilon\, C^1(R_+^4,R)$, and $F_6(E_6)=0$, and $F_6(x,y,z,w)>0$; $\forall(x,y,z,w)\neq E_6$ Now by differentiating F_6 with respect to time t and doing some algebraic handling, given that:

$$\frac{dF_6}{dt}=-(x-x^*)^2+\left(\frac{u_2}{y^*}-1\right)(x-x^*)(y-y^*)-\frac{u_2x}{yy^*}(y-y^*)^2-(u_4-u_{10})\frac{xz}{(u_5+x)}-(u_6-u_{11})yz-(u_6-u_{11})y^*z^*-(u_7-u_{15})yw+\frac{u_4\,x\,z^*}{(u_5+x^*)}+\frac{u_4z\,x^*}{(u_5+x)}-\frac{u_4z^*x^*}{(u_5+x^*)}+(u_6-u_{11})(yz^*+y^*z)+(u_7-u_{15})(y^*w+yw^*)+(w-w^*)^2-\frac{u_{10}xz^*}{u_5+x}-\frac{u_{14}w}{zz^*}(z-z^*)^2+\frac{u_{14}}{z^*}(z-z^*)(w-w^*)-(w-w^*)^2.$$

Now, according to conditions (4.7a),(4.7b) and the biological facts mentioned in theorem (2.1) $u_{10} < u_4$, $u_{11} < u_6$ and $u_{15} < u_7$, we obtain that:

$$\frac{dF_6}{dt} < -\left[(x-x^*) - \sqrt{\frac{u_2 x}{yy^*}}(y-y^*)\right]^2 - \left[\sqrt{\frac{u_{14} w}{zz^*}}(z-z^*) - (w-w^*)\right]^2 \quad \frac{1}{u_5}\big((u_4-u_{10})xz^* + u_4 x^* z\big) + (u_6-u_{11})(yz^*+y^*z) + (u_7-u_{15})(y^*w + yw^*) + (w-w^*)^2 .$$

$$= -\theta_1^* + \theta_2^* .$$

Then $\frac{dF_6}{dt} < 0$ under the condition(4.7a)and the biological acts $u_{10} < u_4$, $u_{11} < u_6$, and . Hence E_6 is a globally asymptotically stable on the sub region $\acute{\omega}_6 \subseteq R_+^4$ and then the proof is complete ■

5 Numerical simulation:

In this section, we confirmed our obtained results in the previous sections numerically by using Runge Kutta method along with predictor corrector method. Note that, we use turbo C++ in programming and matlab in plotting and then discuss our obtained results. The system (2.2) is studied numerically for one set of parameters and different initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior: of system (2.2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point system (2.2) has globally asymptotically stable positive equilibrium point as shown in Fig.(5.1) .

$$u_1 = 0.5, u_2 = 0.2, u_3 = 0.3, u_4 = 0.4, u_5 = 0.4, u_6 = 0.6, u_7 = 0.5,\ u_8 = 0.2,$$

$$u_9 = 0.5, u_{10} = 0.2, u_{11} = 0.1, u_{12} = 0.3, u_{13} = 0.01, u_{14} = 0.002, u_{15} = 0.01 \quad (5.1)$$

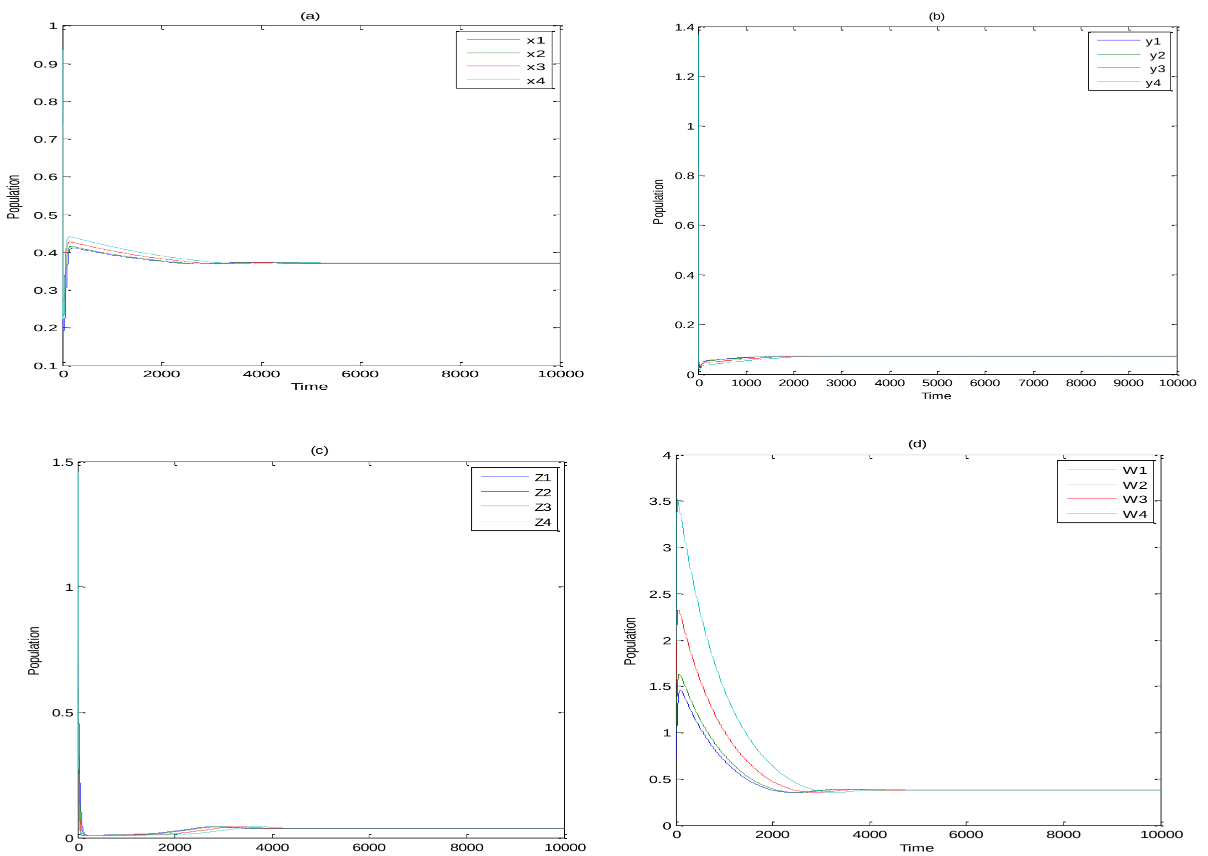


Fig.(5.1) Time series of the solution of system (2.2) that started from four different initial points $(0.2, 0.4, 0.3, 0.7)$, $(0.7, 0.9, 0.6, 1)$, $(0.9, 1, 0.8, 1.5)$ and $(1.3, 1.7, 1.5, 1.9)$ for the data given in (5.1). (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time, (d) trajectories w as a function of time.

Clearly, Fig.(5.1)shows that system (2.2) has a globally asymptotically stable as the solution of system (2.2) approaches asymptotically to the positive equilibrium point $E_6 = (0.354, 0.083, 0.037, 0.375)$ starting from four different initial points .

Now, in order to discuss importance of the parameters values of system (2.2) on the dynamical behavior of the system, the system is solved numerically for the data given in eq.(5.1) with varying one or more than one parameter at each time and the obtained results are given below.

The effect of varying the infection rate u_1 in the range $0.1 \leq u_1 < 2$ and keeping the rest of parameters as data given in (5.1) it is observed that the solution of system (2.2) approaches asymptotically to positive equilibrium point E_6, as shown in Fig.(5.2), for typical value $u_1 = 0.7$

.

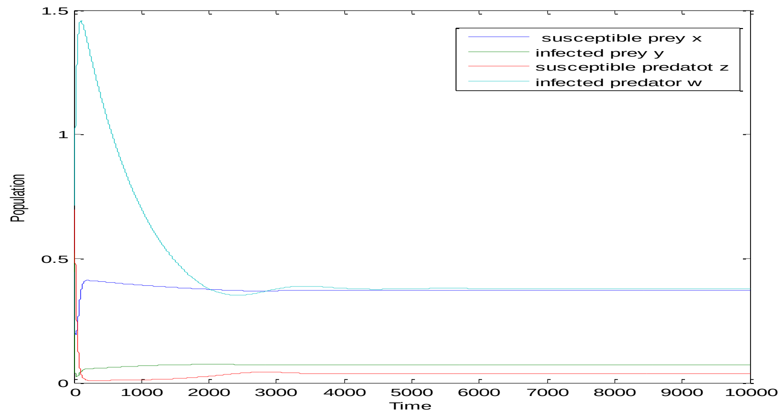


Fig.(5.2) :Time series of the solution of system (2.2) for the data given in eq.(5.1) with u_1=0.7 , which approaches to E_6=(0.335,0.084,0.037,0.364) in the interior of R_+^4 .

Now, varying the external source rate of prey u_2 and keeping the rest of parameters values as data given in (5.1), it is observed that for $0.1 \leq u_2 < 0.651$ the solution of (2.2) approaches to the positive equilibrium point E_6 , as shown in Fig.(5.3) (a) , for typical value $u_2 = 0.4$, while increasing this parameter for $0.651 \leq u_2 < 2$,it is observed that system (2.2) approach asymptotically to equilibrium point E_0=(0,0,0,0), as shown in Fig (5.3) (b) ,for typical value u_2 =0.9 .

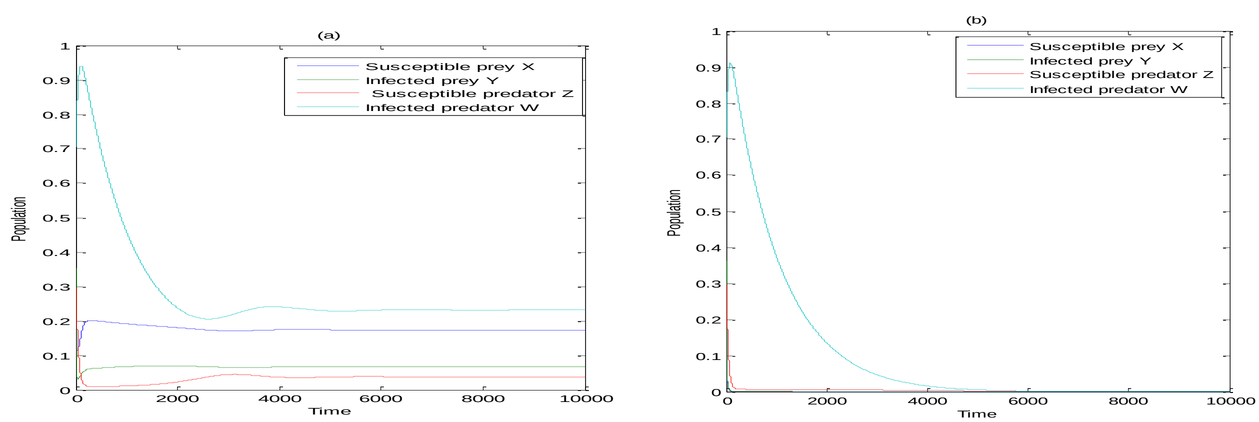


Fig. (5.3) (a): Time series of the solution of system (2.2) for the data given by eq. (5.1) with u_2=0.4 , which approaches to E_6=(0.167,0.070,0.376,0.226) in the interior of R_+^4, and (b) time series of solution of system (2.2) for the data given by (5.1) with u_2=0.9 , which approaches to E_0=(0,0,0,0) in the interior of R_+^4 .

Now, varying the parameter u_3 which represent the death rate of susceptible prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.1 \leq u_3 < 0.8$, the solution of system (2.2) still approaches asymptotically to a positive equilibrium point E_6, as shown in Fig.(5.4) (a) , for typical value $u_3 = 0.6$,while increasing this parameter for $0.8 \leq u_3 < 1$, the solution of system (2.2) approaches to E_0 , as shown in Fig.(5.4) (b) ,for typical value u_3 =0.85.

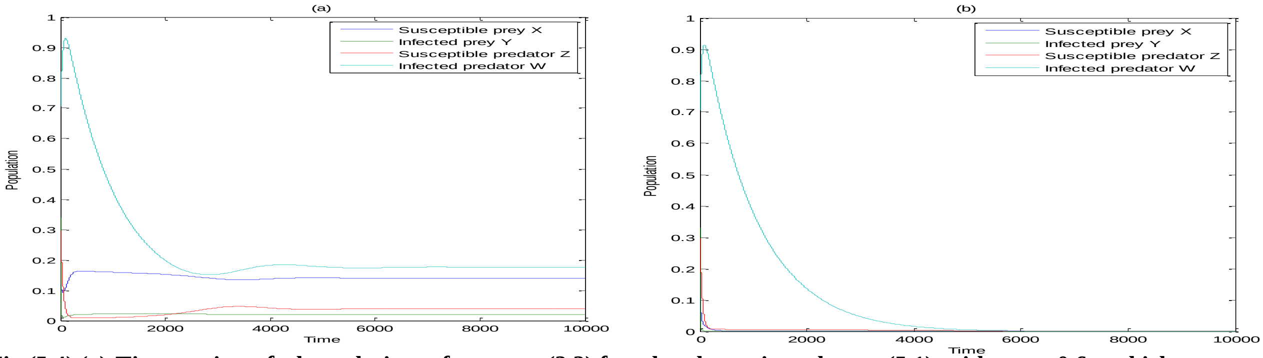


Fig (5.4) (a):Time series of the solution of system (2.2) for the data given by eq.(5.1) with u_3=0.6 , which approaches to E_6=(0.137,0.021,0.039,0.174), and (b) time series of solution of system (2.2) for the data given by eq.(5.1) with u_3=0.85, which approaches to E_0=(0,0,0,0).

Moreover , if we change the two parameters u_2 and u_3 in the same time and keeping the rest of parameters values as data in eq.(5.1) ,it is observed that for 0.001≤ u_2<0.1,and 0.813≤ u_3<1 the solution of system (2.2) approaches to the infected prey free equilibrium point E_5, as shown in Fig(5.5), for typical values u_2=0.002 and u_3=0.814

.

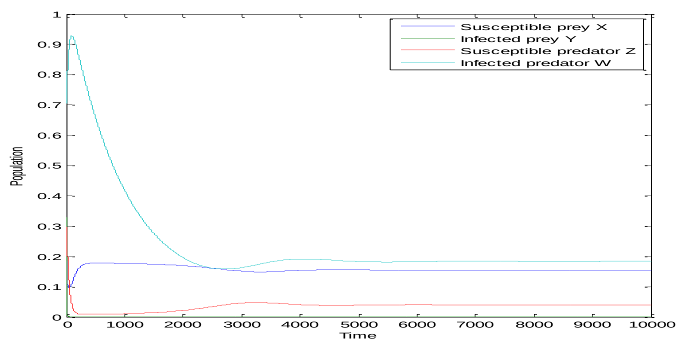


Fig. (5. 5) Time series of the solution of system (2.2) for the data given by eq.(5.1) , with u_2=0.002 and u_3=0.814 , which approaches to E_5=(0.157,0,0.039,0.185).

Now, varying the parameter u_4 which represent the predation rate of susceptible prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for

0.2 ≤ u_4< 2 the solution of system (2.2) still approaches to a positive equilibrium point E_6.

The effect of varying the half saturation rate of susceptible predator u_5 , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.1 ≤ u_5<1.5 the solution of system (2.2) approaches to a positive equilibrium point E_6.

The varying the parameter u_6 which represent the predation rate susceptible predator of infected prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.1 ≤ u_6 < 2 the solution of system (2.2) still approaches to a positive equilibrium point E_6 .

Now the varying of the parameter u_7 which represent the predation rate of infected predator on infected prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.01 ≤ u_7< 2 the solution of system (2.2) still approaches to a positive equilibrium point E_6.

Moreover the varying of the parameter u_8 which represent the death rate of the infected prey due to the disease , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.01 ≤ u_8<1 the solution of system (2.2) still approaches to a positive equilibrium point E_6 .

Now the varying of the parameter u_9 which represent the harvesting rate of infected prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.1 ≤ u_9<1 the solution of system (2.2) approaches to a positive equilibrium point E_6. The varying of the parameter u_{10} which represent the conversion of food rate from susceptible prey to susceptible predator , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for 0.1≤ u_{10}<0.4 the solution of system (2.2) still approaches to a positive equilibrium point E_6 . Moreover the varying the parameter u_{11} which represent the conversion of food rate from infected prey to infected predator , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.1 \leq u_{11} < 0.6$ the solution of system (2.2) still approaches to a positive equilibrium point E_6.

An investigation to the effect of varying the parameter u_{12} which represent the infection rate of predator , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.01 \leq u_{12} < 1.5$ the solution of system (2.2) approaches to a positive equilibrium point E_6 . However the change of the parameters u_2 , u_3 and u_{12} at the same time ,and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0 < u_2 \leq 0.002$,$0.79 \leq u_3 < 1$, and $0 < u_{12} \leq 0.01$ the solution of system (2.2) approaches to disease free equilibrium point E_3. , as shown in Fig (5.6), for typical value u_2 =0.001 , u_3=0.8 , and u_{12} =0.01.

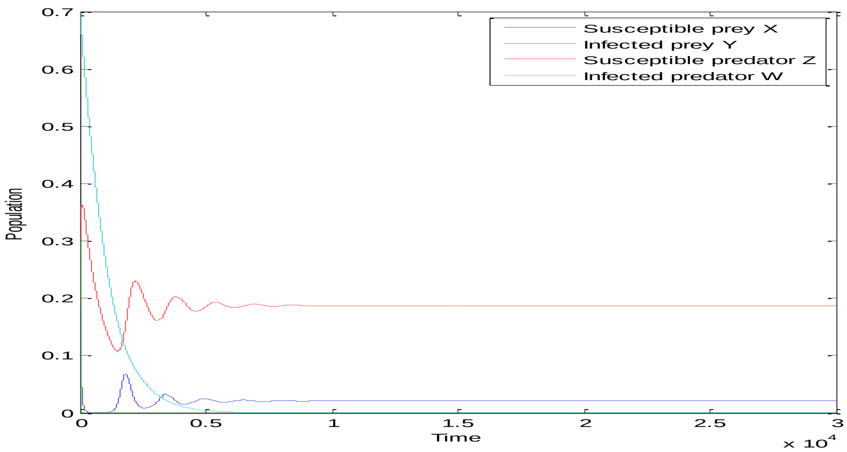
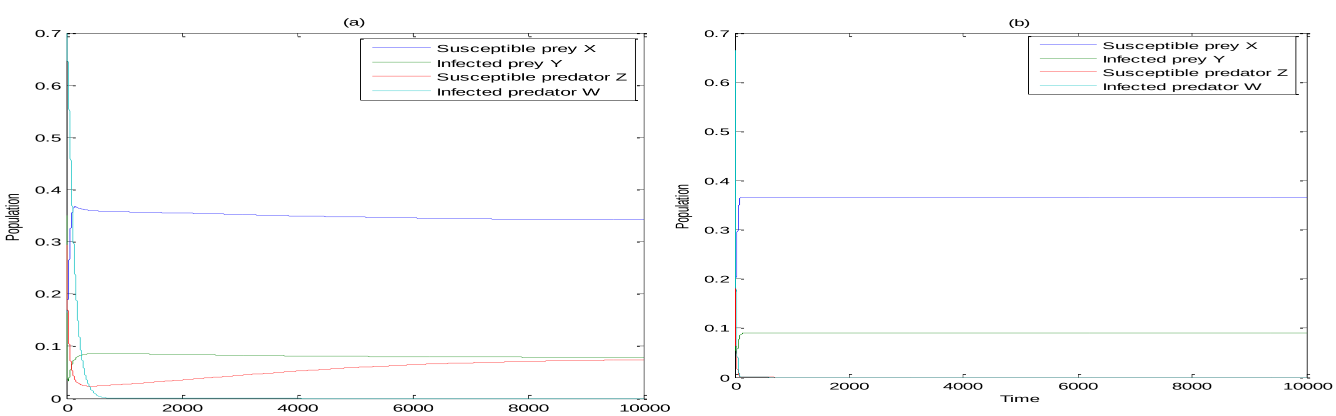


Fig (5.6) : Time series of the solution of system (2.2) for the data given by eq.(5.1) with u_2=0.001 , u_3=0.8 and u_{12}=0.01 which approaches to E_3=(0.020,0 ,0.196,0)

An investigation to the effect of varying the parameter u_{13} which represent the death rate of predator in the absence of prey , and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.01 \leq u_{13} < 0.09$ the solution of system (2.2) approaches to a positive equilibrium point E_6, while increasing this parameter further $0.09 \leq u_{13} < 0.11$ cause extinction in the infected predator and the system will: approach to the infected predator free equilibrium point E_5 , as shown in Fig.(5.7) (a) , for typical value u_{13}=0.1 , while increasing this parameter further $0.11 \leq u_{13} < 1$ cause extinction in the predator and the system will approach to the predator free equilibrium point E_2 , as shown in Fig.(5.7) (b), for typical value u_{13}=0.6.

Moreover change the parameters u_2 , u_3 and u_{13} at the same time, and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0 < u_2 \leq 0.008$, $0.079 \leq u_3 < 1$, and $0 < u_{13} \leq 0.09$ the solution of system (2.2) still approaches to an equilibrium point E_1. , as shown: in Fig.(5.7) (c), for typical values u_{13}=0.08 , u_2=0.007 and u_3= 0.792.



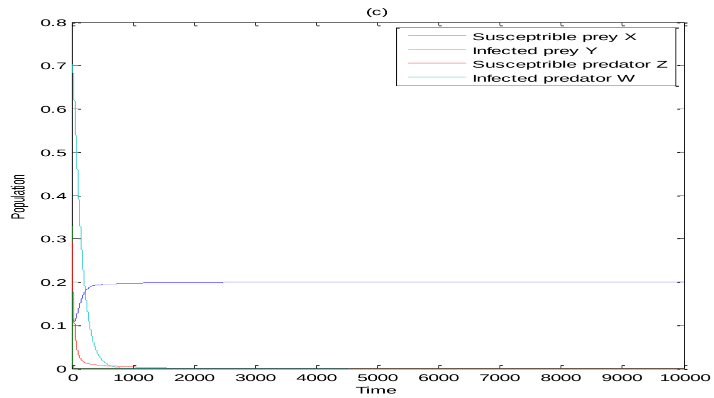
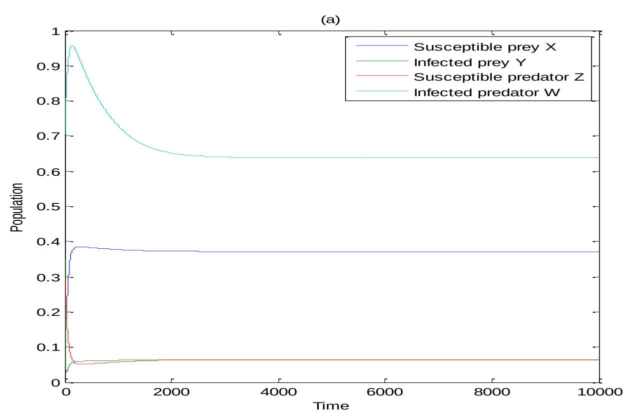


Fig. (5.7) (a): Time series of the solution of system (2.2) for the data given by eq.(5.1) with u_{13}=0.1 , which approaches to E_5=(0.307,0.065,0.167,0), and (b) time series of solution of system: (2.2) for the data given by eq.(5.1) with u_{13}=0.6, which approaches to E_2=(0.365,0.089,0,0). and (c) time series of solution of system (2.2) for the data given by eq.(5.1) with u_2=0.007 , u_3= 0.792 and u_{13}=0.08 which approaches to E_1=(0.207,0,0,0).

Now, varying the recovery rate of predator u_{14} and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.002 \leq u_{14} < 0.14$ the solution of (2.2) approaches to the equilibrium point E_6 , shown in Fig.(5.8)(a) for typical value u_{14}=0.01 as while increasing this parameter further $0.14 \leq u_{14} < 1.5$ the solution of system (2.2)will approach to disease free equilibrium point E_3 , for typical value u_{14}=0.2 as shown in Fig.(5.8)(b).
Finally , varying the parameter u_{15} which represent the conversion rate of food from infected prey to infected predator and keeping the rest of parameters values as data given in (5.1), it is observed that for $0.01 \leq u_{15} < 0.5$ the solution of system (2.2) still approaches to a positive equilibrium point E_6.



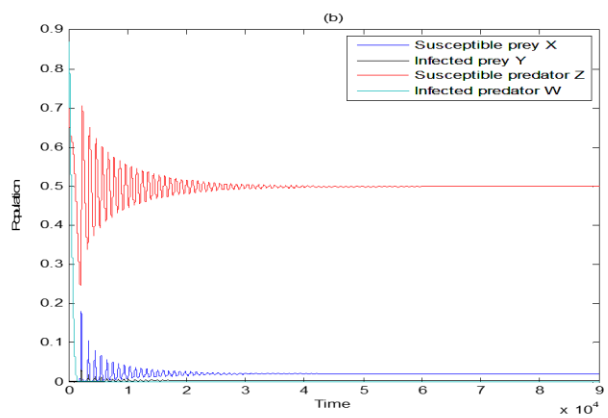


Fig. (5.8) (a): Time series of the solution of system (2.2) for the data given by eq.(5.1) with u_{14}=0.01 , which approaches to E_6=(0.0371,0.063,0.064,0.638). Fig.(5.8) (b): Time series of the solution of system (2.2) for the data given by eq.(5.1) with u_{14}=0.2 , which approaches to E_3=(0.020,0,0.498,0).

CONCLUSIONS AND DISCUSSIONS:

In this paper, an eco - epidemiological prey - predator model is proposed for study. The model includes SI infectious disease in prey which is transmitted by external source and the contact between the susceptible and infected species involving the harvesting on the infected prey only, and SIS disease in predator species which is spread by contact between susceptible individuals and infected individuals. The epidemics cannot transmitted between prey to predator by predation or conversely. Two types of functional response for describing the predation as well as linear incidence for describing the transition are used; the model is proposed and analyzed, and system (2.2) has been solved numerically for different initial points and one hypothetical set of parameters given by eq.(5.1) and the following observation are obtained.

1-For the set of hypothetical parameters values given by eq.(5.1), system (2.2) approaches asymptotically to globally asymptotically stable point E_6= (0.354,0.083,0.037,0.375).

2-For the set of hypothetical parameters values given eq.(5.1), system (2.2) do not have a periodic dynamics.

3-Varying of the parameters u_i , $i = 1,4,5,6,7,8,9,10,11,15$ which represent the infection rate, the predation rate on susceptible prey, the half saturation rate of susceptible predator ,the predation rate of susceptible predator on infected prey ,the predation rate of infected predator on infected prey , the death rate of the infected prey due to disease ,the harvesting rate on infected prey , the conversion of food rate from susceptible prey to susceptible predator , the conversion of food rate from infected prey to susceptible predator and the conversion rate of food from infected prey to infected predator respectively, at each time and keeping other parameters fixed as data given in eq. (5.1) do not have any effect on the dynamical behavior of system (2.2) and the solution of system (2.2) still approaches to a positive equilibrium point E_6=(x*,y*,z*,w*).

4-Varying the external source rate parameter of prey u_2 and u_3 which represent the death rate of susceptible prey , in the range for $0.1\le u_2<0.651$ and $0.1\le u_3<0.8$ and keeping other parameters fixed as data given in eq.(5.1) the solution of system (2.2) still approaches to a positive equilibrium point E_6=(x*,y*,z*,w*). However if $0.651\le u_2<2$ and $0.8\le u_3<1$, the solution of system (2.2) approaches to vanishing equilibrium point E_0=(0,0,0,0), thus u_2=0.651 and u_3=0.8 are bifurcation points.

5-For increasing the death rate of prey u_3 and decreases external source rate of prey: u_2, at the same time in the range $0.001\le u_2<0.1$, and $0.813\le u_3<1$, and keeping other parameters fixed as data given in eq.(5.1) the solution of system (2.2) still approaches to the infected prey free equilibrium point $E_5=(\tilde{x},0,\tilde{z},\widetilde{w})$.

6-By varying the infection rate of predator u_{12} , and external source rate of prey u_2 and the death rate of prey u_3 at the same time in the range $0< u_2 \le 0.002$, $0.79\le u_3<1$, and $0< u_{12}\le 0.01$ and keeping other parameters fixed as data given in eq.(5.1) the solution of system (2.2) approaches to disease free equilibrium point $E_3=(\dot{x},0,\dot{z},0)$.

7-The varying of the parameter u_{13} which represent the natural death rate of predator, and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.01\le u_{13}<0.09$ the solution of system (2.2) approaches to a positive equilibrium point E_6=(x*,y*,z*,w*), while increasing this parameter further $0.09\le u_{13}<0.11$ cause extinction in the infected predator and the system will approach to the infected predator free equilibrium point $E_4=(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}},0)$ while increasing this parameter further $0.11\le u_{13}<1$ cause extinction in the predator and the system will approach to the predator free equilibrium point $E_2=(\bar{x},\bar{y},0,0)$, thus u_{13}= 0.09 , u_{13}=0.11 are bifurcation points.

8-The varying external source rate of prey u_2 and the death rate of susceptible prey u_3 , and natural death rate of predator u_{13} at the same time in the range for $0< u_2\le 0.008$,$0.079\le u_3<1$, and $0< u_{13}\le 0.09$ the solution of system (2.2) approaches to an equilibrium point $E_1=(\hat{x},0,0,0)$.

9-Finally, the varying of the recover rate of predator u_{14} and keeping the rest of parameters values as data given in eq.(5.1), it is observed that for $0.002\le u_{14}<0.14$ the solution of (2.2) approaches to the equilibrium point E_6=(x*,y*,z*,w*), while increasing this parameter further $0.14\le u_{14}<1.5$ the solution of system (2.2) will approach to disease free equilibrium point $E_3=(\dot{x},0,\dot{z},0)$, thus u_{14}=0.14 is bifurcation point.

REFERENCES

1)Frank Hoppensteedt , Courant Institute of Mathematical Sciences, NYU, New York, NY.

2) Ashine . B. A, Global Journal of Science Frontier Research: F , Mathematics and Decision Sciences Volume 17 Issue 2 Version 1.0 Year 2017.

3) Purves, W.K., G.H. Orians and H.C. Heller. Life: The Science of Biology. Sinauer, Sunderland MA.

4) Kermack W. and Mckenderick A. A, Contribution to the mathematics theory of Epidemics . Proceedings of the Roual Socity series A, 115,700-721 ,1972.

5) Anderson, R.M., May, R.M. The population dynamics of microparasites and their inverteparte hosts . of Royal Society of London. Series B, 291,451-463 , 1981

6) Anderson, R.M., May, R.M. infectious diseases of humans dynamics and control . Oxford university Press, Oxford 1991.

7)Majeed .A.A and Shawka.I.I, The stability analysis of eco–epidemiological system with disease, Gen. math52-72P.P(2016)

8) Khalaf K. Q. , Majeed A. A. and Naji R. K., The dynamics of an SIS epidemic disease with contact and external source, Journal of Mathematical Theory and Modeling, Vol.5, 184-197, 2015

9) Naji R. K. and Mustafa A. N., The dynamics of an eco-epidemiological model with Nonlinear Incidence Rate, Journal of Applied Mathematics, Article ID 852631,24 pages, 2012.

10) Bairagi. N, Chaudhuri . S , Chattopadhyay . J , Harvesting as a disease control measure in an eco-epidemiological system – Atheoretical study, Mathematical Biosciences 217 (2009) 134–144

11) Holling, C. S. (May 1959). "The components of predation as revealed by a study of small-mammal predation of the European pine sawfly". The Canadian Entomologist. 91 (5). pp. 293–320.

12) Ali. O.S [M.S.thesis]. Qualitative study of prey – predator Model with harvesting , Department of mathematics , Universtiy of Baghdad , 2017.

13) Guckenheimer . J and Holmes .P , Non-linear oscillations dynamical system and bifurcation of vector fields, Appl. Math. Scien. 42, Springer –Verlag, New York, Inc., 1983.