SOME GENERALIZED HERMITE-HADAMARD INEQUALITIES FOR GEOMETRICALLY-ARITHMETICALLY s-CONVEX FUNCTIONS

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ABSTRACT: *In this paper, we consider the class of geometrically-arithmetically s-convex functions. A generalized integral identity for differentiable functions is obtained. Then using this new integral identity we establish our main results which are Hermite-Hadamard inequalities for geometrically-arithmetically s-convex functions. Some special cases are also discussed.*

Keywords: convex functions, geometrically-arithmetically *s*-convex functions, Hermite-Hadamard inequalities.

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1 INTRODUCTION

Convexity plays an important role in different fields of pure and applied sciences. Consequently classical concepts of convex sets and convex functions have been extended and generalized in different directions using novel and innovative ideas, see [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13,14,15]. It is known that theory of inequalities and theory of convex functions are interrelated to each other, as a result researchers have extended classical inequalities for different generalizations of classical convex functions, see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,16, 17,18]. Shuang et al. [16] introduced the notion of geometrically-arithmetically s-convex functions and obtained various Hermite-Hadamard inequalities for this newly introduced class.

Let $a,b\in I$ with $a<b$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \qquad (1.1)$$

Inequality (1.1) is as a necessary and sufficient condition for a function to be convex. For different generalizations and extensions of Hermite-Hadamard inequalities, see [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14].

In this paper, we consider the class of geometrically-arithmetically s-convex functions and derive some new Hermite-Hadamard inequalities. Some special cases are also discussed. This is the main motivation of this paper.

2 PRELIMINARIES

In this section, we discuss some preliminary concepts and obtain a new result which plays a key role in deriving our main results.

Definition 2.1 [7]. *Let $I\subseteq \mathbb{R}_+$. A geometrically convex set is defined as*

$$x^t y^{1-t}\in I,\quad \forall x,y\in I, t\in[0,1].$$

Definition 2.2 [7]. *A function $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is said to be geometrically-arithmetically convex function, if*

$$f(a^t b^{1-t})\leq tf(a)+(1-t)f(b),$$
$$\forall a,b\in I, t\in[0,1].$$

Definition 2.3 [16]. *For some $s\in(0,1]$, a function $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is said to be geometrically-arithmetically s-convex function, if*

$$f(a^t b^{1-t})\leq t^s f(a)+(1-t)^s f(b),$$
$$\forall a,b\in I, t\in[0,1].$$

Now we derive an identity, which plays an important role in establishing our main results.

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Lemma 2.1 *Let $f:I=[a,b]\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable function on I^0 (the interior of I) with $a<b$. Let $\varphi:[a,b]\to\mathbb{R}_0$ be differentiable function. If $f'\in L[a,b]$ and $\xi\in\mathbb{N}$, then*

$$\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b\varphi'(x)f(x)dx$$
$$=\frac{\ln b-\ln a}{\xi+1}\left\{\begin{array}{l}\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\varphi(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})dt\\+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\varphi(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})dt\end{array}\right\}.$$

Proof. Let

$$\int_a^b\varphi'(x)f(x)dx$$
$$=\int_a^{(ab)^{\frac{\xi}{\xi+1}}}\varphi'(x)f(x)dx+\int_{(ab)^{\frac{\xi}{\xi+1}}}^b\varphi'(x)f(x)dx. \qquad (2.1)$$

Now

$$\int_a^{(ab)^{\frac{\xi}{\xi+1}}} \varphi'(x) f(x)dx$$

$$= -\int_0^1 f(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) d\varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}})$$

$$= \left| -\varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) f(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) \right|_0^1$$

$$-\frac{\ln b - \ln a}{\xi+1} \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} \varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) dt$$

$$= \varphi((ab)^{\frac{\xi}{\xi+1}}) f((ab)^{\frac{\xi}{\xi+1}}) - \varphi(a) f(a)$$

$$-\frac{\ln b - \ln a}{\xi+1} \times \tag{2.2}$$

$$\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} \varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) dt.$$

Also

$$\int_{(ab)^{\frac{\xi}{\xi+1}}}^{b} \varphi'(x) f(x)dx$$

$$= -\int_0^1 f(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) d\varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}})$$

$$= \left| -\varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) f(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) \right|_0^1$$

$$-\frac{\ln b - \ln a}{\xi+1} \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} \varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) dt$$

$$= \varphi((ab)^{\frac{\xi}{\xi+1}}) f((ab)^{\frac{\xi}{\xi+1}}) - \varphi(a) f(a)$$

$$-\frac{\ln b - \ln a}{\xi+1} \times \tag{2.3}$$

$$\int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} \varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) dt.$$

Summation of (2.1), (2.2) and (2.3) completes the proof.

Remark 2.1. *If* $\xi = 1$, *then Lemma 2.1 reduces to Lemma 3.1* [5].

3 MAIN RESULTS

In this Section we, derive our main results and this is main motivation of this paper.

Theorem 3.1 *Let* $f : I = [a,b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* I^0 *(the interior of* I *) with* $a < b$. *Let* $\varphi : [a,b] \to \mathbb{R}_0$ *be differentiable function. If* $f' \in L[a,b]$, $\xi \in \mathbb{N}$ *and* $|f'|^q$ *is geometrically-arithmetically* s*-convex function on* I *where* $s \in (0,1]$ *and* $q \ge 1$, *then, we have*

$$|\varphi(b) f(b) - \varphi(a) f(a) - \int_a^b \varphi'(x) f(x)dx|$$

$$\le \frac{(\ln b - \ln a) \|\varphi\|_\infty}{(\xi+1)^{\frac{s+q}{q}}} [(a^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}])^{1-\frac{1}{q}} \times$$

$$\{\vartheta_1(a,b;\xi;t) |f'(a)|^q + \vartheta_2(a,b;\xi;t) |f'(b)|^q\}^{\frac{1}{q}}$$

$$+(b^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}])^{1-\frac{1}{q}} \times$$

$$\{(\vartheta_3(a,b;\xi;t) |f'(a)|^q + \vartheta_4(a,b;\xi;t) |f'(b)|^q)\}^{\frac{1}{q}}],$$

where $\|\varphi\|_\infty = \sup_{x \in [a,b]} \varphi(x)$ and

$$\vartheta_1(a,b;\xi;t) = \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} (\xi+t)^s dt;$$

$$\vartheta_2(a,b;\xi;t) = \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} (1-t)^s dt;$$

$$\vartheta_3(a,b;\xi;t) = \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} (1-t)^s dt;$$

$$\vartheta_4(a,b;\xi;t) = \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} (\xi+t)^s dt,$$

respectively.

We would like to mention here that one can evaluate the integral values of $\vartheta_1(a,b;\xi;t)$, $\vartheta_2(a,b;\xi;t)$, $\vartheta_3(a,b;\xi;t)$ and $\vartheta_4(a,b;\xi;t)$ using mathematical softwares such as maple.

Proof. Using Lemma 2.1, power-mean inequality and the fact that $|f'|^q$ is geometrically-arithmetically s-convex function, we have

$$|\varphi(b) f(b) - \varphi(a) f(a) - \int_a^b \varphi'(x) f(x)dx|$$

$$=\left| \frac{\ln b - \ln a}{\xi+1} \{ \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} \varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) dt \right.$$

$$\left. + \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} \varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) dt \} \right|$$

$$\le \frac{\ln b - \ln a}{\xi+1} [\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} \varphi(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}) |f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}})| dt$$

$$+ \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} \varphi(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}) |f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}})| dt] \tag{2.6}$$

$$\leq \frac{(\ln b-\ln a)\|\varphi\|_\infty}{\xi+1}\Big[\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} |f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}})|dt + \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} |f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}})|dt\Big]$$

$$\leq \frac{(\ln b-\ln a)\|\varphi\|_\infty}{\xi+1}\times \Big[\Big(\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}} |f'(a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}})|^q dt\Big)^{\frac{1}{q}} + \Big(\int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}} |f'(a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}})|^q dt\Big)^{\frac{1}{q}}\Big]$$

$$\leq \frac{(\ln b-\ln a)\|\varphi\|_\infty}{\xi+1}\times \Big[\Big(\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}dt\Big)^{1-\frac{1}{q}}\Big\{\int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}\Big(\Big(\frac{\xi+t}{\xi+1}\Big)^s |f'(a)|^q + \Big(\frac{1-t}{\xi+1}\Big)^s |f'(b)|^q\Big)dt\Big\}^{\frac{1}{q}} + \Big(\int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}dt\Big)^{1-\frac{1}{q}}\Big\{\int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}\Big(\Big(\frac{1-t}{\xi+1}\Big)^s |f'(a)|^q + \Big(\frac{\xi+t}{\xi+1}\Big)^s |f'(b)|^q\Big)dt\Big\}^{\frac{1}{q}}\Big]$$

$$= \frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^{\frac{s+q}{q}}}\times \Big[\Big(a^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}]\Big)^{1-\frac{1}{q}} \{\vartheta_1(a,b;\xi;t)|f'(a)|^q + \vartheta_2(a,b;\xi;t)|f'(b)|^q\}^{\frac{1}{q}} + \Big(b^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}]\Big)^{1-\frac{1}{q}}\times \{(\vartheta_3(a,b;\xi;t)|f'(a)|^q + \vartheta_4(a,b;\xi;t)|f'(b)|^q)\}^{\frac{1}{q}}\Big].$$

This completes the proof.

Note that, if $q=1$, then Theorem 3.1 reduces to the following result.

Corollary 3.1 *Let $f: I=[a,b]\subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I^0 (the interior of I) with $a<b$. Let $\varphi:[a,b]\to \mathbb{R}_0$ be differentiable function. If $f'\in L[a,b]$, $\xi\in \mathbb{N}$ and $|f'|$ is geometrically-arithmetically s-convex function on I where $s\in(0,1]$, then*

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b \varphi'(x)f(x)dx|$$

$$\leq \frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^{s+1}}\times [\Theta_1(a,b;\xi;t)|f'(a)| + \Theta_2(a,b;\xi;t)|f'(b)|],$$

where $\|\varphi\|_\infty = \sup_{x\in[a,b]} \varphi(x)$,

$\Theta_1(a,b;\xi;t) = \vartheta_1(a,b;\xi;t)+\vartheta_3(a,b;\xi;t)$, and $\Theta_2(a,b;\xi;t) = \vartheta_2(a,b;\xi;t)+\vartheta_4(a,b;\xi;t)$, respectively.

Also, when $s=1$ then, we have

Corollary 3.2 *Let $f: I=[a,b]\subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I^0 (the interior of I) with $a<b$. Let $\varphi:[a,b]\to \mathbb{R}_0$ be differentiable function. If $f'\in L[a,b]$, $\xi\in \mathbb{N}$ and $|f'|^q$ is geometrically-arithmetically convex function on I for $q\geq 1$, then, we have*

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b \varphi'(x)f(x)dx|$$

$$\leq \frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^{\frac{s+q}{q}}}\times \Big[\Big(a^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}]\Big)^{1-\frac{1}{q}}\times \{\vartheta_1(a,b;\xi;t)|f'(a)|^q + \vartheta_2(a,b;\xi;t)|f'(b)|^q\}^{\frac{1}{q}} + \Big(b^{\frac{\xi}{\xi+1}} L[a^{\frac{1}{\xi+1}}, b^{\frac{1}{\xi+1}}]\Big)^{1-\frac{1}{q}}\times \{(\vartheta_3(a,b;\xi;t)|f'(a)|^q + \vartheta_4(a,b;\xi;t)|f'(b)|^q)\}^{\frac{1}{q}}\Big],$$

where $\|\varphi\|_\infty = \sup_{x\in[a,b]} \varphi(x)$ and

$$\vartheta_1(a,b;\xi;t) = \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}(\xi+t)dt;$$

$$\vartheta_2(a,b;\xi;t) = \int_0^1 a^{\frac{\xi+t}{\xi+1}} b^{\frac{1-t}{\xi+1}}(1-t)dt;$$

$$\vartheta_3(a,b;\xi;t) = \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}(1-t)dt;$$

$$\vartheta_4(a,b;\xi;t) = \int_0^1 a^{\frac{1-t}{\xi+1}} b^{\frac{\xi+t}{\xi+1}}(\xi+t)dt,$$

respectively.

We would like to mention here that one can easily evaluate the integral values of $\vartheta_1(a,b;\xi;t)$, $\vartheta_2(a,b;\xi;t)$, $\vartheta_3(a,b;\xi;t)$ and $\vartheta_4(a,b;\xi;t)$ using mathematical softwares such as maple.

Theorem 3.2 *Let $f: I=[a,b]\subseteq \mathbb{R}_+ \to \mathbb{R}$ be a* *differentiable function on* I^0 *(the interior of* I *) with* $a<b$ *and let* $\varphi:[a,b]\to \mathbb{R}_0$ *be differentiable function. If* $f'\in L[a,b]$, $\xi\in\mathbb{N}$ *and* $|f'|^q$ *is geometrically-arithmetically* s *-convex function on* I *where* $s\in(0,1]$ *and* $q>1$*,then, we have*

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b\varphi'(x)f(x)dx|$$
$$\leq\frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^{\frac{s+q}{q}}(s+1)^{\frac{1}{q}}}\times$$
$$[(a^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}\times$$
$$([(\xi+1)^{s+1}-\xi^{s+1}]|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}$$
$$+(b^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}(|f'(a)|^q$$
$$+[(\xi+1)^{s+1}-\xi^{s+1}]|f'(b)|^q)^{\frac{1}{q}}].$$

where $\|\varphi\|_\infty=\sup_{x\in[a,b]}\varphi(x)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is geometrically-arithmetically s-convex function, we have

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b\varphi'(x)f(x)dx|$$
$$=|\frac{\ln b-\ln a}{\xi+1}\{\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\varphi(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})dt$$
$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\varphi(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})dt\}|$$
$$\leq\frac{\ln b-\ln a}{\xi+1}[\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\varphi(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})|f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})|dt$$
$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\varphi(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})|f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})|dt]$$

$$\leq\frac{(\ln b-\ln a)\|\varphi\|_\infty}{\xi+1}[\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}|f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})|dt$$
$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}|f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})|dt]$$
$$\leq\frac{(\ln b-\ln a)\|\varphi\|_\infty}{\xi+1}\times$$
$$[(\int_0^1(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})^p dt)^{\frac{1}{p}}(\int_0^1|f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})|^q\ dt)^{\frac{1}{q}}$$
$$+(\int_0^1(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})^p dt)^{\frac{1}{p}}(\int_0^1|f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})|^q\ dt)^{\frac{1}{q}}]$$
$$\leq\frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^{\frac{s+q}{q}}}[(a^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}\times$$
$$(\frac{(\xi+1)^{s+1}-\xi^{s+1}}{s+1}|f'(a)|^q+\frac{1}{s+1}|f'(b)|^q)^{\frac{1}{q}}$$
$$+(b^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}(\frac{1}{s+1}|f'(a)|^q$$
$$+\frac{(\xi+1)^{s+1}-\xi^{s+1}}{s+1}|f'(b)|^q)^{\frac{1}{q}}].$$

This completes the proof.

Theorem 3.3 *Let* $f:I=[a,b]\subseteq\mathbb{R}_+\to\mathbb{R}$ *be a differentiable function on* I^0 *(the interior of* I *) with* $a<b$*. Let* $\varphi:[a,b]\to\mathbb{R}_0$ *be differentiable function. If* $f'\in L[a,b]$, $\xi\in\mathbb{N}$ *and* $|f'|^q$ *is geometrically-arithmetically* s *-convex function on* I *where* $s\in(0,1]$ *and* $q>1$*,then, we have*

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b\varphi'(x)f(x)dx|$$
$$\leq\frac{(\ln b-\ln a)\|\varphi\|_\infty}{(\xi+1)^2(sq+1)^{\frac{1}{q}}}\times$$
$$[(a^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}\times$$
$$(((\xi+1)^{sq+1}-\xi^{sq+1})^{\frac{1}{q}}|f'(a)|+\|f'(b)|)$$
$$+(b^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}])^{\frac{1}{p}}(|f'(a)|$$
$$+((\xi+1)^{sq+1}-\xi^{sq+1})^{\frac{1}{q}}|f'(b)|)].$$

where $\|\varphi\|_\infty=\sup_{x\in[a,b]}\varphi(x)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is geometrically-arithmetically s-convex function, we have

$$|\varphi(b)f(b)-\varphi(a)f(a)-\int_a^b\varphi'(x)f(x)dx|$$
$$=|\frac{\ln b-\ln a}{\xi+1}\{\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\varphi(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})dt$$
$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\varphi(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})dt\}|$$

$$\leq \frac{\ln b-\ln a}{\xi+1}\Big[\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\varphi(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})\,|\,f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})\,|\,dt$$

$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\varphi(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})\,|\,f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})\,|\,dt\Big]$$

$$\leq \frac{(\ln b-\ln a)\mathrm{P}\varphi\mathrm{P}_\infty}{\xi+1}\Big[\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\,|\,f'(a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})\,|\,dt$$

$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\,|\,f'(a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})\,|\,dt\Big]$$

$$\leq \frac{(\ln b-\ln a)\mathrm{P}\varphi\mathrm{P}_\infty}{\xi+1}\times$$

$$\Big[\int_0^1 a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}}\Big[\Big(\frac{\xi+t}{\xi+1}\Big)^s\,|\,f'(a)\,|+\Big(\frac{1-t}{\xi+1}\Big)^s\,|\,f'(b)\,|\Big]dt$$

$$+\int_0^1 a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}}\Big[\Big(\frac{1-t}{\xi+1}\Big)^s\,|\,f'(a)\,|+\Big(\frac{\xi+t}{\xi+1}\Big)^s\,|\,f'(b)\,|\Big]dt\Big]$$

$$\leq \frac{(\ln b-\ln a)\mathrm{P}\varphi\mathrm{P}_\infty}{\xi+1}\times$$

$$\Big[\Big(\int_0^1 (a^{\frac{\xi+t}{\xi+1}}b^{\frac{1-t}{\xi+1}})^p\,dt\Big)^{\frac{1}{p}}\Big\{\int_0^1\Big[\Big(\frac{\xi+t}{\xi+1}\Big)^{sq}dt\Big]^{\frac{1}{q}}\,|\,f'(a)\,|$$

$$+\int_0^1\Big[\Big(\frac{1-t}{\xi+1}\Big)^{sq}dt\Big]^{\frac{1}{q}}\,|\,f'(b)\,|\Big\}$$

$$+\Big(\int_0^1 (a^{\frac{1-t}{\xi+1}}b^{\frac{\xi+t}{\xi+1}})^p\,dt\Big)^{\frac{1}{p}}\Big\{\int_0^1\Big[\Big(\frac{1-t}{\xi+1}\Big)^{sq}dt\Big]^{\frac{1}{q}}\,|\,f'(a)\,|$$

$$+\int_0^1\Big[\Big(\frac{\xi+t}{\xi+1}\Big)^{sq}dt\Big]^{\frac{1}{q}}\,|\,f'(b)\,|\Big\}\Big]$$

$$\leq \frac{(\ln b-\ln a)\mathrm{P}\varphi\mathrm{P}_\infty}{(\xi+1)^2}\times$$

$$\Big[\Big(a^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}]\Big)^{\frac{1}{p}}\times$$

$$\Big(\Big(\frac{(\xi+1)^{sq+1}-\xi^{sq+1}}{sq+1}\Big)^{\frac{1}{q}}\,|\,f'(a)\,|+\Big(\frac{1}{sq+1}\Big)^{\frac{1}{q}}\,\mathrm{P}\,f'(b)\,|\Big)$$

$$+\Big(b^{\frac{p\xi}{\xi+1}}L[a^{\frac{p}{\xi+1}},b^{\frac{p}{\xi+1}}]\Big)^{\frac{1}{p}}\Big(\Big(\frac{1}{sq+1}\Big)^{\frac{1}{q}}\,|\,f'(a)\,|$$

$$+\Big(\frac{(\xi+1)^{sq+1}-\xi^{sq+1}}{sq+1}\Big)^{\frac{1}{q}}\,|\,f'(b)\,|\Big)\Big].$$

This completes the proof.

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