A NOTE ON RIGHT WEAKLY REGULAR SEMIGROUPS

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ABSTRACT: In this paper, we have characterized right weakly regular semigroups by the properties of their fuzzy right ideals, fuzzy bi-ideals and fuzzy interior ideals.

Key Words: Right weakly regular semigroups, fuzzy right, bi, interior, quasi ideals

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1. INTRODUCTION

There are a lot of uncertainties in many areas of real world problems such as, economics, computer science, medical science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many others. To handle these we need some natural tools such as probability theory and theory of fuzzy sets. Fuzzy set theory incorporates imprecision and subjectivity into the model formulation and solution process. Fuzzy set theory has been used to develop quantitative forecasting models such as time series analysis and regression analysis. This theory is also used in business, medical and related health sciences.

The concept of fuzzy sets was first introduced by Zadeh [1] in 1965. A fuzzy subset is a class of objects with a grade of membership. The concept of fuzzy set was applied in [2] to generalize the basic concepts of general topology. A. Rosenfeld [3] was the first who consider the case of a groupoid in terms of fuzzy sets. Kuroki introduced fuzzy ideals in semigroups [4]. In [5,6] S. Lajos and G. Szasz respectively characterized the intra-regular semigroups in terms of right ideals and left ideals of semigroups. Also, in [7] N. Kehayopulu, S. Lajos and M. Tsingelis characterized the intra-regular ordered semigroups in terms of right ideals and left ideals, and they obtained many characterizations of the intra-regularity in ordered semigroups. Recently, in [8] S. K. Lee characterized the intra-regular semigroups, without order, in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, and bi-ideals and right ideals of semigroups. In [9] Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [10], played a vital role to generate some different types of fuzzy subgroups. It is worth mentioning that Bhakat and Das [11] gave the concept of (α,β)-fuzzy subgroups by using the belongs to relation $\in$ and quasi coincident with relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroups, where $\alpha,\beta\in\{\in,q,\in\vee q,\in\wedge q\}$ and $\alpha\neq\in\wedge q$. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. These fuzzy subgroups are further studied in [12] and [13]. The concept of $(\in,\in\vee q_k)$-fuzzy subgroups is a viable generalization of Rosenfeld's fuzzy subgroups. Davvaz defined $(\in,\in\vee q_k)$-fuzzy sub nearrings and ideals of a nearring in [14]. Jun and Song initiated the study of (α,β)-fuzzy interior ideals of a semigroup in [15] which is the generalization of fuzzy interior ideals. In [16] Yamak et. al defined $(\in,\in\vee q_k)$-fuzzy bi-ideals of semigroups.

In this paper we have characterized right weakly regular semigroups by the properties of their $(\in,\in\vee q_k)$-fuzzy right ideals, $(\in,\in\vee q_k)$-fuzzy bi-ideals and $(\in,\in\vee q_k)$-fuzzy interior ideals.

2. PRELIMINARIES

In this section we recall some basic notions and results on semigroups. Throughout this paper S denotes a semigroup.

- A non-empty subset A of S is called a subsemigroup of S if $A^2\subseteq A$.
- A non-empty subset J of S is called a left (right) ideal of S if $SJ\subseteq J$ $(JS\subseteq J)$.
- J is called a two-sided ideal or simply an ideal of S if it is both left and right ideal of S.
- A non-empty subset Q of S is called a quasi-ideal of S if $QS\cap SQ\subseteq Q$.
- A non-empty subset B of S is called a generalized bi-ideal of S if $BSB\subseteq B$.
- A non-empty subset B of S is called a bi-ideal of S if it is both a subsemigroup and a generalized bi-ideal of S.
- A subsemigroup I of S is called an interior ideal of S if $SIS\subseteq I$.

Obviously every one-sided ideal of S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal.

It is well known that in a regular semigroup the concepts of quasi-ideal, bi-ideal and generalized bi-ideal coincide. Also in a regular semigroup, every interior ideal is an ideal.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \to [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For a fuzzy set f of a semigroup S and $t \in (0,1]$, the crisp set $U(f;t) = \{x \in S$ such that $f(x) \geq t\}$ is called level subset of f.

A fuzzy subset f of a semigroup S of the form

$$f(y) = \begin{cases} t \in (0,1] \text{ if } y = x \\ 0 \quad\quad\quad \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

The product of two fuzzy subsets f and g of a semigroup S denoted by $f \circ g$ and define as

$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz} \{f(y) \wedge g(z)\} & \text{if there exist } y, \\ \quad z \in S, \text{ such that } x = yz, & \\ 0 & \text{otherwise} \end{cases}$$

The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy subsets of a semigroup S.

$(f \wedge g)(a) = \min\{f(a), g(a)\}$ for all a in S.

$(f \vee g)(a) = \max\{f(a), g(a)\}$ for all a in S.

Definition 1. *A fuzzy subset f of a semigroup S is called a fuzzy subsemigroup of S if for all $a, b \in S$*

$f(ab) \geq \min\{f(a), f(b)\}$.

Definition 2. *A fuzzy subset f of a semigroup S is called a fuzzy left (resp: right) ideal of S if for all $a, b \in S$*

$f(ab) \geq f(b)$, (*resp:* $f(ab) \geq f(a)$).

Definition 3. *A fuzzy subset f of a semigroup S is called a fuzzy ideal of S if for all $a, b \in S$.*

$f(ab) \geq \max\{f(a), f(b)\}$.

Definition 4. *A fuzzy subset f of a semigroup S is called a fuzzy generalized bi-ideal of S if for all $a, b, c \in S$*

$f(abc) \geq \min\{f(a), f(c)\}$.

Definition 5. *A fuzzy subset f of a semigroup S is called an fuzzy bi-ideal of S if for all $a, b, c \in S$*

(i) $f(ab) \geq \min\{f(a), f(b)\}$.

(ii) $f(abc) \geq \min\{f(a), (f(c)\}$.

Definition 6. *A fuzzy subset f of a semigroup S is called a fuzzy quasi ideal of S if for all $a, b \in S$*

$f(a) \geq \min\{f \circ \varsigma(a), \varsigma \circ f(a)\}$,

where $f(x) = 1$, *for all* $x \in S$.

Definition 7. *A fuzzy subset f of a semigroup S is called a fuzzy generalized interior ideal of S if*

$f(axb) \geq f(x)$, *for all* $a, x, b \in S$.

Definition 8. *A fuzzy subset f of S is called a fuzzy interior ideal of S if*

(i) $f(ab) \geq \min\{f(a), f(b)\}$.

(ii) $f(axb) \geq f(x)$, *for all* $a, x, b \in S$.

3. CHARACTERIZATION OF RIGHT WEAKLY REGULAR SEMIGROUPS

Theorem 1. *For the semigroup S the following are equivalent*

(i) *S is right weakly regular.*

(ii) $R \cap I \subseteq RI$, *where R is right ideal and I is interior ideal of S.*

(iii) $R[a] \cap I[a] \subseteq R[a]I[a]$, *for some a in S.*

Proof: $(i) \Rightarrow (ii)$: Let S be a right weakly regular semigroup, R a right ideal and I an interior ideal of S. Let $a \in R \cap I$ which implies that $a \in R$ and $a \in I$. Since S is right weakly regular semigroup so for $a \in S$ there exist $x, y \in S$ such that $a = axay$ so we have $a = axay = (ax)(axayy) \in (RS)(SIS) \subseteq RI.$

Therefore $R \cap I \subseteq RI.$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: As $a \cup aS$ is a right ideal and $a \cup a^2 \cup SaS$ is an interior ideal of S generated by a. Then by hypothesis,

$$\begin{aligned} & (a \cup aS) \cap (a \cup a^2 \cup SaS) \\ & \subseteq (a \cup aS)(a \cup a^2 \cup SaS) \\ & = a^2 \cup a^3 \cup aSaS \cup aSa \cup aSa^2 \cup aSSaS \\ & \subseteq a^2 \cup a^3 \cup aSa \cup aSaS. \end{aligned}$$

Therefore $a = a^2$ or $a = a^3$ or $a = axay$ for some x, y in S. If $a = a^2$ then $a = a^2a^2 = aaaa = auav$ where $u = v = a$. If $a = a^3$ then $a = aaaaa = apaq$ where $p = aa$ and $q = a$. Hence S is right weakly regular.

Theorem 2. *For the semigroup S the following are equivalent*

(i) *S is right weakly regular.*

(ii) $B \cap I \subseteq BI$, *where B is bi-ideal and I is interior ideal of S.*

(iii) $B[a] \cap I[a] \subseteq B[a]I[a]$, *for some a in S.*

Proof: $(i) \Rightarrow (ii)$: Let S be a right weakly regular semigroup, B a bi-ideal and I an interior ideal of S. Let $a \in B \cap I$ which implies that $a \in B$ and $a \in I$. Since S

is right weakly regular semigroup so for $a \in S$ there exist $x, y \in S$ such that $a = axay$ so we have

$$a = axay = (axay)(xay)$$
$$= (axa)(yxay) \in (BSB)(SIS) \subseteq BI.$$

Therefore $B \cap I \subseteq BI.$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: As $a \cup a^2 \cup aSa$ is a bi-ideal generated by a and $a \cup a^2 \cup SaS$ is an interior ideal generated by a so by assumption, we have

$$(a \cup a^2 \cup aSa) \cap (a \cup a^2 \cup SaS)$$
$$\subseteq (a \cup a^2 \cup aSa)(a \cup a^2 \cup SaS)$$
$$= a^2 \cup a^3 \cup aSaS \cup a^3 \cup a^4 \cup a^2SaS$$
$$\cup aSa^2 \cup aSa^3 \cup aSaSaS$$
$$= a^2 \cup a^3 \cup a^4 \cup aSaS \cup a^2SaS \cup aSa^2$$
$$\cup aSa^3 \cup aSaSaS$$
$$\subseteq a^2 \cup a^3 \cup a^4 \cup aSaS \cup aSa^2 \cup a^2SaS$$
$$\subseteq a^2 \cup a^3 \cup a^4 \cup aSaS.$$

Therefore $a = a^2$ or $a = a^3$ or $a = a^4$ or $a = axay$ for some x, y in S. If $a = a^2$ then $a = a^2a^2 = aaaa$. If $a = a^3$ then $a = aaaaa = auaa$ where $u = aa$. If $a = a^4 = aaaa$. Hence S is right weakly regular.

Theorem 3. *For a semigroup S the following are equivalent.*

(i) S *is right weakly regular.*

(ii) $R \cap B \cap I \subseteq RBI$ *for every right ideal* R, *bi-ideal* B *and interior ideal* I *of a semigroup* S.

(iii) $R[a] \cap B[a] \cap I[a] \subseteq R[a]B[a]I[a]$, *for some* a *in* S.

Proof: $(i) \Rightarrow (ii)$: Let S be right weakly regular semigroup, R a right ideal, B a bi-ideal and I an interior ideal of S. Let $a \in R \cap B \cap I$ then $a \in R,$ $a \in B$ and $a \in I.$ Since S is right weakly regular semigroup so for a there exist $x, y \in S$ such that

$$a = axay = (axay)(xay)$$
$$= (ax)(ay)(xay) = (ax)(axayy)(xay)$$
$$= (ax)(axa)(yyxay) \in (RS)(BSB)(SIS) \subseteq RBI.$$

Therefore $a \in RBI.$ So $R \cap B \cap I \subseteq RBI.$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: As $a \cup aS,$ $a \cup a^2 \cup aSa$ and $a \cup a^2 \cup SaS$ are right ideal, bi-ideal and interior ideal of S generated by an element a of S respectively. Thus by assumption we have

$$(a \cup aS) \cap (a \cup a^2 \cup aSa) \cap (a \cup a^2 \cup SaS)$$
$$\subseteq (a \cup aS)(a \cup a^2 \cup aSa)(a \cup a^2 \cup SaS)$$
$$= (a^2 \cup a^3 \cup a^2Sa \cup aSa \cup aSa^2 \cup aSaSa)$$
$$\cup (a \cup a^2 \cup SaS)$$
$$= a^3 \cup a^4 \cup a^2Sa^2 \cup aSa^2 \cup aSa^3 \cup aSaSa^2$$
$$\cup a^4 \cup a^5 \cup a^2Sa^3 \cup aSa^3 \cup aSa^4 \cup aSaSa^3$$
$$\cup a^2SaS \cup a^3SaS \cup a^2SaSaS \cup aSaSaS$$
$$\cup aSa^2SaS \cup aSaSaSaS$$
$$\subseteq a^3 \cup a^4 \cup a^5 \cup a^2Sa^2 \cup aSa^2 \cup aSa^3$$
$$\cup aSaSa^2 \cup aSaS$$
$$\subseteq a^3 \cup a^4 \cup a^5 \cup aSa \cup aSaS.$$

Therefore $a = a^3$ or $a = a^4$ or $a = a^5$ or $a = axa$ or $a = auav$ for some x, u, v in S. If $a = a^3$ then $a = aaaaa = auaa$ where $u = aa$. If $a = a^4 = aaaa = apaq$ where $p = q = a$. If $a = a^5$ then $a = aaaaa = aras$ where $r = aa, s = a.$ If $a = axa = axaxa = axat$ where $t = xa.$ Hence S is right weakly regular.

Based on the useful results obtained above, we now characterize right weakly regular semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy right ideals, $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals.

Theorem 4. *For a semigroup S the following conditions are equivalent.*

(i) S *is right weakly regular.*

(ii) $f \wedge_k g \leq f \circ_k g$ *for all* $(\in, \in \vee q_k)$*-fuzzy right ideal* f *and* $(\in, \in \vee q_k)$*-fuzzy interior ideal* g *of* S.

Proof: $(i) \Rightarrow (ii)$: Let f and g be $(\in, \in \vee q_k)$-fuzzy right ideal and $(\in, \in \vee q_k)$-fuzzy interior ideal of S. Since S is right weakly regular, then for each $a \in S$ there exist $x, y \in S$ such that $a = axay = axaxayy$, so we have

$$\begin{aligned}(f\circ_k g)(a) &= (f\circ g)(a)\wedge\frac{1-k}{2}\\ &= \left(\bigvee_{a=pq}\{f(p)\wedge g(q)\}\right)\wedge\frac{1-k}{2}\\ &\geq f(ax)\wedge g(axayy)\wedge\frac{1-k}{2}\\ &\geq f(a)\wedge g(a)\wedge\frac{1-k}{2}\\ &= (f\wedge_k g)(a).\end{aligned}$$

Therefore $f\wedge_k g\leq f\circ_k g$.

$(ii)\Rightarrow(i)$: Let $b\in R[a]\cap I[a]$. Since $R[a]$ and $I[a]$ are right ideal and interior ideal of S generated by a respectively. Then $(C_{R[a]})_k$ and $(C_{I[a]})_k$ are $(\in,\in\vee q_k)$-fuzzy right ideal and $(\in,\in\vee q_k)$-fuzzy interior ideal of S respectively. Then by hypothesis,

$$\begin{aligned}\frac{1-k}{2} &\leq (C_{R[a]\cap I[a]})_k(b) = ((C_{R[a]})_k \wedge_k (C_{I[a]})_k)(b)\\ &\leq ((C_{R[a]})_k \circ_k (C_{I[a]})_k)(b) = (C_{R[a]I[a]})_k(b).\end{aligned}$$

Therefore $b\in R[a]I[a]$. Thus $R[a]\cap I[a]\subseteq R[a]I[a]$. Hence it follows from Theorem 1 that S is right weakly regular.

Theorem 5. *For a semigroup S the following conditions are equivalent.*

(i) *S is right weakly regular.*

(ii) *$f\wedge_k g\leq f\circ_k g$ for all $(\in,\in\vee q_k)$-fuzzy bi-ideal f and $(\in,\in\vee q_k)$-fuzzy interior ideal g of S.*

Proof: $(i)\Rightarrow(ii)$: Let f and g be $(\in,\in\vee q_k)$-fuzzy bi-ideal and $(\in,\in\vee q_k)$-fuzzy interior of S. Since S is right weakly regular, then for each $a\in S$ there exist $x,y\in S$ such that $a=axay=(axay)(xay)=(axa)(yxay)$, so we have

$$\begin{aligned}(f\circ_k g)(a) &= (f\circ g)(a)\wedge\frac{1-k}{2}\\ &= \left(\bigvee_{a=pq}\{f(p)\wedge g(q)\}\right)\wedge\frac{1-k}{2}\\ &\geq f(axa)\wedge g(yxay)\wedge\frac{1-k}{2}\\ &\geq f(a)\wedge g(a)\wedge\frac{1-k}{2}\\ &= (f\wedge_k g)(a).\end{aligned}$$

$(ii)\Rightarrow(i)$: Let $a\in S$ and $b\in B[a]\cap I[a]$. Since $B[a]$ and $I[a]$ are bi-ideal and interior ideal of S generated by a respectively. Then $(C_{B[a]})_k$ and $(C_{I[a]})_k$ are $(\in,\in\vee q_k)$-fuzzy bi-ideal and $(\in,\in\vee q_k)$-fuzzy interior ideal of S. Then by hypothesis,

$$\begin{aligned}\frac{1-k}{2} &\leq (C_{B[a]\cap I[a]})_k(b) = (C_{B[a]})_k \wedge_k (C_{I[a]})_k)(b)\\ &\leq (C_{B[a]})_k \circ_k (C_{I[a]})_k)(b) = (C_{B[a]I[a]})_k(b).\end{aligned}$$

Therefore $b\in B[a]I[a]$. Thus $B[a]\cap I[a]\subseteq B[a]I[a]$. Hence by Theorem 2 S is right weakly regular.

Theorem 6. *For a semigroup S, the following conditions are equivalent.*

(i) *S is right weakly regular.*

(ii) *$f\wedge_k g\wedge_k h\leq f\circ_k g\circ_k h$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f, $(\in,\in\vee q_k)$-fuzzy bi-ideal g and $(\in,\in\vee q_k)$-fuzzy interior ideal h of a semigroup S.*

(iii) *$f\wedge_k g\wedge_k h\leq f\circ_k g\circ_k h$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f, $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal g and $(\in,\in\vee q_k)$-fuzzy interior ideal h of a semigroup S.*

Proof: $(i)\Rightarrow(iii)$: Let f, g and h be any $(\in,\in\vee q_k)$-fuzzy right ideal, $(\in,\in\vee q_k)$ fuzzy generalized bi-ideal and $(\in,\in\vee q_k)$-fuzzy interior ideal of S. Since S is right weakly regular therefore for each $a\in S$ there exists $x,y\in S$ such that

$$\begin{aligned}a &= axay=(axay)(xay)=(ax)(ay)(xay)\\ &= (ax)(axayy)(xay)=(ax)(axa)(yyxay).\end{aligned}$$

Then

$$\begin{aligned}&(f\circ_k g\circ_k h)(a)\\ &= (f\circ g\circ h)(a)\wedge\frac{1-k}{2}\\ &= \left(\bigvee_{a=pq}\{f(p)\wedge(g\circ h)(q)\}\right)\wedge\frac{1-k}{2}\\ &\geq f(ax)\wedge(g\circ h)((axa)(yyxay))\wedge\frac{1-k}{2}\end{aligned}$$

$$\geq f(a) \wedge \left(\bigvee_{(axa)(yyxay)=bc} \{g(b) \wedge h(c)\} \right) \wedge \frac{1-k}{2}$$

$$\geq f(a) \wedge g(axa) \wedge h(yyxay) \wedge \frac{1-k}{2}$$

$$\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}$$

$$\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}.$$

Therefore $f \wedge_k g \wedge_k h \leq f \circ_k g \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$: Let $R[a]$, $B[a]$ and $I[a]$ be right ideal, bi-ideal and interior ideal of S generated by a respectively. Then $(C_{R[a]})_k$, $(C_{B[a]})_k$ and $(C_{I[a]})_k$ be $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal and $(\in, \in \vee q_k)$-fuzzy interior ideal of semigroup S respectively. Let $a \in S$ and $b \in R[a] \cap B[a] \cap I[a]$. Then $b \in R[a]$, $b \in B[a]$ and $b \in I[a]$. Now

$$\frac{1-k}{2} \leq (C_{R[a] \cap B[a] \cap I[a]})_k(b)$$

$$= ((C_{R[a]})_k \wedge_k (C_{B[a]})_k \wedge_k (C_{I[a]})_k))(b)$$

$$\leq ((C_{R[a]})_k \circ_k (C_{B[a]})_k \circ_k (C_{I[a]})_k)(b)$$

$$= (C_{R[a]B[a]I[a]})_k(b).$$

Thus $b \in R[a]B[a]I[a]$.

Therefore $R[a] \cap B[a] \cap I[a] \subseteq R[a]B[a]I[a]$. So by Theorem 3, S is right weakly regular semigroup.

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