THE IMPACT OF REFUGE AND HARVESTING ON THE DYNAMICS OF PREY-PREDATOR SYSTEM

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ABSTRACT: *In this paper a Holling type-II prey-predator model considering Michaelis – Menten type of harvesting function is proposed and investigated. The existence, uniqueness and boundedness are discussed. The local and global stability of all possible equilibrium points are discussed. The persistence conditions of the proposed system are established. Local bifurcation analysis is carried out with the help of Sotomayor's theorem. Hopf bifurcation conditions are presented. Finally numerical simulation is done to verify our obtained analytical results.*

Keywords: Holling type-II prey-predator model; refuge; harvesting; local and global stability; bifurcation

1. INTRODUCTION

Prey–predator interaction and its outcomes are one of the most important topics that studied in ecology. The first well known classical model was given by Lotka-Volterra in 1927, this model was developed by many researchers taking into consideration number of factors affecting the prey- predator system like delayed time, competition, diseases that affect both prey and predator refuge and many other factors [1-5] and the references therein. Predator's functional response, which is defined as "the number of prey eaten per predator per unit of time", plays an important role in population dynamics [6]. In theoretical ecology, there are in general three types of functional responses viz. prey-dependent functional response, ratio-dependent functional response and predator-dependent functional response. The first one assumes that the per capita rate of predation depends on the prey numbers only, for example Lotka-Volterra type and Holling Types I–III [7]. The second one assumes that the per capita rate of predation depends on the ratio of prey numbers to predator number [8]. The third one assumes that the per capita rate predation depends on the numbers of both prey and predator, for example DeAngelis functional response [9].

It is well known that refuge and harvesting are two of the most important factors affecting the dynamics of prey-predator systems. By using refuges the prey population is partially protected against predators. The existence of refuges has great influence on the coexistence of the prey-predator systems [10-13]. Moreover the prey–predator interactions with harvesting have been extensively studied by many authors [14-20]. In fact there are different types of harvesting; the constant-effort harvesting, in which the prey is continuously being harvested at a linear function, and the other type is the nonlinear type harvesting which is more realistic from biological and economic points of view than the first type [21].

Recently, Xiao et al. [22] has been proposed and studied a ratio-dependent prey-predator model involving constant-effort harvesting. They proved that the system has different behaviors for various parameter values. On the other hand, Haque and Sarwardi [23], considered a prey-predator harvesting model with Holling type -II functional response. They assumed that the system incorporates prey refuge proportional to both the species and constant-effort harvesting.

On contrast to the type of harvesting that used in the Haque and Sarwardi model, in this paper we will proposed and stud the dynamics of Holling type-II prey-predator system incorporating constant rate of refuge and Michaelis–Menten type function of harvesting that proposed by Clark [24].

2. Mathematical Model

In this section, a Holling type-II prey-predator model considering Michaelis – Menten type of harvesting function is proposed for study. The model is also considering the effect of refuge on the dynamics of the prey-predator system. Now in order to formulate the mathematical form of the model the following hypotheses are adopted:

1. The density of the prey population at time T is denoted by $X(T)$, while the density of predator population at time T is denoted by $Y(T)$.
2. The prey population, in the absence of the predator, grows logistically while the predator decays exponentially in the absence of their prey.
3. It's assumed that there is a constant number of prey's refuge entering to the reserved area that can't attacked by predator, however the non-refuged prey will be attacked by the predator according to the Holling type-II functional response.
4. Finally the system is assumed to be harvested by an external force according to Michaelis – Menten type of harvesting function.

According to the above hypotheses, the dynamics of such prey-predator system can be represented mathematically with the following set of nonlinear differential equations.

$$\begin{aligned} \frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right) - \frac{a(1-m)XY}{b+(1-m)X} - \frac{c_1EX}{l_1E+l_2X} \\ \frac{dY}{dT} &= -dY + e\frac{a(1-m)XY}{b+(1-m)X} - \frac{c_2EY}{l_3E+l_4Y} \end{aligned} \qquad (1)$$

where $X(0) \geq 0$ and $Y(0) \geq 0$ and the parameters are described as in the table (1):

Now in order to reduce the number of parameters and specify the control set of parameters the following dimensionless variables and parameters are used:

$$\begin{aligned} &t = rT,\ x = \frac{X}{K},\ y = \frac{aY}{rK},\ w_1 = \frac{b}{K} \\ &w_2 = \frac{Ec_1}{rl_2K},\quad w_3 = \frac{El_1}{l_2K},\quad w_4 = \frac{d}{r} \\ &w_5 = \frac{ea}{r},\ w_6 = \frac{c_2aE}{r^2l_4K},\ w_7 = \frac{al_3E}{rl_4K} \end{aligned} \qquad (2)$$

Thus the dimensionless system can be written as follows

$$\begin{aligned} \frac{dx}{dt} &= x\left[1 - x - \frac{(1-m)y}{w_1+(1-m)x} - \frac{w_2}{w_3+x}\right] = f_1(x,y) \\ \frac{dy}{dt} &= y\left[\frac{w_5(1-m)x}{w_1+(1-m)x} - \frac{w_6}{w_7+y} - w_4\right] = f_2(x,y) \end{aligned} \qquad (3)$$

Table 1: The description of the model parameters

Parameter	Description
$r > 0$	intrinsic growth rate
$K > 0$	carrying capacity
$a > 0$	maximum attack rate
$b > 0$	half saturation constant
$m \in (0,1)$	prey's refuge rate, which leave $(1-m)X$ of prey available to be hunted by the predator
$e \in (0,1)$	the efficiency by which predator converts consumed prey into new predator
$d > 0$	predator death rate
$l_i; i = 1,2,3,4$	are suitable positive constants
$c_1 > 0$	catchability coefficient of the prey
$c_2 > 0$	catchability coefficient of the predator
$E > 0$	hunting effort

Clearly the system of differential equations given by (3) is continuous and has a continuous partial derivatives on the domain $R_+^2 = \{(x,y) \in R^2 : x(0) \geq 0, y(0) \geq 0\}$ and hence it has a unique solution in this domain. Farther more the solution of system (3) uniformly bounded R_+^2 as shown in the following theorem.

Theorem 1: All the solutions of system (3) that initiate in the positive quadrant are uniformly bounded.

Proof. According to the first equation of system (3), it's observed that

$$\frac{dx}{dt} \leq x[1-x]$$

Then by solving this differential inequality, it's obtain that $x(t) \leq \frac{x(0)}{e^{-t}[1-x(0)]-x(0)}$, and then for all $t \to \infty$ we have $x(t) \leq 1$. Assume $w = x + \frac{y}{w_5}$, then we get that $\frac{dw}{dt} = \frac{dx}{dt} + \frac{1}{w_5}\frac{dy}{dt}$, which gives that $\frac{dw}{dt} \leq 2 - Lw$, where $L = \min\{1, w_4\}$. Therefore by using again the Grounwall lemma for differential inequality [25], it's observed that $w(t) \leq w(0)e^{-t} - \frac{2}{L}(e^{-t} - 1)$, and hence for $t \to \infty$ we get that

$$\frac{dw}{dt} \leq \frac{2}{L}$$

Hence the proof is complete and all solutions are uniformly bounded. ■

3. Stability Analysis and Persistence

In this section the stability analysis and persistence of system (3) are investigated. It's clear that the system has at most three equilibrium points, which can be described as follow

The trivial equilibrium point that denoted by $E_0 = (0,0)$ always exists. The predator free equilibrium point that denoted by $E_1 = (\bar{x}, 0)$ where

$$\bar{x} = -\frac{w_3 - 1}{2} + \frac{1}{2}\sqrt{(w_3-1)^2 - 4(w_2 - w_3)} \quad (4)$$

Exists uniquely on the positive side of x −axis provided that one set of the following conditions hold

$$w_2 - w_3 < 0 \quad (5a)$$

$$(w_3 - 1)^2 > 4(w_2 - w_3) > 0;\ w_3 - 1 < 0 \quad (5b)$$

Finally the coexistence equilibrium point is denoted by $E_2 = (x^*, y^*)$ where

$$y^* = \frac{[w_1 + (1-m)x^*][(1-x^*)(w_3+x^*) - w_2]}{(1-m)(w_3+x^*)} \quad (6)$$

While x^* represents a unique positive root of the following fourth order polynomial equation

$$A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5 = 0 \quad (7)$$

Here we have

$$A_1 = (1-m)^2(w_5 - w_4)$$

$$A_2 = (1-m)[(1-w_3)(1-m)(w_4 - w_5) - w_1(2w_4 - w_5)]$$

$$A_3 = (1-m)[w_7(1-m)(w_4-w_5) - w_1(1-w_3)(2w_4 - w_5) + (1-m)\left((w_3 - w_2)(w_4 - w_5) + w_6 - \frac{w_1^2 w_4}{(1-m)}\right)]$$

$$A_4 = (1-m)[w_7w_3(1-m)(w_4-w_5) - w_1(w_3 - w_2)(2w_4 - w_5) + w_3w_6(1-m) + w_1w_4w_7 + \frac{w_1^2w_4}{(1-m)}(1-w_3) + w_1w_6]$$

$$A_5 = w_1[(1-m)w_3(w_4w_7 + w_6) - w_1w_4(w_3 - w_2)]$$

Straightforward computation shows that E_2 exists uniquely in the interior of positive quadrant provided that the following sufficient conditions hold

$$(1-x^*)(w_3 + x^*) > w_2 \quad (8)$$

With one of the following sets of conditions

$A_1 < 0$, $A_2 < 0$, $A_4 > 0$ and $A_5 > 0$ (9a)

$A_i < 0$; $i = 1,2,3,4$ and $A_5 > 0$ (9b)

$A_1 < 0$ and $A_i > 0$; $i = 2,3,4,5$ (9c)

$A_1 > 0$, $A_2 > 0$, $A_4 < 0$ and $A_5 < 0$ (9d)

$A_i > 0$; $i = 1,2,3,4$ and $A_5 < 0$ (9e)

$A_1 > 0$ and $A_i < 0$; $i = 2,3,4,5$ (9f)

Now in order to study the local stability of these equilibrium points the Jacobain matrix of system (3) at the point $E = (x,y)$ is computed as follows

$$J(E) = (C_{ij})_{2\times 2} \quad (10)$$

here

$$C_{11} = x\left(-1 + \frac{(1-m)^2y}{[w_1 + (1-m)x]^2} + \frac{w_2}{(w_3+x)^2}\right) + (1-x) - \frac{(1-m)y}{w_1 + (1-m)x} - \frac{w_2}{w_3 + x}$$

$$C_{12} = -\frac{(1-m)x}{w_1 + (1-m)x};\ C_{21} = \frac{w_1w_5(1-m)y}{w_1 + (1-m)x}$$

$$C_{22} = \frac{w_6 y}{(w_7+y)^2} + \frac{w_5(1-m)x}{w_1 + (1-m)x} - \frac{w_6}{w_7 + y} - w_4$$

Consequently the Jacobian matrix at trivial equilibrium point $E_0 = (0,0)$ becomes

$$J_0 = \begin{bmatrix} 1 - \frac{w_2}{w_3} & 0 \\ 0 & -w_4 - \frac{w_6}{w_7} \end{bmatrix} \quad (11)$$

Clearly the eigenvalues of J_0 are given by

$$\lambda_{01} = 1 - \frac{w_2}{w_3} \text{ and } \lambda_{02} = -w_4 - \frac{w_6}{w_7} < 0 \quad (12)$$

Therefore E_0 is locally asymptotically stable if and only if

$$w_3 < w_2 \quad (13)$$

The Jacobian matrix at the predator free equilibrium point $E_1 = (\bar{x}, 0)$ is computed by

$$J_1 = \begin{bmatrix} \bar{x}(-1 + \frac{w_2}{(w_3+\bar{x})^2}) & \frac{-(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} \\ 0 & \frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} - \frac{w_6}{w_7} - w_4 \end{bmatrix} \quad (14)$$

Then the eigenvalues are given by

$$\lambda_{11} = \bar{x}(-1 + \frac{w_2}{(w_3+\bar{x})^2});\ \lambda_{12} = \frac{w_5(1-m)\bar{x}}{w_1 + (1-m)\bar{x}} - \frac{w_6}{w_7} - w_4 \quad (15)$$

Clearly these two eigenvalues are negative and hence E_1 is locally asymptotically stable if and only if the following conditions hold

$\frac{w_2}{(w_3+\bar{x})^2} < 1$ (16a)

$\frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} < w_4 + \frac{w_6}{w_7}$ (16b)

Finally the Jacobian matrix of system (3) at the coexistence equilibrium point $E_2 = (x^*, y^*)$ is computed as follows:

$J_2 = (a_{ij})_{2\times 2}$ (17)

Here

$a_{11} = x^*\left(-1 + \frac{(1-m)^2 y^*}{[w_1+(1-m)x^*]^2} + \frac{w_2}{(w_3+x^*)^2}\right)$

$a_{12} = -\frac{(1-m)x^*}{w_1+(1-m)x^*} < 0$

$a_{21} = \frac{w_1 w_5 (1-m) y^*}{w_1+(1-m)x^*} > 0$

$a_{22} = \frac{w_6 y^*}{(w_7+y^*)^2} > 0$

Accordingly the eigenvalues of J_2 are the roots of the characteristic equation given by:

$\lambda_2{}^2 + A\lambda_2 + B = 0$ (18)

here

$A = x^* - \frac{(1-m)^2 x^* y^*}{R_1^{*2}} - \frac{w_2 x^*}{R_2^{*2}} - \frac{w_6 y^*}{R_3^{*2}}$

$B = -\frac{w_6 x^* y^*}{R_3^{*2}} + \frac{(1-m)^2 w_6 x^* y^{*2}}{R_1^{*2} R_3^{*2}} + \frac{w_2 w_6 x^* y^*}{R_2^{*2} R_3^{*2}} + \frac{(1-m)^2 w_1 w_5 x^* y^*}{R_1^{*2}}$

with $R_1^* = w_1 + (1-m)x^*$, $R_2^* = w_3 + x^*$ and $R_3^* = w_7 + y^*$. Now according to the Routh–Hurwitz criterion the equation (18) has two roots with negative real parts provided that $A > 0$ and $B > 0$. Straightforward computation shows that these two requirements are satisfied and hence $E_2 = (x^*, y^*)$ is locally asymptotically stable if and only if the following two conditions hold:

$x^* > \frac{(1-m)^2 x^* y^*}{R_1^{*2}} + \frac{w_2 x^*}{R_2^{*2}} + \frac{w_6 y^*}{R_3^{*2}}$ (19a)

$$(1-m)^2 w_6\, y^* R_2^{*2} + w_2 w_6 R_1^{*2} + (1-m)^2 w_1 w_5 R_2^{*2} R_3^{*2} > w_6 R_1^{*2} R_2^{*2} \quad (19b)$$

In the following theorem, the global stability of the coexistence equilibrium point under certain conditions is investigated with the help of suitable Lyapunov function.

Theorem 2. Let $E_2 = (x^*, y^*)$ be locally asymptotically stable then it's a globally asymptotically stable in the interior of R_+^2 provided that the following conditions are satisfied

$\frac{N^2 y^*}{R_1 R_1^*} + \frac{w_2}{R_2 R_2^*} < 1$ (20a)

$(x-x^*)^2\left[1 - \frac{N^2 y^*}{w_1 R_1^*} - \frac{w_2}{w_3 R_2^*}\right] > \frac{w_6 R_1^*}{w_7 w_1 w_5 R_3^*}(y-y^*)^2$ (20b)

here N, R_1, R_2, R_3 are given in the proof while R_1^*, R_2^* and R_3^* are given in (18).

Proof. Consider the following real valued function

$$V = c_1\left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right] + c_2\left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right]$$

where $c_i; i = 1,2$ are positive constants to be determined. Clearly the function V is positive definite so that $V(x^*, y^*) = 0$ and $V(x, y) > 0$; for all $(x, y) \neq (x^*, y^*)$. Moreover,

$$\frac{dV}{dt} = c_1(x-x^*)\left[-(x-x^*) - \frac{N}{R_1 R_1^*}[R_1^*(y-y^*) - Ny^*(x-x^*)] + \frac{w_2}{R_2 R_2^*}(x-x^*)\right] + c_2(y-y^*)\left[\frac{w_1 w_5 N}{R_1 R_1^*}(x-x^*) + \frac{w_6}{R_3 R_3^*}(y-y^*)\right]$$

here $N = 1 - m$, $R_1 = w_1 + Nx$, $R_2 = w_3 + x$ and $R_3 = w_7 + y$. Then it's obtain

$$\frac{dV}{dt} = -c_1(x-x^*)^2\left[1 - \frac{N^2 y^*}{R_1 R_1^*} - \frac{w_2}{R_2 R_2^*}\right] + \frac{N}{R_1}(x-x^*)(y-y^*)\left[c_1 - c_2\frac{w_1 w_5}{R_1^*}\right] + c_2\frac{w_6}{R_3 R_3^*}(y-y^*)^2$$

So, by choosing $c_1 = 1$ and $c_2 = \frac{R_1^*}{w_1 w_5}$ we get after some manipulation that

$$\frac{dV}{dt} \leq -(x-x^*)^2\left[1 - \frac{N^2 y^*}{w_1 R_1^*} - \frac{w_2}{w_3 R_2^*}\right] + \frac{w_6}{w_7 R_3^*}\frac{R_1^*}{w_1 w_5}(y-y^*)^2$$

So if the conditions (20a) and (20b) hold then $\frac{dV}{dt} < 0$ and hence due to Lyapunov first theorem E_2 is globally asymptotically stable in the interior of first quadrant. Hence the proof is complete. ■

Now before we go further to study the local bifurcation of the system the persistence of system (3), which indicates to coexistence of all the species in system (3) for all $t > 0$, is studied in the following theorem using the average Lyapunov function as shown in the following theorem.

Theorem 3. The system (3) is persistent provided that

$w_2 < w_3$ (21a)

$\frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} > \frac{w_6}{w_7} + w_4$ (21b)

Proof. Consider the following function

$\varphi(x, y) = x^\alpha y^\beta$,

Where α and β are positive constants, clearly $\varphi(x, y) > 0$ for all $(x, y) \in Int.\ R_+^2$ and $\varphi(x, y) \to 0$ when x or $y \to 0$. Furthermore, it's clear that

$$\frac{\varphi'}{\varphi} = \frac{\alpha}{x}\frac{dx}{dt} + \frac{\beta}{y}\frac{dy}{dt} = \alpha\left[1 - x - \frac{(1-m)y}{w_1+(1-m)x} - \frac{w_2}{w_3+x}\right] + \beta\left[\frac{w_5(1-m)x}{w_1+(1-m)x} - \frac{w_6}{w_7+y} - w_4\right]$$

Now the proof follows if $\frac{\varphi'}{\varphi} > 0$ for all the boundary equilibrium points, for suitable choice of constants $\alpha > 0$ and $\beta > 0$.

$\frac{\varphi'}{\varphi}(E_0) = \alpha\left[1 - \frac{w_2}{w_3}\right] + \beta\left[-\frac{w_6}{w_7} - w_4\right]$

$\frac{\varphi'}{\varphi}(E_1) = \beta\left[\frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} - \frac{w_6}{w_7} - w_4\right]$

Clearly $\frac{\varphi'}{\varphi}(E_0) > 0$ under condition (21a) with suitable choice of positive constants α sufficiently large with respect to the constant $\beta > 0$. While $\frac{\varphi'}{\varphi}(E_1) > 0$ under condition (21b). Hence the proof is complete. ■

4. The local bifurcation study

In the following the occurrence of local bifurcation types is investigated through applying the Sotomayor's theorem [26]. It is well known that the nonhyperbolic equilibrium point property is a necessary but not sufficient condition for the occurrence of bifurcation around that point. Accordingly, in the following theorem the candidate bifurcation parameter is selected so that the equilibrium point under study will be a nonhyperbolic point.

Theorem 4. As the parameter w_2 passes through the value of $w_2^* \equiv w_3$, the system (3), around the trivial equilibrium point, has no saddle node bifurcation. However it has a transcritical bifurcation provided that

$w_3 \neq 1$ (22)

Otherwise the system undergoes a pitchfork bifurcation.

Proof. It is easy to verify that as the parameter w_2 passes through the value of w_2^* the Jacobain matrix of system (3) at E_0 , Eq. (11), becomes

$$J_0^* = \begin{bmatrix} 0 & 0 \\ 0 & -w_4 - \frac{w_6}{w_7} \end{bmatrix}$$

Clearly J_0^* has simple zero eigenvalue $\lambda_{01} = 0$, then $E_0 = (0,0)$ is nonhyperbolic point.

Let $U_0 = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}$ and $W_0 = \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix}$ denote the eigenvectors corresponding to the simple zero eigenvalue $\lambda_{01} = 0$ of the matrix J_0^* and J_0^{*T} respectively. Then we obtain that $U_0 = \begin{bmatrix} u_{01} \\ 0 \end{bmatrix}$ while $W_0 = \begin{bmatrix} w_{01} \\ 0 \end{bmatrix}$, where $u_{01} \neq 0$ and $w_{01} \neq 0$. Moreover, by rewrite system (3) in the form $X' = F(X)$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $F(X) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ then it observed that $\frac{\partial F}{\partial w_2} = F_{w_2} = \begin{bmatrix} -\frac{x}{w_3+x} \\ 0 \end{bmatrix}$, then $F_{w_2}(E_0, w_2^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives that $W_0^T F_{w_2}(E_0, w_2^*) = 0$. Thus saddle node bifurcation cannot occur.

Now since the derivative of F_{w_2} with respect to X at (E_0, w_2^*) is computed by

$$DF_{w_2}(E_0, w_2^*) = \begin{pmatrix} -\frac{1}{w_3} & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$W_0^T\left[DF_{w_2}(E_0, w_2^*)U_0\right] = -\frac{u_{01}w_{01}}{w_3} \neq 0$$

Also, since the second derivative of F with respect to X at (E_0, w_2^*) is computed by

$$D^2F(E_0, w_2^*)(U_0, U_0) = \begin{pmatrix} -2u_{01}^2 + \frac{2}{w_3}u_{01}^2 \\ 0 \end{pmatrix}$$

Hence

$$W_0^T[D^2F(E_0, w_2^*)(U_0, U_0)] = 2u_{01}^2 w_{01}\left[-1 + \frac{1}{w_3}\right]$$

Then $W_0^T[D^2F(E_0, w_2^*)(U_0, U_0)] \neq 0$ under condition (22), and hence the system (3) experience a transcritical bifurcation at the equilibrium point E_0 with $w_2 = w_2^*$. However, if the condition (22) violates then straightforward computation shows that the third derivative of F with respect to X at (E_0, w_2^*) is computed by

$$D^3F(E_0, w_2^*)(U_0, U_0, U_0) = \begin{pmatrix} -\frac{6}{w_3^2}u_{01}^3 \\ 0 \end{pmatrix}$$

Thus

$$W_0^T[D^3F(E_0, w_2^*)(U_0, U_0, U_0)] = -\frac{6}{w_3^2}u_{01}^3 w_{01} \neq 0$$

Thus system (3) undergoes a pitchfork bifurcation. ■

Theorem 5. Assume that condition (16a) holds, then as the parameter w_4 passes through the value of $w_4^* \equiv \frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} - \frac{w_6}{w_7}$, the system (3), around the predator free equilibrium point $E_1 = (\bar{x}, 0)$, has no saddle node bifurcation. However it has a transcritical bifurcation provided that

$$\frac{w_1w_5Np_1}{(w_1+N\bar{x})^2} + \frac{w_1w_5Np_1}{w_1+N\bar{x}} + \frac{2w_6}{w_7^2} \neq 0 \qquad (23)$$

Otherwise the system undergoes a pitchfork bifurcation

Proof. It's clear that as the parameter w_4 passes through the $w_4{}^* \equiv \frac{w_5(1-m)\bar{x}}{w_1+(1-m)\bar{x}} - \frac{w_6}{w_7}$ then the equilibrium point $E_1 = (\bar{x}, 0)$ becomes a nonhyperbolic point and the Jacobain matrix of system (3) at this point will be

$$J_1^* = \begin{bmatrix} \bar{x}(-1 + \frac{w_2}{(w_3+\bar{x})^2}) & \frac{-N\bar{x}}{w_1+N\bar{x}} \\ 0 & 0 \end{bmatrix}$$

Clearly J_1^* has simple zero eigenvalue $\lambda_{12} = 0$, with the other eigenvalue given by $\lambda_{11} = \bar{x}(-1 + \frac{w_2}{(w_3+\bar{x})^2})$, which is negative due to condition (16a).

Let $U_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ and $W_1 = \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix}$ denote the eigenvectors corresponding to the simple zero eigenvalue $\lambda_{12} = 0$ of the matrix J_1^* and J_1^{*T} respectively. Then we obtain that $U_1 = \begin{bmatrix} p_1u_{12} \\ u_{12} \end{bmatrix}$, where $u_{12} \neq 0$ and $p_1 = \frac{N(w_3+\bar{x})^2}{[w_2-(w_3+\bar{x})^2][w_1+N\bar{x}]} < 0$ under condition (16a). While $W_1 = \begin{bmatrix} 0 \\ w_{12} \end{bmatrix}$, where $w_{12} \neq 0$. Moreover, by rewrite system (3) in the form $X' = F(X)$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $F(X) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ then it observed that:

$\frac{\partial F}{\partial w_4} = F_{w_4} = \begin{bmatrix} 0 \\ -y \end{bmatrix}$, then $F_{w_4}(E_1, w_4^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives that $W_1^T F_{w_4}(E_1, w_4^*) = 0$. Thus saddle node bifurcation cannot occur.

Now since the derivative of F_{w_4} with respect to X at (E_1, w_4^*) is computed by

$$DF_{w_4}(E_1, w_4^*) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$W_1^T\left[DF_{w_4}(E_1, w_4^*)U_1\right] = -w_{12}u_{12} \neq 0.$$

Also, since the second derivative of F with respect to X at (E_1, w_4^*) is computed by

$$D^2F(E_1, w_4^*)(U_1, U_1) = \begin{pmatrix} -2p_1^2u_{12}^2 + \frac{2w_2w_3p_1^2u_{12}^2}{(w_3+\bar{x})^3} - \frac{2w_1Np_1u_{12}^2}{(w_1+N\bar{x})^2} \\ \frac{w_1w_5Np_1u_{12}^2}{(w_1+N\bar{x})^2} + \frac{w_1w_5Np_1u_{12}^2}{w_1+N\bar{x}} + \frac{2w_6u_{12}^2}{w_7^2} \end{pmatrix}$$

Therefore

$$W_1^T[D^2F(E_1, w_4^*)(U_1, U_1)] = \left[\frac{w_1w_5Np_1}{(w_1+N\bar{x})^2} + \frac{w_1w_5Np_1}{w_1+N\bar{x}} + \frac{2w_6}{w_7^2}\right]u_{12}^2w_{12}$$

Then $w_1^T[D^2F(E_1, w_4^*)(U_1, U_1)] \neq 0$ under condition (23a), and hence the system (3) undergoes a transcritical bifurcation at the equilibrium point E_1 with $w_4 = w_4^*$. However, if the condition (23) violates then straightforward computation shows that the third derivative of F with respect to X at (E_1, w_4^*) is computed by

$$D^3F(E_1, w_4^*)(U_1, U_1, U_1) = \begin{pmatrix} -\frac{6w_2w_3p_1^3u_{12}^3}{(w_3+\bar{x})^4} + \frac{6w_1N^2p_1^3u_{12}^3}{(w_1+N\bar{x})^3} \\ -\frac{2w_1w_5N^2p_1^2u_{12}^3}{(w_1+N\bar{x})^3} - \frac{2w_1w_5N^2p_1^2u_{12}^3}{(w_1+N\bar{x})^2} - \frac{6w_6u_{12}^3}{w_7^3} \end{pmatrix}$$

Thus

$$W_1^T[D^3F(E_1, w_4^*)(U_1, U_1, U_1)] = -2\left[\frac{w_1w_5N^2p_1^2}{(w_1+N\bar{x})^3} + \frac{w_1w_5N^2p_1^2}{(w_1+N\bar{x})^2} + \frac{3w_6}{w_7^3}\right]u_{12}^3w_{12} \neq 0$$

Thus system (3) undergoes a pitchfork bifurcation. Hence the proof is complete. ■

Theorem 6. Assume that conditions (19a) and the following condition hold

$$-2q_1^2q_2 + \frac{2w_1N^2y^*q_1^2q_2}{R_1^{*3}} + \frac{2w_2w_3q_1^2q_2}{R_2^{*3}} - \frac{2w_1Nq_1q_2}{R_1^{*2}} - \frac{w_1w_5^*N^2y^*q_1^2}{R_1^{*2}} + \frac{w_1w_5^*Nq_1}{R_1^{*2}} + \frac{w_1w_5^*Nq_1}{R_1^*} + \frac{2w_6w_7}{R_3^{*3}} \neq 0 \qquad (24)$$

here q_1 and q_2 are given in the proof. Then as the parameter w_5 passes through the value of $w_5^* \equiv$

$\frac{R_1^{*2}}{N^2 w_1}\left[\frac{w_6}{R_3^{*2}} - \frac{N^2 w_6 y^*}{R_1^{*2} R_3^{*2}} - \frac{w_2 w_6}{R_2^{*2} R_3^{*2}}\right]$, the system (3), around the positive equilibrium point $E_2 = (x^*, y^*)$, has a saddle node bifurcation. However the transcritical and pitchfork bifurcations can't occur.

Proof. It's clear that as the parameter w_5 passes through the $w_5{}^*$ then the positive equilibrium point $E_2 = (x^*, y^*)$ becomes a nonhyperbolic point and the Jacobain matrix of system (3) at this point will be $J_2^* = (a_{ij})_{2\times 2}$, where a_{ij} as given in Eq. (17) with $a_{21} = \frac{R_1^* y^*}{N}\left[\frac{w_6}{R_3^{*2}} - \frac{N^2 w_6 y^*}{R_1^{*2} R_3^{*2}} - \frac{w_2 w_6}{R_2^{*2} R_3^{*2}}\right]$. Clearly J_2^* has simple zero eigenvalue $\lambda_{22} = 0$, with the other eigenvalue given by $\lambda_{21} = a_{11} + a_{22}$, which is negative due to condition (19a).

Let $U_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ and $W_2 = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix}$ denote the eigenvectors corresponding to the simple zero eigenvalue $\lambda_{22} = 0$ of the matrix J_2^* and J_2^{*T} respectively. Then we obtain that $U_2 = \begin{bmatrix} q_1 u_{22} \\ u_{22} \end{bmatrix}$, where $u_{22} \neq 0$ and $q_1 = -\frac{a_{12}}{a_{11}} < 0$ under condition (19a). While $W_2 = \begin{bmatrix} q_2 w_{22} \\ w_{22} \end{bmatrix}$, where $w_{22} \neq 0$ and $q_2 = -\frac{a_{21}}{a_{11}} > 0$ under condition (19a). Moreover, by rewrite system (3) in the form $X' = F(X)$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $F(X) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ then it observed that:

$\frac{\partial F}{\partial w_5} = F_{w_5} = \begin{bmatrix} 0 \\ \frac{Nxy}{w_1 + Nx} \end{bmatrix}$, then $F_{w_5}(E_2, w_5^*) = \begin{bmatrix} 0 \\ \frac{Nx^* y^*}{R_1^*} \end{bmatrix}$,

which gives that:

$$W_2^T F_{w_5}(E_2, w_5^*) = \frac{Nx^* y^*}{R_1^*} w_{22} \neq 0.$$

Thus transcritical and pitchfork bifurcations can't occur while the first condition of saddle node bifurcation is satisfied. Now since the second derivative of F with respect to X at (E_2, w_5^*) is computed by

$$D^2F(E_2, w_5^*)(U_2, U_2) = \begin{pmatrix} -2q_1^2 u_{22}^2 + \frac{2w_1 N^2 y^* q_1^2 u_{22}^2}{R_1^{*3}} + \frac{2w_2 w_3 q_1^2 u_{22}^2}{R_2^{*3}} - \frac{2w_1 N q_1 u_{22}^2}{R_1^{*2}} \\ -\frac{w_1 w_5^* N^2 y^* q_1^2 u_{22}^2}{R_1^{*2}} + \frac{w_1 w_5^* N q_1 u_{22}^2}{R_1^{*2}} + \frac{w_1 w_5^* N q_1 u_{22}^2}{R_1^*} + \frac{2w_6 w_7 u_{22}^2}{R_3^{*3}} \end{pmatrix}$$

Therefore

$$W_2^T[D^2F(E_2, w_5^*)(U_2, U_2)] = \left[-2q_1^2 q_2 + \frac{2w_1 N^2 y^* q_1^2 q_2}{R_1^{*3}} + \frac{2w_2 w_3 q_1^2 q_2}{R_2^{*3}} - \frac{2w_1 N q_1 q_2}{R_1^{*2}} - \frac{w_1 w_5^* N^2 y^* q_1^2}{R_1^{*2}} + \frac{w_1 w_5^* N q_1}{R_1^{*2}} + \frac{w_1 w_5^* N q_1}{R_1^*} + \frac{2w_6 w_7}{R_3^{*3}}\right] u_{22}^2 w_{22}$$

Clearly $W_2^T[D^2F(E_2, w_5^*)(U_2, U_2)] \neq 0$ under the condition (24). Hence the proof is complete. ■

Now, before we go further to study the global dynamics of system (3) numerically, the occurrence of Hopf bifurcation in system (3) around the positive equilibrium point is investigated. It's well known that the Hopf bifurcation refers to the local birth or death of a periodic solution from equilibrium as a parameter crosses a critical value. Consequently the planar dynamical system (3) undergoes a Hopf bifurcation around the positive equilibrium point $E_2 = (x^*, y^*)$ as a specific parameter (say w_6 as shown in the following theorem) passes through a critical value (say $w_6{}^*$) so that the eigenvalues of the linearized system about the equilibrium point E_2 be given by $\lambda(w_6) = \alpha(w_6) \pm i\beta(w_6)$ with the following two conditions are satisfied [26]:

1. $\alpha(w_6{}^*) = 0$ and $\beta(w_6{}^*) \neq 0$ (non-hyperbolicity condition)
2. $\left.\frac{d\alpha}{dw_6}\right|_{w_6 = w_6{}^*} \neq 0$ (transversality condition)

Theorem 7. Assume that conditions (19b) along with the following condition are satisfied.

$$\frac{(1-m)^2 y^*}{R_1^{*2}} + \frac{w_2}{R_2^{*2}} < 1 \quad (25)$$

Then system (3) undergoes a Hopf bifurcation around the positive equilibrium point E_2 as the parameter w_6 passes through the value $w_6{}^* = \frac{R_3^{*2} x^*}{y^*}\left[1 - \frac{(1-m)^2 y^*}{R_1^{*2}} - \frac{w_2}{R_2^{*2}}\right]$.

Proof. Clearly, due to the characteristic equation of system (3) at $E_2 = (x^*, y^*)$ that given by Eq. (18), the eigenvalues of the Jacobian matrix $J_2 = (a_{ij})_{2\times 2}$, which is given by Eq. (17) are written as $\lambda(w_6) = -\frac{A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4B} \equiv \alpha(w_6) \pm i\beta(w_6)$, where A and B are given in Eq. (18). It is easy to verify that $\alpha(w_6{}^*) = -\frac{A(w_6{}^*)}{2} = 0$. While $B(w_6{}^*) > 0$ under conditions (19b) and (25) and hence $\beta(w_6{}^*) = \sqrt{B} \neq 0$, which means the nonhyperbolicity condition holds. Moreover, since we have that $\left.\frac{d\alpha}{dw_6}\right|_{w_6 = w_6{}^*} = \frac{y^*}{2R_3^{*2}} \neq 0$, hence the second condition of Hopf bifurcation (transversality condition) is satisfied. Thus the system (3) undergoes a Hopf bifurcation and the proof is complete. ■

Note that since the form of $w_6{}^*$ in the above theorem depends on many other parameters, then the Hopf bifurcation may occurs for different parameters values too.

5. Numerical Simulation

In this section the global dynamics of system (3) is investigated numerically in order to verify the obtained analytical results in addition to specify the control set of parameters. For the following set of hypothetical parameters, system (3) is solved numerically and the obtained numerical solutions are drawn in the form of phase portrait and time series.

$$\begin{array}{c} w_1 = 0.2, w_2 = 0.5, w_3 = 0.75, w_4 = 0.01 \\ w_5 = 0.9, w_6 = 0.4, w_7 = 0.75,\ m = 0.4 \end{array} \quad (26)$$

It's observed that for the set of data (26), system (3) has a globally asymptotically stable positive equilibrium point $E_2 = (0.35, 0.13)$ starting from four different initial points as shown in the phase portrait given in Fig. (1) and time series given in Fig. (2). Clearly these two figures show the existence of a globally asymptotically stable point of system (3) as obtained analytically.

Now in order to discover the control set of parameters and specify the bifurcation point whenever it exist, the system is solved numerically with varying one parameter, in the set of data (26), each time and then the obtained solutions are drawn in number of figures as shown below.

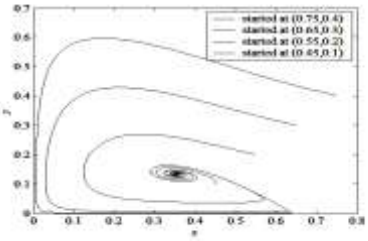


Fig.1: Globally asymptotically stable positive equilibrium point for system (3) using data (26).

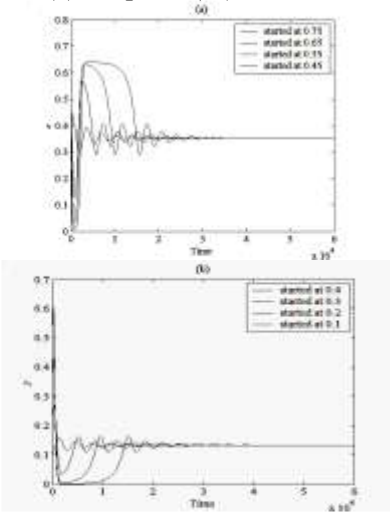


Fig. 2: Time series of the solution given in Fig. (1). (a) The solution of x as a function of time. (b) The solution of y as a function of time.

It's observed that for the set of data (26) with $0.16 < w_1 < 0.26$, system (3) approaches asymptotically to the positive equilibrium point. However, increasing the parameter w_1 in the range $w_1 \geq 0.26$ with the other parameters fixed as in (26), the solution approaches asymptotically to the predator free equilibrium point $E_1 = (0.64,0)$ as shown in Fig. (3) when $w_1 = 0.4$. While decreasing this parameter in the range $w_1 \leq 0.16$ with the other parameters as given in (26), the positive equilibrium point became unstable and the solution of system (3) approaches asymptotically to the periodic solution as shown in Fig. (4) for the typical values $w_1 = 0.15, 0.1$, which indicates to occurrence of Hopf bifurcation.

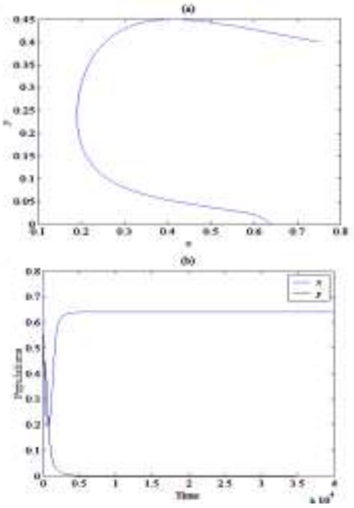


Fig. 3: (a) System (3) approaches to $E_1 = (0.64, 0)$ for $w_1 = 0.4$ with other parameters fixed as in (26). (b) Time series of the solution given by (a).

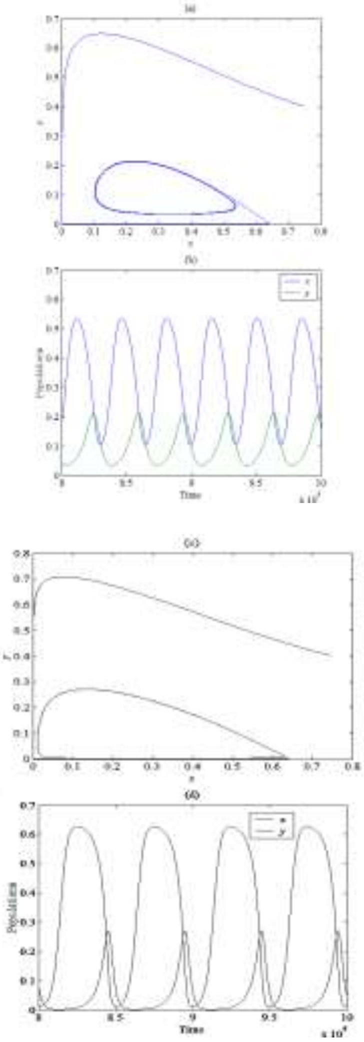


Fig. 4: (a) System (3) approaches to periodic attractor for $w_1 = 0.15$ with other parameters fixed as in (26). (b) Time series of the solution given by (a). (c) System (3) approaches to large periodic attractor for $w_1 = 0.1$ with other parameters fixed as in (26). (d) Time series of the solution given by (c).

On the other hand it's observed that for the data (26) with $0.51 < w_3 < 0.86$ the solution of system (3) still approaches to the positive equilibrium point, however increasing this value of the parameter in the range $w_3 \geq 0.86$ leads to occurrence of Hopf bifurcation and the system approaches asymptotically to periodic dynamics as shown in Fig. (5) for the values $w_3 = 0.9, 0.95$. Moreover decreasing the value of w_3 in the range $0.5 < w_3 \leq 0.51$, the solution of system (3) approaches asymptotically to predator free equilibrium point $w_3 = (0.51,0)$ as shown in the typical figure given by Fig. (6c-6d) when $w_3 = 0.51$. Finally, further decreasing in the range $w_3 < 0.5$ leads to approaching of the solution to trivial equilibrium point $E_0 = (0,0)$ as shown in the typical figure given by Fig. (6a-6b) when $w_3 = 0.4$.

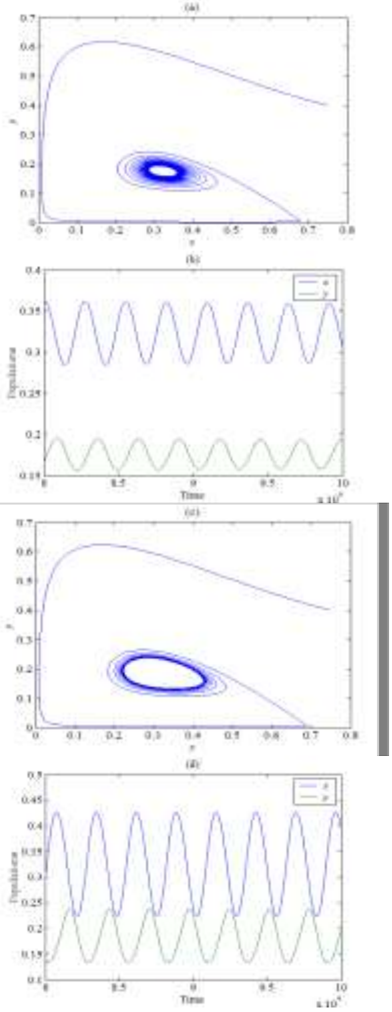
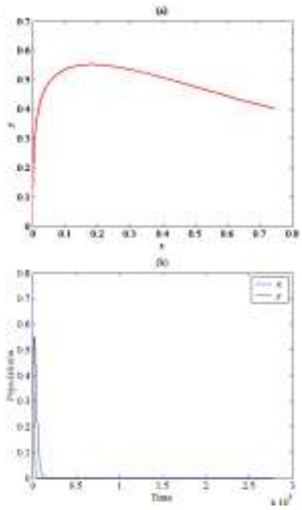


Fig. 5: (a) System (3) approaches to periodic attractor for $w_3 = 0.9$ with other parameters fixed as in (26). (b) Time series of the solution given by (a). (c) System (3) approaches to large periodic attractor for $w_3 = 0.95$ with other parameters fixed as in (26). (d) Time series of the solution given by (c).



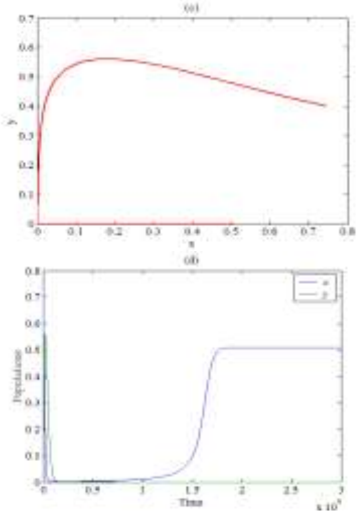


Fig. 6: (a) System (3) approaches to $E_0 = (0, 0)$ for $w_3 = 0.4$ with other parameters fixed as in (26). (b) Time series of the solution given by (a). (c) System (3) approaches to $E_1 = (0.51, 0)$ for $w_3 = 0.51$ with other parameters fixed as in (26). (d) Time series of the solution given by (c).

For the range of parameter $w_4 < 0.06$ with the rest of parameters as in (26) the solution of system (3) still approaches asymptotically to the positive equilibrium point. However increasing this parameter in the range $w_4 \geq 0.06$ leads to approaching of the system to $E_1 = (0.64,0)$ as shown in the typical figure given by Fig. (7) when $w_4 = 0.2$.

For the parameters set (26) with $0.82 < w_5 < 0.94$, it's observed that the system approaches asymptotically to the positive equilibrium point, however increasing this parameter as $w_5 \geq 0.94$ the solution will approaches asymptotically to the periodic dynamics and the Hopf bifurcation take place as shown in Fig. (8) when $w_5 = 0.95, 0.99$ respectively. Now decreasing the parameter w_5 in the range $w_5 \leq 0.82$ leads to approaching of the solution to the predator free point $E_1 = (0.64,0)$ as in Fig. (9) when $w_5 = 0.7$.

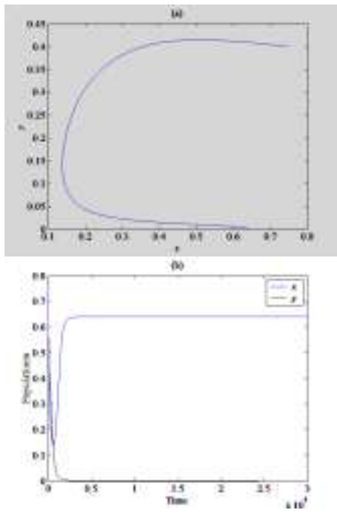


Fig. 7: (a) System (3) approaches to $E_1 = (0.64, 0)$ for $w_4 = 0.2$ with other parameters fixed as in (26). (b) Time series of the solution given by (a).

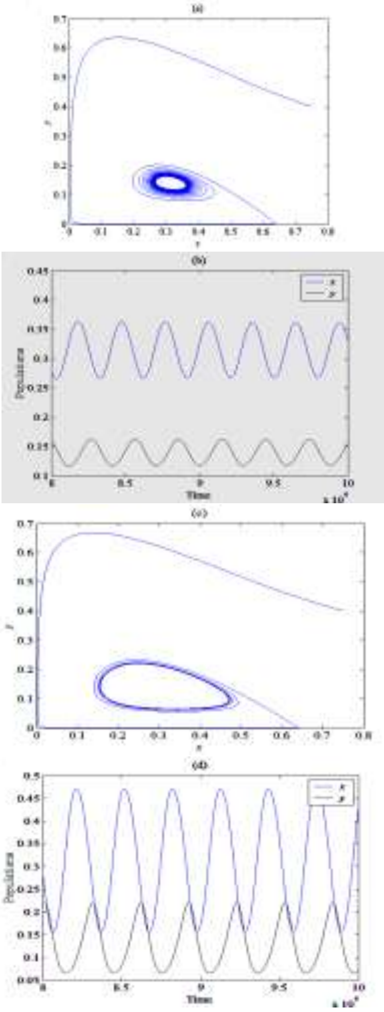


Fig. 8: (a) System (3) approaches to periodic attractor for $w_5 = 0.95$ with other parameters fixed as in (26). (b) Time series of the solution given by (a). (c) System (3) approaches to large periodic attractor for $w_5 = 0.99$ with other parameters fixed as in (26). (d) Time series of the solution given by (c).

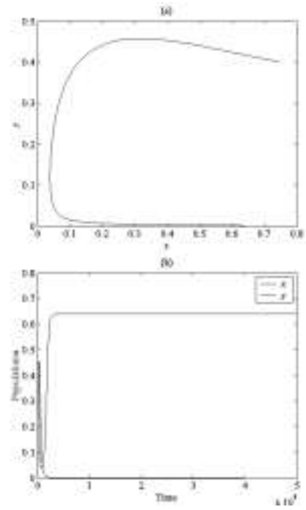


Fig. 9: (a) System (3) approaches to $E_1 = (0.64, 0)$ for $w_5 = 0.7$ with other parameters fixed as in (26). (b) Time series of the solution given by (a).

Finally, an investigation for the effect of other parameters on the dynamical behavior of system (3) is also studied and the obtained results were similar to those given above and hence they summarized in the following table.

Table 2: The dynamical behavior of system (3) as a parameter varying

Parameter varying		Dynamical behavior
w_2	$w_2 \geq 0.62$	E_1 is asymptotically stable
	$0.47 < w_2 < 0.62$	E_2 is asymptotically stable
	$w_2 \leq 0.47$	Periodic dynamic
w_6	$w_6 \geq 0.45$	E_1 is asymptotically stable
	$0.38 < w_6 < 0.45$	E_2 is asymptotically stable
	$w_6 \leq 0.38$	Periodic dynamic
w_7	$w_7 \geq 0.8$	Periodic dynamic
	$0.63 < w_7 < 0.8$	E_2 is asymptotically stable
	$w_7 \leq 0.63$	E_1 is asymptotically stable
m	$m \geq 0.57$	E_1 is asymptotically stable
	$0.33 < m < 0.57$	E_2 is asymptotically stable
	$m \leq 0.33$	Periodic dynamic

6. DISCUSSION AND CONCLUSIONS

In this paper, an ecological system consisting of Holling type-II prey-predator model with a Michaelis–Menten type of harvesting function is proposed and study. The model is assumed to be considering the effect of prey refuge on the dynamics of the prey-predator system. The existence, uniqueness and boundedness of the solution of the proposed model are discussed. All possible equilibrium points with their local stability conditions are obtained. Suitable Lyapunov functions are used to investigate the global dynamics of the equilibrium points. The persistence of the system is investigated with the help of average Lyapunov method. The Local bifurcation analysis around these equilibrium points is carried out depending on the Sotomayor's theorem. Finally the occurrence of Hopf bifurcation around the positive equilibrium point is also investigated. For the suitable set of biologically feasible hypothetical data, the proposed system is solved numerically in order to verify the obtained analytical results and specify the control set of parameters too. The obtained numerical results depending on the data given by (26) can be summarized as follows.

1. Increasing the parameter w_1 that related to the half saturation constant above a critical value (bifurcation point) leads to destabilizing the positive equilibrium point and the system approaches asymptotically to the predator free equilibrium point. However decreasing this parameter leads to persistence of the system in the form of periodic dynamics and a Hopf bifurcation takes place.
2. Increasing the parameter related to the hunting effort (w_3) above a specific value (bifurcation point) destabilize the positive equilibrium point, but the system still persist in the form of periodic dynamics representing by occurrence of Hopf bifurcation. However decreasing this parameter under a specific value leads to losing the persistence of the system and the system approaches asymptotically to the predator free equilibrium point. Further decreasing this parameter so that it becomes below the parameter related to product of catchability coefficient of prey and hunting effort (w_2) cases extinction in both the species and the system approaches to the trivial point.
3. Increasing the natural death rate of the predator species (w_4) leads to losing the persistence of the system and

the solution approaches asymptotically to the predator free equilibrium point.

4. Increasing the parameter related to the conversion rate (w_5) above a specific value (bifurcation point) destabilize the positive equilibrium point, but the system still persist in the form of periodic dynamics representing by occurrence of Hopf bifurcation. However decreasing this parameter under a specific value leads to losing the persistence of the system and the system approaches asymptotically to the predator free equilibrium point.
5. Finally it's observed that the parameters w_2, w_6 and m have similar effect on the dynamics of the system (3) as that shown by the parameter w_1. While the effect of the varying the parameter w_7 on the dynamics of system (3) is similar to that given by w_5.

According to the numerical results described above all the obtained analytical results are verified and they coincide with the real life in the environment.

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