FUZZY DIFFERENTIAL SUBORDINATION PROPERTIES OF ANALYTIC FUNCTIONS INVOLVING GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT: *In this paper, by making use of fuzzy differential subordination results of G. I. Oros and Gh. Oros [5,6], we study certain suitable classes of admissible functions and investigate properties of analytic functions in the open unit disk involving generalized differential operator.*

KEYWORDS. Fuzzy set, Fuzzy differential subordination, Analytic functions, Admissible functions, Fuzzy best dominant.

1. INTRODUCTION

Let $\mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$.

For a positive integer number n and $a \in \mathbb{C}$, we denote by

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U): f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(U): f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots, z \in U\}.$$

Let $\mathcal{A}$ denote the class of functions f that are analytic in U and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).$$

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $\alpha, \delta \ge 0, \mu, \lambda, \beta > 0$ and $\alpha \ne \lambda$, we consider the differential operator $A^m_{\mu,\lambda,\delta}(\alpha,\beta) : \mathcal{A} \longrightarrow \mathcal{A}$ was introduced by Amourah and Darus [2], where

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = z + \sum_{n=2}^{\infty} \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta + n\delta]}{\mu+\lambda}\right]^m a_n z^n . \quad (1.1)$$

It is readily verified from (1.1) that

$$\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta} A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right) A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta} \times \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta+n\delta]}{\mu+\lambda}\right]^{m+1} a_n z^n$$

$$+ \sum_{n=2}^{\infty} \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right) \times \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta+n\delta]}{\mu+\lambda}\right]^m a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta} \frac{\mu+\lambda+(n-1)[(\lambda-\alpha)\beta+n\delta]}{\mu+\lambda} + \frac{(\lambda-\alpha)\beta+n\delta-\mu-\lambda}{(\lambda-\alpha)\beta+n\delta}\right) \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta+n\delta]}{\mu+\lambda}\right]^m a_n z^n$$

$$= z + \sum_{n=2}^{\infty} n \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta+n\delta]}{\mu+\lambda}\right]^m a_n z^n .$$

Thus, we get

$$z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)' = \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta} A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right) A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z). \quad (1.2)$$

We may point out here that some of the special cases of the operator defined by (1.1) can be found in [1,3,7,8].

Definition (1.1): [4] Denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\xi \in \partial U : \lim_{z\to\xi} q(z) = \infty\right\}$$

and are such that $q'(\xi) \ne 0$ for $\xi \in \partial U \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a), Q(0) = Q_0$ and $Q(1) = Q_1$.

Definition (1.2): [9] Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and

$$A = \{x \in X : 0 < F_A(x) \le 1\} = supp(A, F_A)$$

is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition (1.3): [5] Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x), x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Definition (1.4): [5] Let $D \subseteq \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1- $f(z_0) = g(z_0)$,
2- $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), z \in D$.

where

$$f(D) = supp(f(D), F_{f(D)})$$

$$= \{f(z) : 0 < F_{f(D)}(f(z)) \leq 1, z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)})$$

$$= \{g(z) : 0 < F_{g(D)}(g(z)) \leq 1, z \in D\}.$$

Definition (1.5): [6] Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination:

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z)) \leq F_{h(U)}(h(z)), \quad (1.3)$$

i.e.

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), z \in U,$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_F q(z), z \in U$ for all p satisfying (1.3). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in U$ for all fuzzy dominant q of (1.3) is said to be the fuzzy best dominant of (1.3).

Definition (1.6): [6] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$F_\Omega(\psi(r, s, t; z)) = 0,$$

whenever

$$r = q(\xi), \ s = k\xi q'(\xi)$$

and

$$Re\left\{\frac{t}{s} + 1\right\} \geq k\, Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Lemma (1.7): [6] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z)) \leq F_\Omega(z), \ z \in U,$$

then

$$F_{p(U)}(p(z)) \leq F_{q(U)}(q(z))$$

i.e. $p(z) \prec_F q(z), \ z \in U$.

2. MAIN RESULTS

Definition (2.1): Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mathcal{H}$. The class of admissible functions $\Phi_A[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $F_\Omega(\phi(u, v, w; z)) = 0$, whenever

$$u = q(\xi), \ v = \frac{k\xi q'(\xi) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)q(\xi)}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}}$$

and

$$Re\left\{\frac{(\mu+\lambda)^2 w + ((\lambda-\alpha)\beta+n\delta-\mu-\lambda)^2 u}{((\lambda-\alpha)\beta+n\delta)[(\mu+\lambda)(v-u) + ((\lambda-\alpha)\beta+n\delta)u]} + 2\left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)\right\} \geq k\, Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem (2.2): Let $\phi \in \Phi_A[\Omega, q]$. If $f \in \mathcal{A}$ satisfies

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z)\right) \leq F_\Omega(z), \quad (2.1)$$

then

$$F_{(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q(U)}(q(z))$$

i.e.

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

Proof: Assume that

$$p(z) = A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) . \quad (2.2)$$

In view of the relation (1.2), it follows from (2.2) that

$$A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = \frac{zp'(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)p(z)}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}}. \quad (2.3)$$

Further computations show that

$$A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = \frac{z^2p''(z) - \left(1 - \frac{2(\mu+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)zp'(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2 p(z)}{\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2}. \quad (2.4)$$

Now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{s - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)r}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}},$$

$$w = \frac{t - \left(1 - \frac{2(\mu+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)s - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2 r}{\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2}. \quad (2.5)$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)r}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}}, \frac{t - \left(1 - \frac{2(\mu+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)s - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2 r}{\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2}; z\right). \quad (2.6)$$

The proof shall make use of Lemma (1.7) Using equations (2.2), (2.3) and (2.4), it follows from (2.6) that

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right).$$

Therefore, by making use (2.1), we obtain

$$F_{\psi(\mathbb{C}^3\times U)}\left(\psi(p(z), zp'(z), z^2p''(z); z)\right) \le F_{\Omega}(z).$$

A computation using (2.5) yields

$$\frac{t}{s} + 1 = \frac{(\mu+\lambda)^2 w + \left((\lambda-\alpha)\beta + n\delta - \mu - \lambda\right)^2 u}{\left((\lambda-\alpha)\beta+n\delta\right)\left[(\mu+\lambda)(v-u) + \left((\lambda-\alpha)\beta+n\delta\right)u\right]} + 2\left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right).$$

Thus the admissibility condition for $\phi \in \Phi_A[\Omega, q]$ in Definition (2.1) is equivalent to the admissibility condition for ψ as given in Definition (1.6). Hence $\psi \in \Psi[\Omega, q]$ and by Lemma (1.7),

$$F_{p(U)}\left(p(z)\right) \le F_{q(U)}\left(q(z)\right),$$

or equivalently,

$$F_{\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \le F_{q(U)}\left(q(z)\right)$$

i.e.

$$A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

We consider the special situation when $\Omega \ne \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω and the class $\Phi_A[h(U), q]$ is written as $\Phi_A[h, q]$. The following result is an immediate consequence of Theorem (2.2).

Theorem (2.3): Let $\phi \in \Phi_A[h, q]$. If $f \in \mathcal{A}$, $\phi\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right)$ is analytic in U and

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right)\right) \le F_{h(U)}\left(h(z)\right), \quad (2.7)$$

then

$$F_{\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \le F_{q(U)}\left(q(z)\right)$$

i.e.

$$A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

By taking $\phi(u,v,w;z) = 1 + \frac{v}{u}$ in Theorem (2.3), we obtain the following corollary :

Corollary (2.4): Let $\phi \in \Phi_A[h, q]$. If $f \in \mathcal{A}$, $2 + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda}\left(\frac{z\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} - 1\right)$ is analytic in U and

$$2 + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda}\left(\frac{z\left(A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} - 1\right) \prec_F h(z),$$

then

$$A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

Our result is an extension of Theorem (2.2) to the case in which the behavior of q on the boundary of U is unknown.

Corollary (2.5): Let $\Omega \in \mathbb{C}$ and q be univalent in U with $q(0) = 0$. Let $\phi \in \Phi_A[h, q_\rho]$ for some $\in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right)\right) \leq F_\Omega(z),$$

then

$$F_{\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q(U)}\left(q(z)\right)$$

i.e.

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

Proof: From Theorem (2.1), we obtain

$$F_{\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q_\rho(U)}\left(q_\rho(z)\right).$$

Since $q_\rho(z) = q(\rho z)$, we have $F_{q_\rho(U)}\left(q_\rho(z)\right) = F_{q(\rho U)}\left(q(\rho z)\right)$ and $q_\rho(0) = q(0)$. Hence,

$$F_{\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q(\rho U)}\left(q(\rho z)\right)$$

i.e.

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(\rho z).$$

By letting $\rho \longrightarrow 1$, we obtain $A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z)$.

Theorem (2.6): Let h and q be univalent in U with $q(0) = 0$ and set $h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ satisfy one of the following conditions:

1- $\phi \in \Phi_A[h, q_\rho]$ for some $\rho \in (0,1)$,
2- there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_A[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$,

$$\phi\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right)$$

is analytic in U and if f satisfies (2.7), then

$$F_{\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q(U)}\left(q(z)\right)$$

i.e.

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

Proof:

Case (1): The proof is similar to that of Corollary (2.2) and hence we omit it.

Case (2): Let $p(z) = A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)$ and $p_\rho(z) = p(\rho z)$. Then

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z\right)\right)$$
$$= F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(p(\rho z), zp'(\rho z), z^2p''(\rho z); \rho z\right)\right)$$
$$\leq F_{h_\rho(U)}\left(h_\rho(z)\right).$$

By using Theorem (2.2) and the comment associated with

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(p(z), zp'(z), z^2p''(z); w(z)\right)\right) \leq F_\Omega(z),$$

where w is any function mapping U into U, with $w(z) = \rho z$, we obtain $p_\rho(z) \prec_F q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \longrightarrow 1$, we get $p(z) \prec_F q(z)$.

Therefore

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z).$$

Our next theorem yields the best dominant of the fuzzy differential subordination (2.7).

Theorem (2.7): Let h be univalent in U and $\phi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. Suppose that the differential equation

$$\phi\left(q(z), \frac{zq'(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)q(z)}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}}, \frac{z^2q''(z) - \left(1 - \frac{2(\mu+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)zq'(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2 q(z)}{\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2}; z\right)$$

$$= h(z) \qquad (2.8)$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

1- $q \in Q_0$ and $\phi \in \Phi_A[h, q]$,
2- q is univalent in U and $\phi \in \Phi_A[h, q_\rho]$ for some $\rho \in (0,1)$,
3- q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_A[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$,

$$\phi\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z), A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z); z\right)$$

is analytic in U and if f satisfies (2.7), then

$$F_{\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f\right)(U)}\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right) \leq F_{q(U)}\left(q(z)\right)$$

i.e.

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) \prec_F q(z)$$

and q is the fuzzy best dominant.

Proof: By using Theorem (2.3) and Theorem (2.6), we deduce that q is a fuzzy dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7) and hence q will be dominated by all fuzzy dominants of (2.7). Therefore q is the fuzzy best dominant of (2.7).

Definition (2.8): Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{A^*}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $F_\Omega\big(\phi(u, v, w; z)\big) = 0$, whenever

$$u = q(\xi),\ \ v = q(\xi) + \frac{[(\lambda-\alpha)\beta + n\delta]k\xi q'(\xi)}{(\mu+\lambda)q(\xi)}$$

and

$$Re\left\{\frac{(\mu+\lambda)[vw - u(3v-2u)]}{(v-u)[((\lambda-\alpha)\beta+n\delta)]}\right\} \geq k\ Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem (2.9): Let $\phi \in \Phi_{A^*}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}, \frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}, \frac{A^{m+3}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}; z\right)\right) \leq F_\Omega(z), \tag{2.9}$$

then

$$F_{\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f}\right)(U)}\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\right) \leq F_{q(U)}\big(q(z)\big)$$

i.e.

$$\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} \prec_F q(z).$$

Proof: Assume that

$$p(z) = \frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}. \tag{2.10}$$

By making use of (1.2)and (2.10), we have

$$\frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} = p(z) + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda}\,\frac{zp'(z)}{p(z)}. \tag{2.11}$$

Further computations show that

$$\frac{A^{m+3}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} = p(z) + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda} \times \left[\frac{zp'(z)}{p(z)} + \frac{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}zp'(z) + \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2}{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}p(z) + \frac{zp'(z)}{p(z)}}\right]. \tag{2.12}$$

Now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\qquad v = r + \frac{[(\lambda-\alpha)\beta+n\delta]s}{(\mu+\lambda)r},$$

$$w = r + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda} \times \left[\frac{s}{r} + \frac{\frac{(\mu+\lambda)s}{(\lambda-\alpha)\beta+n\delta} + \frac{s+t}{r} - \left(\frac{s}{r}\right)^2}{\frac{(\mu+\lambda)r}{(\lambda-\alpha)\beta+n\delta} + \frac{s}{r}}\right].$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

$$= \phi\left(r, r + \frac{[(\lambda-\alpha)\beta+n\delta]s}{(\mu+\lambda)r}, r + \frac{(\lambda-\alpha)\beta+n\delta}{\mu+\lambda} \times \left[\frac{s}{r} + \frac{\frac{(\mu+\lambda)s}{(\lambda-\alpha)\beta+n\delta} + \frac{s+t}{r} - \left(\frac{s}{r}\right)^2}{\frac{(\mu+\lambda)r}{(\lambda-\alpha)\beta+n\delta} + \frac{s}{r}}\right]; z\right). \tag{2.13}$$

Using equations (2.10), (2.11) and (2.12), it follows from (2.13) that

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}, \frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}, \frac{A^{m+3}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}; z\right).$$

Hence (2.9) becomes

$$F_{\psi(\mathbb{C}^3\times U)}\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \leq F_\Omega(z).$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{A^*}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition (1.6). For this purpose, note that

$$\frac{s}{r} = \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}(v-u),$$

$$\frac{t}{r} = v(w-v)\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2 - \frac{s}{r}\left(\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\delta}v - \frac{2s}{r} + 1\right)$$

and $$\frac{t}{s}+1=\frac{(\mu+\lambda)[vw-u(3v-2u)]}{(v-u)\left[\left((\lambda-\alpha)\beta+n\delta\right)\right]}.$$

Hence $\psi\in\Psi[\Omega,q]$ and by Lemma (1.7),

$$F_{p(U)}\big(p(z)\big)\le F_{q(U)}\big(q(z)\big),$$

or equivalently,

$$F_{\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f}\right)(U)}\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\right)\le F_{q(U)}\big(q(z)\big)$$

i.e.

$$\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\prec_F q(z).$$

In the case $\Omega\neq\mathbb{C}$ is a simply connected domain with $\Omega=h(U)$ for some conformal mapping h of U onto Ω, the class $\Phi_{A^*}[h(U),q]$ is written as $\Phi_{A^*}[h,q]$. The following result is an immediate consequence of Theorem (2.9).

Theorem (2.10): Let $\phi\in\Phi_{A^*}[h,q]$. If $f\in\mathcal{A}$, $\phi\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)},\frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)},\frac{A^{m+3}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)};z\right)$ is analytic in U and

$$F_{\phi(\mathbb{C}^3\times U)}\left(\phi\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)},\frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)},\frac{A^{m+3}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)};z\right)\right)$$
$$\le F_{h(U)}\big(h(z)\big),$$

then

$$F_{\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f}\right)(U)}\left(\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\right)\le F_{q(U)}\big(q(z)\big)$$

i.e.

$$\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\prec_F q(z).$$

By taking $\phi(u,v,w;z)=uv$ in Theorem (2.10), we obtain the following corollary:

Corollary (2.3): Let $\phi\in\Phi_{A^*}[h,q]$. If $f\in\mathcal{A}$, $\frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}$ is analytic in U and

$$\frac{A^{m+2}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\prec_F h(z),$$

then

$$\frac{A^{m+1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{A^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}\prec_F q(z).$$

REFERENCES

[1] F. M. Al-Oboudi, "On univalent functions defined by a generalized Salagean operator", Int. J. Math. Math. Sci., **27**, 1429–1436 (2004).

[2] A. Amourah and M. Darus, "Some properties of a new class of univalent functions involving a new generalized differential operator with negative coefficients", Indian J. Sci. Tech., **9**(36), 1–7 (2016).

[3] M. Darus and R. W. Ibrahim, "On subclasses for generalized operators of complex order", Far East J. Math. Sci., **33**(3), 299–308 (2009).

[4] S. S. Miller and P. T. Mocanu, "Differential Subordinations: Theory and Applications", Series on Monographs and Textbooks in Pure and Applied Mathematics Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[5] G. I. Oros and Gh. Oros, "The notion of subordination in fuzzy set theory", General Mathematics, **19**(4), 97-103 (2011).

[6] G. I. Oros and Gh. Oros, "Fuzzy differential subordination", Acta Universitatis Apulensis, **30**, 55-64 (2012).

[7] G. S. Salagean, "Subclasses of univalent functions", Lecture Notes in Math., Springer Verlag, Berlin, **1013**, 362-372 (1983).

[8] S. R. Swamy, "Inclusion properties of certain subclasses of analytic functions", Int. Math. Forum Universitatis Apulensis, **7**(36), 1751–1760 (2012).

[9] L. A. Zadeh, "Fuzzy sets", Information and control, **8**, 338-353 (1965).