A QUARTIC B-SPLINE COLLOCATION TECHNIQUE FOR THE SOLUTION OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH A WEAKLY SINGULAR KERNEL

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ABSTRACT: *A quartic B-spline collocation method is proposed for numerical solution of partial integro-differential equations with a weakly singular kernel. The scheme is developed by discretization of the time derivative using backward difference formula while the spatial derivatives are approximated using quartic B-spline functions. Numerical solution of five test problems is provided to validate the scheme. Accuracy of the method is assessed in terms of L_∞ and L_2 error norms. Excellent accuracy is obtained and the results are compared with cubic B-spline collocation method.*

Keywords: Collocation method, Quartic B-Spline function, Partial integro-differential equation, Weakly singular kernel.

1. INTRODUCTION

Partial integro-differential equations with weakly singular kernels are widely used in various fields of science and engineering such as fluid dynamics, geophysics, plasma physics, viscoelastic mechanics, civil and aerospace engineering [1-7]. This paper is concerned with the solution of a parabolic type partial integro-differential equation (PIDE) having a weakly singular kernel. Several methods have been introduced in the literature for the approximate solution of such equations (see [5, 7-13]). Yanik and Fairweather [13] used finite element methods to obtain the solution of parabolic type integro-differential equations. Le Roux and Thomee [12] solved parabolic PIDEs using finite element methods with non-smooth data. Lin and Zhang [9] used finite element methods for the stability of Ritz-Volterra projections and error estimates for a class of integro-differential equations. Pani et al. [11] obtained the numerical solutions of parabolic and hyperbolic integro-differential equations using energy methods. Tang [5] solved a second order PIDE by finite difference method. Lopez-Marcos [8] solved a nonlinear PIDE by a finite difference technique. Dehghan et al. [8, 15] obtained the numerical solution of one-dimensional and three-dimensional advection-diffusion equations using fully explicit, fully implicit and weighted finite difference methods. Nawaz et al. [10] used variational iteration method and Homotophy perturbation method for the solution of fourth-order fractional integro-differential equation. Yang et al. [7, 16] used quasi-wavelet based numerical methods for approximate solution of fourth-order partial integro-differential equations with weakly singular kernel.

In this paper we use quartic B-spline collocation method for the numerical solution of a class of partial integro-differential equations with a weakly singular kernel. Spline was introduced by Schoenberg [17] in 1946 for smooth, piecewise polynomial approximation with applications in the aircraft and shipbuilding industries. Isaac Jacob Schoenberg used the term B-spline as short for basis spline. In 1975, Prenter [18] provided a theoretical background for stable computation using polynomial B-splines and their derivatives. The main feature of a B-spline function is its minimal support with respect to a given degree, smoothness, and domain partition [18-19]. This feature leads to a band structure of matrices which appears in interpolation and collocation problems using B-splines. Furthermore some of the characteristics of B-splines give total positivity. As a result, Gaussian elimination without pivoting using computers can be applied for invertibility of the matrices obtained from B-splines methods. Recently, the B-spline techniques are extended for the solution of several partial differential equations and PIDEs including regularized long wave (RLW) equation [20], equal width wave (EW) equation [21], modified regularized long wave (MRLW) equation [22], fourth order PIDE [23], parabolic type PIDE [24] and PIDEs using sixth-degree B-spline function technique [25].

2. CONSTRUCTION OF PROPOSED METHOD

Consider the following partial integro- differential equation [4]:

$$u_t(x,t) + mu_x(x,t) - wu_{xx}(x,t) = \int_0^t K(t-s)u(x,s)ds + f(x,t),\ x \in [a,b],\ t > 0, \tag{1}$$

subject to the initial condition:

$$u(x,0) = g(x), \quad x \in [a,b], \tag{2}$$

and boundary conditions:

$$u(a,t) = \beta_1(t),$$
$$u_x(a,t) = \beta_2(t), \tag{3}$$
$$u(b,t) = \beta_3(t),\ t > 0.$$

Eq. (1) is used to describe convection-diffusion phenomena. The coefficients m and w are positive constants which are used to quantify the advection (convection) and diffusion process, the integral term is called memory and the weakly singular kernel is given by

$K(t-s) = (t-s)^{-\alpha},\ 0 < \alpha < 1.$

The time derivative in Eq. (1) is discretized by first order forward difference formula at time level $n+1$ and quartic B-splines will be used to approximate the space derivatives. Let $t^n = n\delta t$, where δt is the time step and $t^{n+1} = t^n + \delta t, n = 0,1,2 \ldots$ Approximation of time derivative in Eq. (1) yields

$$u_t(x, t^{n+1}) \approx \frac{u(x,\ t^{n+1}) - u(x,\ t^n)}{\delta t} \tag{4}$$

Substituting Eq. (4) in Eq. (1) we get

$$\frac{u(x,\ t^{n+1}) - u(x, t^n)}{\delta t} + mu_x(x, t^{n+1}) - wu_{xx}(x, t^{n+1}) = \int_0^{t^{n+1}} K(t^{n+1} - s)u(x,s)ds + f(x, t^{n+1}). \tag{5}$$

The integral part in Eq. (5) is evaluated as follows [4-5]:

$$\int_0^{t^{n+1}} K(t^{n+1}-s)u(x,s)ds$$
$$= \int_0^{t^{n+1}} s^{-\alpha}\, u(x,t^{n+1}-s)ds$$
$$=\sum_{j=0}^{n}\int_{t^j}^{t^{j+1}} s^{-\alpha} u(x,t^{n+1}-s)ds$$
$$\approx \sum_{j=0}^{n} u(x,t^{n-j+1})\int_{t^j}^{t^{j+1}} s^{-\alpha}\, ds. \quad (6)$$

Substituting Eq. (6) in Eq. (5), we obtain

$$\frac{u(x,t^{n+1})-u(x,t^n)}{\delta t} + mu_x(x,t^{n+1}) - wu_{xx}(x,t^{n+1}) = \frac{(\delta t)^{1-\alpha}}{1-\alpha}\sum_{j=0}^{n} u(x,t^{n-j+1})[(j+1)^{1-\alpha}-j^{1-\alpha}] + f(x,t^{n+1}). \quad (7)$$

Eq. (7) can be rearranged as

$$u^{n+1}(x) + m\delta t {u_x}^{n+1}(x) - w\delta t {u_{xx}}^{n+1}(x) - \frac{(\delta t)^{2-\alpha}}{1-\alpha}u^{n+1}(x) = u^n(x) + \frac{(\delta t)^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n} b_j u^{n-j+1}(x) + \delta t f^{n+1}(x), \quad n \geq 1, \quad (8)$$

where $u^{n+1}(x) = u(x,t^{n+1})$, $f^{n+1}(x) = f(x,t^{n+1})$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 1,2,3,\ldots$

To find the numerical solution of problem (1)-(3) using the scheme (8), the interval $[a,b]$ is partitioned into N finite elements of consistently equal length h by the grid points x_i, $i = 1,2,3,\ldots N$, such that $a = x_0 < x_1 < x_2 \ldots\ldots < x_N = b$ and $h = \frac{b-a}{N}$. The Quartic B-splines function $B_i(x), i = -2,-1,0,\ldots N+1$ at these knots are given by

$$B_i(x) = \frac{1}{h^4}\begin{cases} a_1 = (x-x_{i-2})^4, & x \in [x_{i-2},x_{i-1}], \\ a_2 = a_1 - 5(x-x_{i-1})^4, & x \in [x_{i-1},x_i], \\ a_3 = a_2 + 10(x-x_i)^4, & x \in [x_i,x_{i+1}], \\ (x_{i+3}-x)^4 - 5(x_{i+2}-x)^4, & x \in [x_{i+1},x_{i+2}], \\ (x_{i+3}-x)^4, & x \in [x_{i+2},x_{i+3}], \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

The set of quartic B-splines $\{B_{-2}, B_{-1}, B_0, \ldots, B_N, B_{N+1}\}$ forms a basis for the functions over the interval $[a,b]$. A global approximation $U(x,t)$ to the exact solution $u(x,t)$ takes the form

$$U(x,t) = \sum_{i=-2}^{N+1}\gamma_i(t)B_i(x), \quad (10)$$

where $\gamma_i(t)$ are unknown time dependent quantities to be determined from collocation, boundary and initial conditions. The nodal values U_i, the first and second derivatives U_i' and U_i'' at the knots are obtained from Eqs. (9) and (10) in the following form

$$\left.\begin{aligned} U_i &= U(x_i) = \gamma_{i-2} + 11\gamma_{i-1} + 11\gamma_i + \gamma_{i+1}, \\ U_i' &= U'(x_i) = \frac{4}{h}(-\gamma_{i-2} - 3\gamma_{i-1} + 3\gamma_i + \gamma_{i+1}), \\ U_i'' &= U''(x_i) = \frac{12}{h^2}(\gamma_{i-2} - \gamma_{i-1} - \gamma_i + \gamma_{i+1}), \end{aligned}\right\} \quad (11)$$

where dashes represent differentiation with respect to space variable. To develop the general scheme consider Eq. (8) and substituting the approximate values of $u^{n+1}(x), {u_x}^{n+1}(x)$and ${u_{xx}}^{n+1}(x)$ at $x = x_i$, $i = 0,1,2,\ldots,N$, using Eq. (11), we get

$$C_1\gamma_{i-2}^{n+1} + C_2\gamma_{i-1}^{n+1} + C_3\gamma_i^{n+1} + C_4\gamma_{i+1}^{n+1} = F_i, \quad (12)$$

where

$$C_1 = 1 - \frac{4}{h}m\delta t - \frac{12}{h^2}w\delta t - \frac{(\delta t)^{2-\alpha}}{1-\alpha},$$
$$C_2 = 11 - \frac{12}{h}m\delta t + \frac{12}{h^2}w\delta t - 11\frac{(\delta t)^{2-\alpha}}{1-\alpha},$$
$$C_3 = 11 + \frac{12}{h}m\delta t + \frac{12}{h^2}w\delta t - \frac{(\delta t)^{2-\alpha}}{1-\alpha},$$
$$C_4 = 1 + \frac{4}{h}m\delta t - \frac{12}{h^2}w\delta t - \frac{(\delta t)^{2-\alpha}}{1-\alpha},$$
$$F_i = u^n(x_i) + \frac{(\delta t)^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n} b_j u^{n-j+1}(x_i) + \delta t f^{n+1}(x_i).$$

Eq. (12) represent a system of $N+1$ equations in $N+4$ unknowns $\gamma_i, i = -2,-1,\ldots,N+1$. In order to get a unique solution; we eliminate the parameters $\{\gamma_{-2},\gamma_{-1},\gamma_{N+1}\}$. The values of these parameters can be calculated from the boundary conditions (3) and (11). The linear system can be solved by a four-diagonal solver successively once initial time solution is obtained from Eqs. (2)-(3). Finally the the approximate solution will be obtained from Eq. (10).

3. NUMERICAL TESTS AND PROBLEMS

In this section we present some examples in order to test the scheme (12) for the solution of the problem defined in Eqs. (1)-(3). All the computations are carried out through Matlab using Intel Core i5 processor with 4 GB RAM. For the sake of comparison all the examples are taken from those given in reference [4].

Example 1: Consider Eq. (1) with the parameters $m = 0.05, w = 0.4, \alpha = \frac{1}{2}$ and choose $f(x,t)$ so that the exact solution is

$$u(x,t) = (t+1)^2 \sin \pi x.$$

The initial and boundary conditions are given as:

$u(x,0) = \sin \pi x$, $0 \leq x \leq 1$,

$u(0,t) = 0$, $u_x(0,t) = \pi(t+1)^2$ and $u(1,t) = 0$.

Numerical simulations are performed with $h = 0.01, 0.02, \delta t = 0.0001, 0.00001, m = 0.05$, $w = 0.4$ and the error norms L_∞ and L_2 at various time levels are recorded in Tables 1-2 along with the results of cubic B-spline collocation method given in [4]. Better accuracy of the present method than cubic B-spline collocation method [4] is evident from Tables 1-2. Fig.1 presents exact and approximate solutions obtained through the present method corresponding to $M = 500$ whereas Fig. 2 shows the error plot at $M = 500$. Fig. 3 represents the quartic B-spline solutions at various time levels.

Example 2: In this example we consider Eq. (1) with the parameters $m = 0.005, w = 0.5, \alpha = \frac{1}{3}$ and $f(x,t)$ is chosen so that the exact solution of Eq. (1) is given by

$$u(x,t) = (t+1)^2(1-\cos 2\pi x) + 2\pi^2 x(1-x). \quad (13)$$

The initial and boundary conditions are extracted from Eq. (13). Computations are carried out using the parameters values $h = 0.01,\ 0.02,\ \delta t = 0.0001,\ 0.00001,\ m = 0.005$, $w = 0.5$and the error norms L_∞ and L_2 at various time levels are noted in Tables 3-4 along with the results of cubic B-spline collocation method given in [4]. Higher accuracy of the present method than cubic B-spline collocation method is observed form Tables 3-4. Fig. 4 shows exact and approximate solution obtained through the present method at $M = 500$.

Table 1: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-1 for $m = 0.05, w = 0.4, \alpha = \frac{1}{2}, \delta t = 0.0001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.01	10	9.30×10^{-8}	6.51×10^{-9}	9.43×10^{-6}	1.32×10^{-7}
	50	5.27×10^{-7}	3.69×10^{-8}	1.14×10^{-5}	3.31×10^{-7}
	100	1.09×10^{-6}	7.66×10^{-8}	1.19×10^{-5}	5.73×10^{-7}
	500	5.92×10^{-6}	4.15×10^{-7}	3.53×10^{-5}	2.43×10^{-6}
0.02	10	9.32×10^{-8}	9.14×10^{-9}	2.64×10^{-5}	5.62×10^{-7}
	50	5.28×10^{-7}	5.18×10^{-8}	3.98×10^{-5}	1.08×10^{-6}
	100	1.09×10^{-6}	1.07×10^{-7}	4.40×10^{-5}	1.39×10^{-6}
	500	5.94×10^{-6}	5.82×10^{-7}	5.36×10^{-5}	2.86×10^{-6}

Example 3: Consider Eq. (1) with the parameters $m = 0.5, w = 0.005, \alpha = \frac{1}{3}$. The initial and boundary conditions for the given problem are defined as:

$u(x,0) = \cos \pi x,\ 0 \le x \le 1,$

$u(0,t) = t+1, u_x(0,t) = 0, u(1,t) = -(t+1).$

The exact solution of the problem is given by

$u(x,t) = (t+1)\cos \pi x.$

Numerical simulations are done using $h = 0.01, \delta t = 0.0001,\ 0.00001, m = 0.5, w = 0.00$. The error norms $L_\infty,\ L_2$ at various time levels are reported in Tables (5)-(6) along with the results of cubic B-spline collocation method. Excellent accuracy of the present method than cubic B-spline collocation method [4] is seen form Tables (5)-(6).

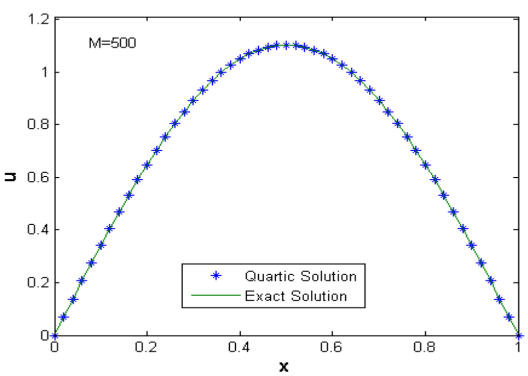

Figure 1: Quartic B-spline and exact solution for Example-1 for $h = 0.02, \delta t = 0.0001$.

Example 4: Consider Eq. (1) with the parameters $m = 0.1, w = 0.1, \alpha = \frac{1}{4}$. The initial and boundary conditions for the given problem are defined as

$u(x,0) = 0,\ 0 \le x \le 4\pi,$

$u(0,t) = 0, u_x(0,t) = 0, u(1,t) = 0.$

The exact solution of the problem is given by

$u(x,t) = 2(t+1)\sin^2 x,\ \ 0 \le x \le 4\pi.$

Numerical simulations are done using $N = 10{,}50,$ $\delta t = 0.0001, 0.00001$. The error norms $L_\infty,\ L_2$ at various time levels are reported in Tables (7)-(8) along with the results of cubic B-spline collocation method. Better accuracy of the present method than cubic B-spline collocation method is seen form Tables (7)-(8).

Example 5: In this example we take parameters $m = 0.5, w = 0.001, \alpha = \frac{1}{4}$. The initial and boundary conditions for the problem are given as

$u(x,0) = 2\sin^2 \pi x,\ 0 \le x \le 1,$

$u(0,t) = 0,\ u_x(0,t) = 0$ and $u(1,t) = 0.$

The exact solution of the problem is given by

$u(x,t) = 2(t^2 + t + 1)\sin^2 \pi x.$

Computations are performed using $h = 0.01, 0.1, 0.02,$ $\delta t = 0.0001, 0.00001$. The error norms L_∞, L_2 at different time levels are recorded in Tables (9)-(10) along with the results of cubic B-spline collocation method. From Tables (9)-(10) comparable accuracy of the present method than cubic B-spline collocation method can be observed. Fig. 7 shows exact and approximate solution using the present method at $M = 500$.

Table 2: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-1 for $m = 0.05,\ w = 0.4, \alpha = \frac{1}{2}, \delta t = 0.00001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.01	10	9.11×10^{-10}	6.38×10^{-11}	4.37×10^{-6}	4.41×10^{-8}
	50	5.03×10^{-9}	3.52×10^{-10}	8.25×10^{-6}	9.59×10^{-8}
	100	1.02×10^{-8}	7.20×10^{-10}	9.55×10^{-6}	1.24×10^{-7}
	500	5.40×10^{-8}	3.78×10^{-9}	1.15×10^{-5}	2.25×10^{-7}
0.02	10	9.32×10^{-10}	9.14×10^{-11}	6.79×10^{-6}	1.37×10^{-7}
	50	5.15×10^{-9}	5.05×10^{-10}	2.00×10^{-5}	4.11×10^{-7}
	100	1.05×10^{-8}	1.03×10^{-9}	2.69×10^{-5}	5.87×10^{-7}
	500	5.52×10^{-8}	5.41×10^{-9}	4.02×10^{-5}	1.25×10^{-6}

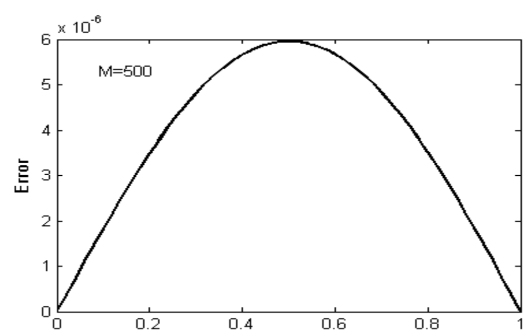

Figure 2: Error Plot of Example-1 for $h = 0.02,\ \delta t = 0.0001$.

Table 3: Error norms produced by quartic B-spline collocation method along with the results of [4] corresponding to Example-2 for $m = 0.005, w = 0.5, \alpha = \frac{1}{3}, \delta t = 0.0001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.02	10	7.17×10^{-7}	4.32×10^{-7}	2.42×10^{-5}	2.00×10^{-6}
	50	3.60×10^{-6}	2.18×10^{-6}	1.00×10^{-4}	8.88×10^{-6}
	100	7.21×10^{-6}	4.38×10^{-6}	1.71×10^{-4}	1.57×10^{-5}
	500	3.50×10^{-5}	2.17×10^{-5}	4.69×10^{-4}	4.17×10^{-5}
0.01	10	6.98×10^{-7}	4.31×10^{-7}	6.62×10^{-6}	5.07×10^{-7}
	50	3.51×10^{-6}	2.17×10^{-6}	3.13×10^{-5}	2.41×10^{-6}
	100	7.03×10^{-6}	4.36×10^{-6}	5.92×10^{-5}	4.63×10^{-6}
	500	3.43×10^{-5}	2.15×10^{-5}	2.28×10^{-4}	1.93×10^{-5}

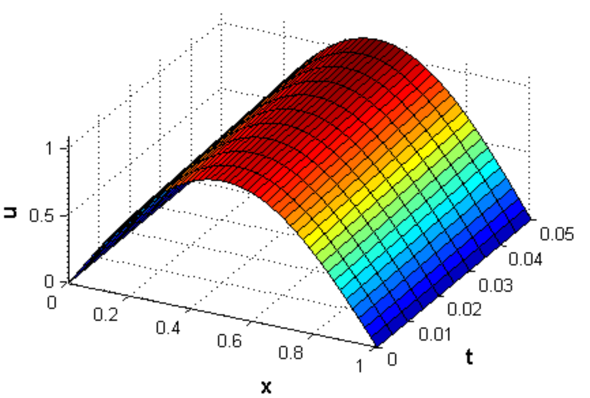

Figure 3: Quartic B-spline solutions at various time levels corresponding to Example-1 for $h = 0.02, \delta t = 0.0001$.

Table 4: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-2 for $m = 0.005, w = 0.5, \alpha = \frac{1}{3}, \delta t = 0.00001$

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.01	10	1.01×10^{-12}	1.20×10^{-13}	3.75×10^{-7}	1.46×10^{-8}
	50	1.62×10^{-11}	6.52×10^{-13}	5.00×10^{-6}	2.54×10^{-7}
	100	5.25×10^{-11}	1.31×10^{-12}	1.68×10^{-5}	9.20×10^{-7}
	500	7.67×10^{-10}	6.83×10^{-12}	2.84×10^{-4}	1.90×10^{-5}

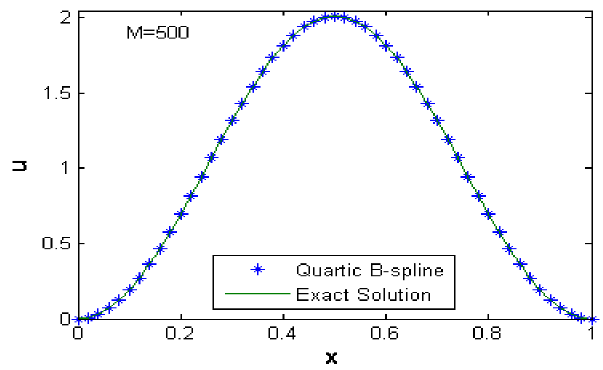

Figure 4: Plot of quartic B-spline and exact solution corresponding to Example-2 for $h = 0.02, \delta t = 0.0001$.

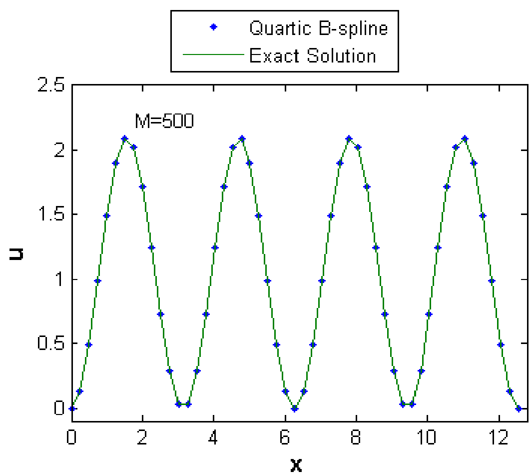

Figure 5: Plot of Quartic B-spline and exact solution corresponding to Example-4

Table 5: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-3 for $m = 0.5, w = 0.005, \alpha = \frac{1}{3}, \delta t = 0.0001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.02	10	8.99×10^{-9}	4.33×10^{-9}	2.92×10^{-6}	2.48×10^{-7}
	50	4.50×10^{-8}	2.18×10^{-8}	1.25×10^{-5}	1.19×10^{-6}
	100	9.01×10^{-8}	4.39×10^{-8}	2.47×10^{-5}	2.35×10^{-6}
	500	4.49×10^{-7}	2.25×10^{-7}	1.19×10^{-4}	1.06×10^{-5}
0.01	10	7.07×10^{-9}	4.29×10^{-9}	1.90×10^{-6}	4.54×10^{-8}
	50	3.54×10^{-8}	2.15×10^{-8}	3.09×10^{-6}	2.01×10^{-7}
	100	7.10×10^{-8}	4.32×10^{-8}	5.96×10^{-6}	3.91×10^{-7}
	500	3.57×10^{-7}	2.18×10^{-7}	2.61×10^{-5}	1.75×10^{-6}

Table 6: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-3 for $= 0.5, w = 0.5, \alpha = \frac{1}{3}, \delta t = 0.00001$.

		Quartic B-spline		Cubic B-spline [4]	
N	M	L_∞	L_2	L_∞	L_2
10	10	1.14×10^{-4}	6.88×10^{-6}	1.80×10^{-4}	3.65×10^{-5}
	50	5.74×10^{-4}	3.50×10^{-5}	8.17×10^{-4}	1.66×10^{-4}
	100	1.15×10^{-3}	7.08×10^{-5}	1.50×10^{-3}	3.01×10^{-4}
50	10	1.12×10^{-6}	6.12×10^{-7}	6.39×10^{-6}	7.52×10^{-7}
	50	4.22×10^{-6}	2.35×10^{-6}	1.78×10^{-4}	1.48×10^{-5}
	100	7.11×10^{-6}	4.04×10^{-6}	6.82×10^{-4}	5.61×10^{-5}

Table 7: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-4 for $m = 0.1, w = 0.1, \alpha = \frac{1}{4}, \delta t = 0.0001$.

		Quartic B-spline		Cubic B-spline [4]	
N	M	L_∞	L_2	L_∞	L_2
10	10	1.14×10^{-5}	7.29×10^{-7}	1.90×10^{-5}	3.78×10^{-6}
	50	5.72×10^{-5}	3.65×10^{-6}	9.32×10^{-5}	1.86×10^{-5}
	100	1.14×10^{-4}	7.32×10^{-6}	1.82×10^{-4}	3.68×10^{-5}
50	500	1.34×10^{-8}	1.07×10^{-8}	9.46×10^{-7}	7.29×10^{-8}
	10	5.99×10^{-8}	4.11×10^{-8}	4.22×10^{-6}	2.89×10^{-7}
	50	1.17×10^{-7}	7.05×10^{-8}	6.13×10^{-6}	6.35×10^{-7}

Table 8: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-4 for $m = 0.1, w = 0.1, \alpha = \frac{1}{4}, \delta t = 0.00001$.

		Quartic B-spline		Cubic B-spline [4]	
H	M	L_∞	L_2	L_∞	L_2
0.01	10	4.48×10^{-10}	9.89×10^{-13}	2.92×10^{-5}	2.48×10^{-6}
	50	7.34×10^{-9}	2.18×10^{-12}	1.25×10^{-4}	1.19×10^{-5}
	100	2.26×10^{-8}	4.39×10^{-11}	2.47×10^{-3}	2.35×10^{-5}
	500	3.00×10^{-7}	2.25×10^{-9}	1.19×10^{-2}	1.06×10^{-3}

Table 9: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example 5 for $m = 0.5, w = 0.001, \alpha = \frac{1}{4}, \delta t = 0.0001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.02	10	1.03×10^{-6}	5.16×10^{-7}	7.06×10^{-7}	5.87×10^{-8}
	50	3.74×10^{-6}	1.87×10^{-6}	3.53×10^{-6}	2.94×10^{-7}
	100	6.14×10^{-6}	3.07×10^{-6}	7.08×10^{-6}	5.89×10^{-7}
	500	1.75×10^{-5}	8.77×10^{-6}	3.59×10^{-5}	2.98×10^{-6}
0.01	10	1.63×10^{-6}	4.81×10^{-7}	3.53×10^{-6}	7.50×10^{-7}
	50	6.81×10^{-6}	1.71×10^{-6}	1.77×10^{-5}	3.76×10^{-6}
	100	1.22×10^{-5}	2.80×10^{-6}	3.56×10^{-5}	7.54×10^{-6}
	500	5.25×10^{-5}	8.00×10^{-6}	1.82×10^{-4}	3.84×10^{-5}

Table 10: Error norms produced by Quartic B-spline collocation method along with the results of [4] corresponding to Example-5 for $m = 0.5, w = 0.001, \alpha = \frac{1}{4}, \delta t = 0.00001$.

		Quartic B-spline		Cubic B-spline [4]	
h	M	L_∞	L_2	L_∞	L_2
0.02	10	1.99×10^{-8}	9.96×10^{-9}	6.61×10^{-9}	3.94×10^{-10}
	50	7.44×10^{-8}	3.72×10^{-8}	3.30×10^{-8}	1.97×10^{-9}
	100	1.25×10^{-7}	6.25×10^{-8}	6.61×10^{-7}	3.94×10^{-9}
	500	3.95×10^{-7}	1.97×10^{-7}	3.31×10^{-7}	1.97×10^{-8}
0.01	10	1.99×10^{-8}	9.96×10^{-9}	7.17×10^{-9}	5.89×10^{-10}
	50	7.43×10^{-8}	3.71×10^{-8}	3.51×10^{-8}	2.94×10^{-9}
	100	1.25×10^{-7}	6.25×10^{-8}	7.18×10^{-8}	5.89×10^{-9}
	500	3.95×10^{-7}	1.97×10^{-7}	3.59×10^{-7}	2.95×10^{-8}

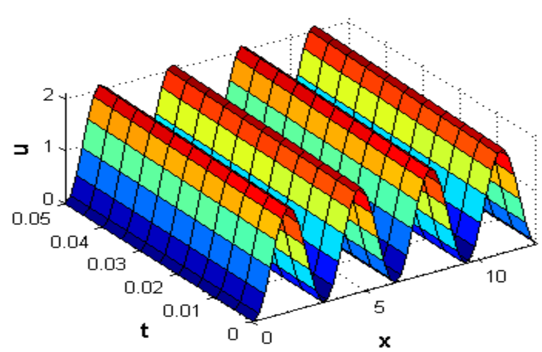

Figure 6: Plot of Quartic B-spline solutions at various times in the interval $[0, 0.05]$ for Example-4 using $N = 50, \delta t = 0.0001$.

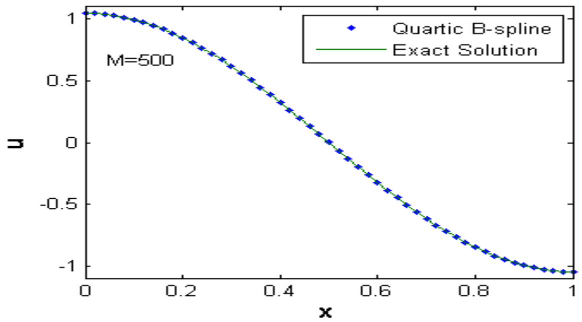

Figure 7: Quartic B-spline and exact solution for Example-5 using $h = 0.02, \delta t = 0.0001$.

4. CONCLUSION

Quartic B-spline collocation method is used to obtain the approximate solution of parabolic type partial integro-differential equations with a weakly singular kernel. The proposed method is implemented with five test problems from literature for its validity. The accuracy of the method is examined through two error norms L_∞, L_2 and by comparison with cubic B-spline collocation method. It has been observed that the errors are sufficiently small. Simple applicability and excellent accuracy of the quartic B-Spline collocation method provide that this method can be employed for numerical approximation of integral equations, partial differential equations and partial integro-differential equations of such type.

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