QUASI-SUBORDINATION CONDITIONS ON BI-UNIVALENT FUNCTIONS INVOLVING HURWITZ-LERCH ZETA FUNCTIONS

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ABSTRACT: *We introduce some new subclasses of the function class Σ of bi-univalent functions defined in unit disk $\mathfrak{U}$, which are involving Hurwitz-Lerch Zeta function and satisfying quasi-subordination conditions. Also we determine estimates of the coefficient $|a_3|$ and $|a_2|$ for functions of these subclasses. Many of the new and well-known consequences are shown to follow as particular cases of our outcomes.*

Keywords: Holomorphic function, bi-univalent functions, subordination, Hurwitz-Lerch Zeta function, quasi-subordination.

INTRODUCTION

Let $\mathcal{A}$ denote the class of all holomorphic functions f defined on the unit disk $\mathfrak{U}$.

$\mathfrak{U} = \{z: |z| < 1\}$,

which are normalized by $f(0) = 0$ and $f'(0) = 1$. The Taylors expansion of $f \in \mathcal{A}$ is

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ... (1.1)

Let S be the class of all holomorphic and univalent functions in the open unit disk. These univalent functions are invertible but the inverse function may not be defined on the entire disk $\mathfrak{U}$. For f in $\mathcal{A}$, the well-known Koebe-one quarter theorem [1] stats that the image of the disk $\mathfrak{U}$ under each univalent function f in $\mathcal{A}$ contains disk with the radius $\frac{1}{4}$. Its well-known that each univalent function f has an inverse f^{-1} such that

$f^{-1}(f(z)) = z, (z \in \mathfrak{U})$

and

$f(f^{-1}(\omega)) = \omega,\ (|\omega| < r_\circ(f), r_\circ(f) \geq \frac{1}{4})$,

A holomorphic function f is called bi-univalent in $\mathfrak{U}$ if both f and C are univalent in $\mathfrak{U}$. Let Σ denote the class of all bi-univalent functions in $\mathfrak{U}$. Since f in Σ has the form (1.1), a computation shows that the invers $g = f^{-1}$ has the following expansion

$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega 2 + (2a_2^2 - a_3)\omega^3 - \cdots$.

The class of bi-univalent functions was introduced by Lewin [2] and proved that $|a_2| \leq 1.51$ for the function of the form (1.1).Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Later Netanyahu in [4] proved that $max_{f\in\Sigma}|a_2| = \frac{4}{3}$. Also several authors studied classes of bi-univalent holomorphic functions and found estimates of the coefficients $|a_3|$ and $|a_2|$ for functions in these classes [5-11].

For functions $f \in \mathcal{A}$ of the form (1.1) and $F \in \mathcal{A}$ defined in the following form

$F(z) = z + \sum_{n=2}^{\infty} b_n z^n$ ... (1.2)

The convolution of the functions f and F denoted by $f(z) * F(z)$ and defined as

$(f * F)(z) = z + \sum_2^{\infty} a_n\, b_n z^n, (z \in \mathfrak{U})$

The general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ given by

$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$ ... (1.3)

$(a \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, s \in \mathbb{C}, Re\ s > 1$ and $|z| = 1)$. Choi and Srivastava [12] found several interesting properties of Hurwitz-Lerch Zeta function. In [13], Srivastava and Attiya introduced the following operator:

$\mathcal{J}_{\mu,b}: \mathcal{A} \longrightarrow \mathcal{A}$

where

$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b}(z) * f(z)$, ... (1.4)

$(b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \mu \in \mathbb{C}, z \in \mathfrak{U}, f \in \mathcal{A})$, where for convenience

$\mathcal{G}(z) := (1 + b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}]$,

$(z \in \mathfrak{U})$. ... (1.5)

By using (1.1), (1.4) and (1.5), we have

$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n\, z^n$ ... (1.6)

In [14] Murugusundaramoorthy studied the following integral operator

$\mathcal{J}_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b, \mu) a_n z^n$, ... (1.7)

where

$C_n^m(b, \mu) = \left|\left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!}\right|$, ... (1.8)

μ, b are constrained as $b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $\mu \in \mathbb{C}$, $k \geq 2$ and $m > -1$.

Consider the integral of the second type $G_{\alpha}(z)$, $\alpha \in \mathbb{C}$ which is defined as the following

$G_{\alpha}(z) = \int_0^z [f'(t)]^{\alpha} dt, \alpha \in \mathbb{C}$, (see [15]). … (1.9)

The relation in (1.9) can be written as follows:

For $f \in \mathcal{A}, z \in \mathfrak{U}$, then

$G_{\alpha} f(z) = \int_0^z f'(t)dt = \int_0^z [1 + \sum_{n=2}^{\infty} n a_n t^{n-1}]^{\alpha} dt = \int_0^z \sum_{j=0}^{\infty} \binom{\alpha}{j} [\sum_{n=2}^{\infty} n a_n t^{n-1}]^j dt = \int_0^z \{[\binom{\alpha}{0} + \binom{\alpha}{1} \sum_{n=2}^{\infty} n a_n t^{n-1}] + \binom{\alpha}{2}[(\sum_{n=2}^{\infty} n a_n t^{n-1})^2 + \cdots]\} dt = \binom{\alpha}{0} z + \sum_{n=2}^{\infty} \binom{\alpha}{1} a_n z^n$

$G_{\alpha} f(z) = z + \sum_{n=2}^{\infty} \binom{\alpha}{1} a_n z^n$, … (1.10)

corresponding to the integral operator defined in (1.7) and the second integral type G_{α} in (1.10), we consider the linear operator

$\mathcal{M}_{\mu,b}^{m,k,\alpha} f(z) = \mathcal{J}_{\mu,b}^{m,k}(G_{\alpha} f(z)) = z + \sum_{n=2}^{\infty} \Psi_{n,\alpha}\, a_n z^n$, … (1.11)

where

$\Psi_{n,\alpha} = \left|\left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!}\right| \binom{\alpha}{1}$.

Note that $\mathcal{M}_{\mu,b}^{1,2,1} \equiv j_{\mu,b}^{1,2}$, is the Srivastava-Attiya operator and $\mathcal{M}_{0,b}^{m,k,1} \equiv j_{0,b}^{m,k}$ is known Choi-Saigo-Srivastave operator [16].

It is interesting to note that special values to m, k, μ, b and α in $\mathcal{M}_{\mu,b}^{m,k,\alpha}$, we can obtain assorted operators studied in [17-20].

For two holomorphic functions f and g, f is quasi-subordination to g, written as follows:

$f(z) \prec_q g(z)\ (z \in \mathfrak{U})$, … (1.12)

if there exist holomorphic functions h and $\mathscr{k}$ with $|h(z)| \leq 1, \mathscr{k}(0) = 0$ and $|\mathscr{k}(z)| < 1$, such that

$f(z) = h(z) g(\mathscr{k}(z)). (z \in \mathfrak{U})$.

Note that if $h(z)=1$, then $f(z)=g(\mathscr{k}(z))$, hence $f(z)\prec g(z)$ in $\mathfrak{U}$. [21].

In [22], Ma and Minda studied the unified classes:

$S^*(\phi)=\{f\in\mathcal{A}: \frac{zf'(z)}{f(z)}\prec\phi(z);\ z\in\mathfrak{U}\}$,

$K(\phi)=\{f\in\mathcal{A}: 1+\frac{zf'''(z)}{f'(z)}\prec\phi(z);\ z\in\mathfrak{U}\}$,

where $\phi(z)$ is a holomorphic and univalent function with positive real part in the unit disk $\mathfrak{U}$, satisfying $\phi(0)=1$, $\phi'(0)>0$ and $\phi(\mathfrak{U})$ is a starlike region with the respect to 1 and symmetric with the respect to the real axis. The functions in the classes $S^*(\phi)$ and $K(\phi)$, are called starlike of Ma-Minda type or convex of Ma-Minda type respectively.

By $S^*_\Sigma(\phi)$ and $K_\Sigma(\phi)$, we denote to bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type respectively. (see [23,24]).

In this investigation, we assume that

$$h(z)=h_\circ+h_1z+h_2z2+h_3z^3+\cdots, \quad \ldots(1.13)$$

and

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots, \qquad B1>0. \quad \ldots(1.14)$$

The aim of this paper is to introduce new subclasses of the class Σ and determine estimates of bounds on the coefficient $|a_3|$ and $|a_2|$ for the functions in above subclasses.

2. Coefficient Estimates For The class $\sum_{\mu,b}^{m,k,\alpha}(\delta,\gamma,\lambda,\phi)$

Definition (2.1): A function $f\in\Sigma$ defined in (1.1) is said to be in the class $\sum_{\mu,b}^{m,k,\alpha}(\delta,\gamma,\lambda,\phi)$ if it satisfies the following quasi-subordination conditions:

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}+\delta\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)}\right\}-1\right]\prec_q \phi(z)-1 \quad \ldots(2.1)$$

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}+\delta\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)}\right\}-1\right]\prec_q \phi(\omega)-1 \quad \ldots(2.2)$$

$(0\le\lambda<1, 0\le\delta\le1, \gamma\in\mathbb{C}\setminus\{0\}, z,\omega\in\mathfrak{U})$.

For special values to parameters $\delta,\alpha,\ \mu,\gamma,m$ and k, we get new and well-known classes.

Remark (2.2): For $\delta=0$ and $\gamma\in\mathbb{C}\setminus\{0\}$, a function $f\in\Sigma$, defined in (1.1) is said to be in the class $\sum_{\mu,b}^{m,k,\alpha}(\gamma,\lambda,\phi)$, if it satisfies the following quasi-subordination conditions:

$$\frac{1}{\gamma}\left[\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}-1\right]\prec_q \phi(z)-1$$

$$\frac{1}{\gamma}\left[\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}-1\right]\prec_q \phi(\omega)-1,$$

where

$0\le\lambda<1, \gamma\in\mathbb{C}\setminus\{0\}$, z and $\omega\in\mathfrak{U}$ with g is the invers function of f.

In particular for $h(z)\equiv1$, and $\alpha=1$, we obtain the following

$\sum_{\mu,b}^{m,k,1}(\delta,\gamma,\lambda,\phi):=\sum_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$,

which was introduced and studied in [25]. If we take

k=2, m =1, μ=0 and b=0, in the class $\sum_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$, then we get

$\sum_{0,0}^{1,2,1}(\gamma,\lambda,\phi):=S_\Sigma^*(\gamma,\lambda,\phi)$,

this class was introduced and studied in [25].

Remark (2.3): For $\delta=1$ and $\gamma\in\mathbb{C}\setminus\{0\}$, a function $f\in\Sigma$, defined in (1.1) is said to be in the class $\sum_{\mu,b}^{m,k,\alpha}(\alpha,\gamma,\lambda,\phi)$, if the following conditions are satisfied

$$\frac{1}{\gamma}\left[\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)}-1\right]\prec_q \phi(z)-1$$

$$\frac{1}{\gamma}\left[\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)}-1\right]\prec_q \phi(\omega)-1,$$

where

$\gamma\in\mathbb{C}\setminus\{0\}$, z and $\omega\in\mathfrak{U}$ with g is the invers function of f.

In particular for γ=1, k=2, μ=0, m =1, b=0 and α=1, we obtain the class

$\sum_{0,0}^{1,2,1}(1,\gamma,\lambda,\phi):=\wp_\Sigma^{*,q}(\phi)$,

which was introduced and studied in [26]. If we take α=1, in the class $\sum_{\mu,b}^{m,k}(1,\gamma,\lambda,\phi)$, then we get $\sum_{\mu,b}^{m,k}(\gamma,\phi)$, this class was introduced and studied in [25].

Remark (2.4): For μ =0, b=0, m =1, α=1, k=2 and $\gamma\in\mathbb{C}\setminus\{0\}$, a function $f\in\Sigma$, defined in (1.1) is said to be in the class $\Sigma(\delta,\gamma,\lambda,\phi)$ if the next quasi-subordination conditions are satisfied

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+\delta\frac{zf'(z)}{f(z)}\right\}-1\right]\prec_q \phi(z)-1$$

and

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{\omega g'(\omega)}{(1-\lambda)g(\omega)+\lambda\omega g'(\omega)}+\delta\frac{\omega g'(z)}{g(\omega)}\right\}-1\right]\prec_q \phi(\omega)-1.$$

In next we find estimates on the coefficient $|a_2|$ and $|a_3|$ for the function in class $\sum_{\mu,b}^{m,k,\alpha}(\delta,\gamma,\lambda,\phi)$.

We need the following lemma in our investigation.

Lemma (2.5): [27] If $p\in\mathcal{P}$, then $|p_i|\le2$ for each i, where $\mathcal{P}$ is the family of all functions p, holomorphic in $\mathfrak{U}$, for which $Re\{p(z)\}>0, (z\in\mathfrak{U})$,

where

$$p(z)=1+p_1z+p_2\,z2+p_3z^3+\cdots, (z\in\mathfrak{U}).$$

Theorem (2.6): Let $f(z)$ given in (1.1) be in the class $\sum_{\mu,b}^{m,k,\alpha}(\delta,\gamma,\lambda,\phi)$. Then

$$|a_2|\le\frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{\left|[\gamma\{\lambda^2(1+\delta)-1\}h_\circ{}^2B_1^2+\{1+\lambda(\delta-1)\}^2(B_1-B_2)]\Psi_{2,\alpha}^2+2\gamma\{1+\lambda(\delta-1)\}h_\circ{}^2B_1^2\Psi_{3,\alpha}\right|}} \quad \ldots(2.3)$$

$$|a_3|\le\frac{|\gamma|^2|h_\circ|^2B_1^2}{\{1+\lambda(\delta-1)\}^2]\Psi_{2,\alpha}^2}+\frac{|\gamma||h_\circ|B_1}{2\{1+\lambda(\delta-1)\}\Psi_{3,\alpha}}+\frac{|\gamma||h_1|B_1}{2\{1+\lambda(\delta-1)\}\Psi_{3,\alpha}}, B1>1, \quad \ldots(2.4)$$

where

$$\Psi_{2,\alpha}=C_2^m\,(b,\mu)\binom{\alpha}{1}=\left|\left(\frac{1+b}{2+b}\right)^\mu\frac{m!k!}{(k-2)!(1+m)!}\right|\binom{\alpha}{1}$$

$$\Psi_{3,\alpha}=C_3^m\,(b,\mu)\binom{\alpha}{1}=\left|\left(\frac{1+b}{3+b}\right)^\mu\frac{m!(1+k)!}{(k-2)!(2+m)!}\right|\binom{\alpha}{1}$$

Proof: Since $f\in\sum_{\mu,b}^{m,k,\alpha}(\delta,\gamma,\lambda,\phi)$, then there exist holomorphic functions r,s in $\mathcal{A}$ and $r,s:\mathfrak{U}\to\mathfrak{U}$, such that $r(0)=s(0)=0$, satisfying

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}+\delta\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)}\right\}-1\right]=h(z)(\phi(r(z))-1), \quad \ldots(2.5)$$

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}+\delta\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)}\right\}-1\right]=h(\omega)(\phi(\omega)-1). \quad ...(2.6)$$

Define the functions u and v by

$$u(z)=\frac{1+r(z)}{1-r(z)}=1+u_1z+u_2z^2+u_3z^3+\cdots$$

and

$$v(z)=\frac{1+s(z)}{1-s(z)}=1+v_1z+v_2z^2+v_3z^3+\cdots \quad (2.7)$$

or equivalently,

$$r(z)=\frac{u(z)-1}{u(z)+1}=\frac{1}{2}[u_1z+(u_2-\frac{u_1^2}{2})z^2+\cdots \quad (2.8)$$

$$s(z)=\frac{v(z)-1}{v(z)+1}=\frac{1}{2}[v_1z+(v_2-\frac{v_1^2}{2})z^2+\cdots \quad (2.9)$$

Using (2.8), (2.9) in (2.5) and (2.6), we have

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}+\delta\frac{z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)}\right\}-1\right]=h(z)[\phi(\frac{u(z)-1}{u(z)+1})-1], \quad ...(2.10)$$

$$\frac{1}{\gamma}\left[\left\{(1-\delta)\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}+\delta\frac{\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)}\right\}-1\right]=h(\omega)[\phi\left(\frac{v(\omega)-1}{v(\omega)+1}\right)-1]. \quad ...(2.11)$$

Utilize (2.8), (2.9) together with (1.13) it is evident that

$$h(z)\left(\phi\left(\frac{u(z)-1}{u(z)+1}\right)-1\right)=\frac{1}{2}h_\circ B_1u_1z+\left\{\frac{1}{2}h_1B_1u_1+\frac{1}{2}h_\circ B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_\circ B_2u_1^2\right\}z^2+\cdots \quad ...(2.12)$$

$$h(\omega)\left(\phi\left(\frac{v(z)-1}{v(z)+1}\right)-1\right)=\frac{1}{2}h_\circ B_1v_1\omega+\left\{\frac{1}{2}h_1B_1v_1+\frac{1}{2}h_\circ B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_\circ B_2v_1^2\right\}\omega^2+\ldots \quad ...(2.13)$$

It follows from (2.10), (2.11), (2.12) and (2.13) ,we get

$$1+\lambda(\delta-1)\Psi_{2,\alpha}a_2=\frac{1}{2}h_\circ B_1u_1, \quad \ldots(2.14)$$

$$\frac{2\{1+\lambda(\delta-1)\}\Psi_{3,\alpha}}{\gamma}a_3-\frac{1-\lambda^2(1+\delta)}{\gamma}\Psi_{2,\alpha}^2a_2^2=\frac{1}{2}h_1B_1u_1+\frac{1}{2}h_\circ B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_\circ B_2u_1^2, \quad \ldots(2.15)$$

$$-\{1+\lambda(\delta-1)\}\Psi_{2,\alpha}a_2=\frac{1}{2}h_\circ B_1v_1 \quad ...(2.16)$$

and

$$\frac{1}{\gamma}\left[4\{(1+\lambda(\delta-1)\}\Psi_{3,\alpha}-\left\{1-\lambda^2(1+\delta)\right\}\right]a_2^2-\frac{2}{\gamma}\{1+\lambda(\delta-1)\}\Psi_{3,\alpha}a_3=\frac{1}{2}h_1B_1v_1+\frac{1}{2}h_\circ B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_\circ B_2v_1^2 \quad ...(2.17)$$

From (2.14) and (2.16), we obtain

$$a_2=\frac{\gamma B_1h_\circ u_1}{2\{1+\lambda(\delta-1)\}\Psi_{2,\alpha}}=-\frac{\gamma B_1h_\circ v_1}{2\{1+\lambda(\delta-1)\}\Psi_{2,\alpha}}, \quad ...(2.18)$$

it follows that

$$u_{1=}-v_1 \quad ...(2.19)$$

and

$$8\{1+\lambda(\delta-1)\}\Psi_{2,\alpha}^2a_2^2=\gamma^2h_\circ^{\,2}B_1^2({u_1}^2+{v_1}^2). \quad ...(2.20)$$

Adding (2.15) and (2.17), in light of (2.18) and (2.19), we have

$$\frac{1}{\gamma}\left[4\{(1+\lambda(\delta-1)\}\Psi_{3,\alpha}-2\{1-\lambda^2(1+\delta)\}\Psi_{2,\alpha}^2\right]a_2^2=\frac{1}{2}h_\circ B_1(u_2+v_2)+\frac{h_\circ(B_2-B_1)}{4}(u_1^2+v_1^2). \quad ...(2.21)$$

Put (2.18) and (2.19) in (2.21), we get

$$a_2^2=\frac{\gamma^2h_\circ^{\,2}B_1^3(u_2+v_2)}{4[\{\gamma(\lambda^2(1+\delta)-1)h_\circ^{\,2}B_1^2+(1+\lambda(\delta-1))^2(B_1-B_2)\}\Psi_{2,\alpha}^2+2\gamma(1+\lambda(\delta-1))h_\circ^{\,2}B_1^2\Psi_{3,\alpha}]}. \quad \ldots(2.22)$$

Applying Lemma (2.5) in (2.22), we get (2.3).

Subtracting (2.17) from (2.15) and computation using (2.19), we obtain

$$a_3=\frac{\gamma^2h_\circ^{\,2}B_1^2({u_1}^2+{v_1}^2)}{8(1+\lambda(\delta-1))^2\Psi_{2,\alpha}^2}+\frac{\gamma h_1B_1u_1}{8(1+\lambda(\delta-1))\Psi_{3,\alpha}}+\frac{\gamma h_\circ B_1(u_2-v_2)}{8(1+\lambda(\delta-1))\Psi_{3,\alpha}}.$$

Apply Lemma (2.5) yields the estimate in (2.4).The proof is complete.

In virtue of Remarks (2.2-2.4), we have the following results:

Corollary (2.7): Let f be in the class $\sum_{\mu,b}^{m,k,\alpha}(\gamma,\lambda,\phi)$, and $\gamma\in\mathbb{C}\setminus\{0\}$. Then

$$|a_2|\le\frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|[\gamma\{\lambda^2-1\}h_\circ^{\,2}B_1^2+\{1-\lambda\}^2(B_1-B_2)]\Psi_{2,\alpha}^2+2\gamma\{1-\lambda)h_\circ^{\,2}B_1^2\Psi_{3,\alpha}|}}$$

and

$$|a_3|\le\frac{|\gamma|^2|h_\circ|^2B_1^2}{\{1-\lambda\}^2]\Psi_{2,\alpha}^2}+\frac{|\gamma||h_\circ|B_1}{2(1-\lambda)\Psi_{3,\alpha}}+\frac{|\gamma||h_1|B_1}{2(1-\lambda)\Psi_{3,\alpha}},\ B_1>1.$$

Corollary (2.8): Let f be in the class $\sum_{\mu,b}^{m,k}(\alpha,\gamma,\lambda,\phi)$, and $\gamma\in\mathbb{C}\setminus\{0\}$. Then

$$|a_2|\le\frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|[\gamma\{2\lambda^2-1\}h_\circ^{\,2}B_1^2+(B_1-B_2)]\Psi_{2,\alpha}^2+2\gamma h_\circ^{\,2}B_1^2\Psi_{3,\alpha}|}}$$

and

$$|a_3|\le\frac{|\gamma|^2|h_\circ|^2B_1^2}{\Psi_{2,\alpha}^2}+\frac{|\gamma||h_\circ|B_1}{2\Psi_{3,\alpha}}+\frac{|\gamma||h_1|B_1}{\Psi_{3,\alpha}}.$$

Corollary (2.9): Let f be in the class $\Sigma(\delta,\gamma,\lambda,\phi)$, and $\gamma\in\mathbb{C}\setminus\{0\}$. Then

$$|a_2|\le\frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|[\gamma\{\lambda^2(1+\delta)-1\}h_\circ^{\,2}B_1^2+\{1+\lambda(\delta-1)\}^2(B_1-B_2)]+2\gamma\{1+\lambda(\delta-1)\}\,h_\circ^{\,2}B_1^2|}},$$

$$|a_3|\le\frac{|\gamma|^2|h_\circ|^2B_1^2}{\{1+\lambda(\delta-1)\}^2]}+\frac{|\gamma||h_\circ|B_1}{2\{1+\lambda(\delta-1)\}}+\frac{|\gamma||h_1|B_1}{2\{1+\lambda(\delta-1)\}}.$$

3. Coefficients Estimates for the Function class $\mathcal{H}_{\mu,b}^{m,k}\mathcal{M}^{\alpha}(\mathcal{S},\gamma,\lambda,\phi)$

Definition (3.1): A function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{H}_{\mu,b}^{m,k,\alpha}(\mathcal{S},\gamma,\lambda,\phi)$ if the next quasi-subordination conditions are satisfied

$$\frac{1}{\gamma}\left[\frac{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{z}+\delta z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)''-1\right]\prec_q\phi(z)-1 \quad ...(3.1)$$

$$\frac{1}{\gamma}\left[\frac{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\omega}+\delta z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)''-1\right]\prec_q\phi(\omega)-1 \quad ...(3.2)$$

$(0\le\lambda<1,0\le\delta\le1,\gamma\in\mathbb{C}\setminus\{0\},z,\omega\in\mathfrak{U})$.

For special values to parameters $\delta,\alpha,\mu,\gamma,m,b,k$ and λ, we get new and known classes.

Remark (3.2): For $\delta=0$ and $\gamma\in\mathbb{C}\setminus\{0\}$, a function $f\in\Sigma$, defined in (1.1) is said to be in the class $\mathcal{H}_{\mu,b}^{m,k,\alpha}(\gamma,\lambda,\phi)$, if the next quasi-subordination conditions are satisfied

$$\frac{1}{\gamma}\left[\frac{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)+\lambda z\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}f(z)\right)'}{z}-1\right]\prec_q\phi(z)-1$$

$\frac{1}{\gamma}\left[\frac{(1-\lambda)\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)+\lambda\omega\left(\mathcal{M}_{\mu,b}^{m,k,\alpha}g(\omega)\right)'}{\omega}-1\right] \prec_q \phi(\omega)-1,$

where

$(0 \leq \lambda < 1, \gamma \in \mathbb{C}\setminus\{0\}, \mathcal{S} \geq 0, z, \omega \in \mathfrak{U})$.

In particular for h(z)≡1, and α=1, we get the class

$\mathcal{H}_{\mu,b}^{m,k,1}(\gamma,\lambda,\phi) := \mathcal{H}_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$,

this was introduced and studied in [25]. Also, we note that for λ=0, we obtain the class

$\mathcal{H}_{\mu,b}^{m,k}(\gamma,0,\phi) := \mathcal{H}_{\mu,b}^{m,k}(\gamma,\phi)$,

and for λ=1, we obtain the class

$\mathcal{H}_{\mu,b}^{m,k}(\gamma,1,\phi) := \mathcal{G}_{\mu,b}^{m,k}(\gamma,\phi)$.

The previous classes were introduced and studied in [25].

Remark (3.3): If μ =0, b=0, m=1, α=1, k=2, and γ∈$\mathbb{C}\setminus\{0\}$, then a function $f \in \Sigma$, defined in (1.1) is said to be in the class $\mathcal{H}(\mathcal{S},\gamma,\lambda,\phi)$, if the next quasi-subordination conditions are satisfied

$\frac{1}{\gamma}\left[\frac{(1-\lambda)f(z)+\lambda zf'(z)}{z}+\delta zf''(z)-1\right] \prec_q \phi(z)-1$

$\frac{1}{\gamma}\left[\frac{(1-\lambda)g(\omega)+\lambda\omega g'(\omega)}{\omega}+\delta\omega g''(\omega)-1\right] \prec_q \phi(\omega)-1,$

where

$(0 \leq \lambda < 1, 0 \leq \delta \leq 1, \gamma \in \mathbb{C}\setminus\{0\}, z, \omega \in \mathfrak{U})$.

Theorem (3.4): If $f \in \mathcal{H}_{\mu,b}^{m,k,\alpha}(\mathcal{S},\gamma,\lambda,\phi)$ is defined in (1.1), then

$|a_2| \leq \frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|\gamma\{6\delta+2\lambda+1\}h_\circ B_1^2\Psi_{3,\alpha}-(2\delta+\lambda+1)^2(B_2-B_1)\Psi_{2,\alpha}^2|}}$ ... (3.3)

and

$|a_3| \leq \frac{|\gamma|^2|h_\circ|^2B_1^2}{(2\delta+\lambda+1)^2\Psi_{2,\alpha}^2}+\frac{|\gamma||h_\circ|B_1}{\{6\delta+2\lambda+1\}\Psi_{3,\alpha}}+\frac{|\gamma||h_1|B_1}{\{6\delta+2\lambda+1\}\Psi_{3,\alpha}}, \quad B_1 > 1.$ ... (3.4)

Where

$$\Psi_{2,\alpha} = C_2^m(b,\mu)\binom{\alpha}{1} = \left|\left(\frac{1+b}{2+b}\right)^\mu \frac{m!k!}{(k-2)!(1+m)!}\right|\binom{\alpha}{1},$$

$$\Psi_{3,\alpha} = C_3^m(b,\mu)\binom{\alpha}{1} = \left|\left(\frac{1+b}{3+b}\right)^\mu \frac{m!(1+k)!}{(k-2)!(2+m)!}\right|\binom{\alpha}{1}.$$

Proof: Proceeding as in the proof of Theorem (2.6), we can get the relations as follow

$$\frac{1}{\gamma}(2\delta+\lambda+1)\Psi_{2,\alpha}a_2 = \frac{1}{2}h_\circ B_1u_1, \quad ... (3.5)$$

$\frac{1}{\gamma}\{6\delta+2\lambda+1\}\Psi_{3,\alpha}a_3 = \frac{1}{2}h_1B_1u_1+\frac{1}{2}h_\circ B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_\circ B_2u_1^2$ ... (3.6)

$-\frac{1}{\gamma}(2\delta+\lambda+1)\Psi_{2,\alpha}a_2 = \frac{1}{2}h_\circ B_1v_1$ ... (3.7)

and

$\frac{1}{\gamma}\{6\delta+2\lambda+1\}\Psi_{3,\alpha}(2a_2^2-a_3) = \frac{1}{2}h_1B_1v_1+\frac{1}{2}h_\circ B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_\circ B_2v_1^2.$ ... (3.8)

From (3.5) and (3.7), we obtain

$u_1 = -v_1$ ... (3.9)

and

$8(2\delta+\lambda+1)^2\Psi_{2,\alpha}^2a_2^2 = \gamma^2h_\circ^2B_1^2(u_1^2+v_1^2).$ ... (3.10)

Also by adding (3.6) and (3.8), in light of (3.9) and (3.10), we have

$a_2^2 = \frac{\gamma^2h_\circ^2B_1^3(u_2+v_2)}{4[\gamma(6\delta+2\lambda+1)h_\circ B_1^2\Psi_{3,\alpha}-(2\delta+\lambda+1)^2(B_2-B_1)\Psi_{2,\alpha}^2]}.$ ... (3.11)

Applying Lemma (2.5) in (3.11), we obtain the required result (3.3).

By subtracting (3.8) from (3.6), further computation using (3.9) and (3.10), we obtain

$a_3 = \frac{\gamma^2h_\circ^2B_1^2(u_1^2+v_1^2)}{8(2\delta+\lambda+1)^2\Psi_{2,\alpha}^2}+\frac{\gamma h_1B_1u_1}{2(6\delta+2\lambda+1)\Psi_{3,\alpha}}+\frac{\gamma h_\circ B_1(u_2-v_2)}{4(6\delta+2\lambda+1)\Psi_{3,\alpha}}.$

Apply Lemma (2.5) yields the estimate in (3.4).The proof is complete.

In virtue of Remarks (3.2) and (3.3), we have the following results:

Corollary (3.5): If $f \in \mathcal{H}_{\mu,b}^{m,k,\alpha}(\gamma,\lambda,\phi)$, and γ∈$\mathbb{C}\setminus\{0\}$, then

$|a_2| \leq \frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|\gamma\{2\lambda+1\}h_\circ B_1^2\Psi_{3,\alpha}-(\lambda+1)^2(B_2-B_1)\Psi_{2,\alpha}^2|}}$

and

$|a_3| \leq \frac{|\gamma|^2|h_\circ|^2B_1^2}{(\lambda+1)^2\Psi_{2,\alpha}^2}+\frac{|\gamma||h_\circ|B_1}{\{2\lambda+1\}\Psi_{3,\alpha}}+\frac{|\gamma||h_1|B_1}{\{2\lambda+1\}\Psi_{3,\alpha}}, B_1 > 1.$

Corollary (3.6): Let f be in the class $\mathcal{H}(\mathcal{S},\gamma,\lambda,\phi)$, and γ∈$\mathbb{C}\setminus\{0\}$. Then

$|a_2| \leq \frac{|\gamma||h_\circ|B_1\sqrt{B_1}}{\sqrt{|\gamma\{6\delta+2\lambda+1\}h_\circ B_1^2-(2\delta+\lambda+1)^2(B_2-B_1)|}}$

and

$|a_3| \leq \frac{|\gamma|^2|h_\circ|^2B_1^2}{(2\delta+\lambda+1)^2}+\frac{|\gamma||h_\circ|B_1}{\{6\delta+2\lambda+1\}}+\frac{|\gamma||h_1|B_1}{\{6\delta+2\lambda+1\}}.$

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