THE DYNAMICS ANALYSIS OF A DISEASED PREY-PREDATOR SYSTEM WITH HERD BEHAVIOR

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ABSTRACT: *An eco-epidemiological model consisting of prey-predator system with disease in prey has been proposed and analyzed. It is assumed that the prey species exhibits herd behavior as a defensive strategy against the predation. The existence of all possible equilibrium points are carried out. The local stability analysis and the basin of attractions for these equilibrium points are discussed. The occurrences of local bifurcations as well as Hopf bifurcation are studied. Finally numerical simulation is used to investigate the global dynamics of the system.*

Keywords: Prey-predator system; SI epidemics disease; Herd behavior; Stability; Local bifurcation.

1. INTRODUCTION

The prey-predator system is one of the most important systems in ecology, which describe the nutrient interaction between the species. It has been studied mathematically by many researchers in literatures [1–5] and the references therein. In nature, the prey–predator systems involve different types of factors, such as age structure, switching, refuge, harvesting, group defense etc. The dynamics of these systems have been widely studied using mathematical models [6-8]. On the other hand, the mathematical modeling of epidemics has become a very important subject of research since the pioneer work of Kermac-McKendric (1927) on SIRS (susceptible-infected-removed-susceptible) systems. In these systems the evolution of a disease that gets transmitted upon contact is described and investigated. Number of models in the framework of Kermac-McKendric model has been proposed and studied, which are aimed to controlling the effects of diseases and prevent their spread [9-13] and the reference therein.

Although, ecology and epidemiology are two distinct and major fields, in nature species do not exist alone and it is more reasonable to study the interaction between them subjected to the disease. The mathematical models that companied between these two fields are known as eco-epidemiological models. In the last two decades many researches have been proposed and studied variety of eco-epidemiological models [14-18] and the reference therein.

It is well known that the mass action predation term and Holling family of functional responses are the most types of functional responses used in literatures. These types of functional responses have been built as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. However, in nature, most of the species live in groups in order to help and protect each other. Animals demonstrating this behavior, which known as herd behavior, stay in very large groups and are often in open spaces, by sticking together in a herd, they reduce the chance of being singled out and becoming the one animal that gets killed by the predator. Consequently, Cosner et al. [19] made the assumption that the number of predators in a shape is proportional to the area of the group in two dimensions, and to its volume in three dimensions. Therefore, the encounter rate between the prey X and its predator Y, say $f(X,Y)=aXY$ should change its form to $f(X,Y)=a\sqrt{X}\,Y$ in two dimension, and in three dimension it should be $f(X,Y)=aX^{\frac{2}{3}}Y$. Recently Ajraldi et al. [20] argued that when prey population exhibit herd behavior; the functional response should be in terms of square root of prey population. Number of researchers have been investigated this type of prey's behavior mathematically and discussed there effects on the dynamical behavior of the prey-predator systems [21-25].

Keeping the above in view, in this paper, an eco-epidemiological model consisting of prey-predator system with disease in prey has been proposed and studied. The property of prey's herd behavior is also included in the model. The stability analysis and bifurcation analysis of the proposed model are investigated.

2. The model formulation

In this section a mathematical model, which describes the dynamical behavior of a prey-predator system having infectious disease in prey and prey's herd behavior as a defensive strategy, is proposed and analyzed. Consequently, in order to formulates this model the following hypotheses are considered

1. Due to the existence of infectious disease the prey population is divided into two classes, susceptible that denotes to its population size at time T by $X(T)$ and infected, which denotes to its population at time T by $Y(T)$. It is assumed that in the absence of the predation the susceptible population grows Logistically with intrinsic growth rate $r>0$ and carrying capacity $K>0$, while the infected prey can't reproduce but it still has the capability to compete for the resources.
2. The disease is of SI-type that means whenever the susceptible individual infected with the disease it will transfer to the infected class and can't be recovering again. The disease transmitted from infected individual to the susceptible individual by contact with infected rate $b>0$. Further the infected individual facing death due to the disease with disease death rate $d_1>0$.
3. It is assumed that the susceptible prey exhibits herd behavior as defense strategy against the predation and hence the predator, which denotes to its population at time T by $Z(T)$, will be capable to attacks only the extreme individual of susceptible prey that will be represented by

the square root of susceptible prey in the form of functional response. However the infected prey losses this strategy due to the effect of disease. Finally the predator consumes the prey according to Lotka-Volterra functional response with maximum attack rates $a_1 > 0$ and $a_2 > 0$ for susceptible and infected prey respectively.

4. The food will be converted to the predator from the susceptible and infected prey with conversion rates $0 < e_1 < 1$ and $0 < e_2 < 1$ respectively. Finally in the absence of the prey species the predator will be decaying exponentially with natural death rate $d_2 > 0$.Accordingly the dynamics of the prey-predator system having infectious disease in prey and prey's herd behavior can be represented mathematically by the following set of nonlinear ordinary differential equations:

$$\begin{aligned}\frac{dX}{dT} &= rX\left(1-\frac{X+Y}{K}\right) - a_1\sqrt{X}Z - bXY\\ \frac{dY}{dT} &= bXY - a_2YZ - d_1Y \qquad (1)\\ \frac{dZ}{dT} &= e_1a_1\sqrt{X}Z + e_2a_2YZ - d_2Z\end{aligned}$$

with $X(0) \ge 0,\ Y(0) \ge 0,\ Z(0) \ge 0$. Clearly, due to the biological meaning of the variables given in system (1) then the system defines on the following domain $R_+^3 = \{(X,Y,Z) \in R^3 : X \ge 0, Y \ge 0, Z \ge 0\}$ and is bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1) those initiate in the R_+^3 are uniformly bounded.

Proof: Let $(X(t),Y(t),Z(t))$ be any solution initiate in R_+^3. Since we have that

$$\frac{dX}{dT} \le rX\left(1-\frac{X}{K}\right)$$

Then straightforward computation shows that $X \le K$ as $T \to \infty$. Now consider the function $W = X + Y + Z$, then we obtain that

$$\frac{dW}{dT} \le \mu_1 - \mu_2 W$$

here $\mu_1 = \frac{r+1}{r}K$ and $\mu_2 = \min.\{1, d_1, d_2\}$. So, straightforward computation gives that $W \le \frac{\mu_2}{\mu_1}$ as $T \to \infty$. Hence all solutions are uniformly bounded.

■

Now in order to simplify the form of system (1) and discard the square root from it the following transformation is used. Let $P = \sqrt{X}$ then we obtain

$$\begin{aligned}\frac{dP}{dT} &= \frac{r}{2}P\left(1-\frac{P^2+Y}{K}\right) - \frac{a_1}{2}Z - \frac{b}{2}PY\\ \frac{dY}{dT} &= bPY - a_2YZ - d_1Y \qquad (2)\\ \frac{dZ}{dT} &= e_1a_1PZ + e_2a_2YZ - d_2Z\end{aligned}$$

Further simplification of system (2) is used to reduce the number of parameters and generalized the analytical results by applying the following dimensionless variables:

$$s = \frac{P}{\sqrt{K}};\ i = \frac{Y}{K};\ z = \frac{Z}{K};\ t = \frac{r}{2}T$$

Now straightforward computation on system (2) gives the following dimensionless system

$$\begin{aligned}\frac{ds}{dt} &= s(1-s^2-i) - w_1z - w_2si = f_1(s,i,z)\\ \frac{di}{dt} &= w_3s^2i - w_4iz - w_5i = f_2(s,i,z) \qquad (3)\\ \frac{dz}{dt} &= w_6sz + w_7iz - w_8z = f_3(s,i,z)\end{aligned}$$

with $s(0) \ge 0, i(0) \ge 0,\ z(0) \ge 0$ and the dimensionless parameters are given by:

$$\begin{aligned}&w_1 = \frac{a_1}{r}\sqrt{K},\ w_2 = \frac{bK}{r},\ w_3 = \frac{2bK}{r},\ w_4 = \frac{2a_2K}{r},\\ &w_5 = \frac{2d_1}{r},\ w_6 = \frac{2}{r}e_1a_1\sqrt{K},\ w_7 = \frac{2}{r}e_2a_2K,\ w_8 = \frac{2d_2}{r}\end{aligned} \qquad (4)$$

Clearly the interaction functions in the right hand side of system (3) are continuous and have continuous partial derivatives, and hence they are Liptchazian functions. Hence system (3) has a unique solution. It is also bounded due to the boundedness of system (1).

3. Local stability analysis of the equilibrium points

In this section the existence conditions of all possible equilibrium point of system (3) are established and then the local stability analysis of each of them is discussed. There are at most five non-negative equilibrium points of system (3), these are describing as follows:

The vanishing equilibrium point that denoted by $E_0 = (0,0,0)$ and the disease-predator free equilibrium point, say $E_1 = (1,0,0)$, which full down on the $s-$axis are always exist. The predator free equilibrium point $E_2 = (\bar{s},\bar{i},0)$ exists uniquely in the interior of $si-$plane ($Int.R^2_{+(si)}$) if and only if the following condition holds

$$w_3 > w_5 \qquad (5a)$$

Where

$$\bar{s} = \sqrt{\frac{w_5}{w_3}};\ \bar{i} = \frac{1-(\frac{w_5}{w_3})}{1+w_2} \qquad (5b)$$

The disease free equilibrium point $E_3 = (\hat{s},0,\hat{z})$ exists uniquely in the interior of $sz-$plane ($Int.R^2_{+(sz)}$) if and only if the following condition holds

$$\left(\frac{w_8}{w_6}\right)^2 < 1 \qquad (6a)$$

Where

$$\hat{s} = \frac{w_8}{w_6},\ \hat{z} = \frac{w_8}{w_6w_1}\left[1-\left(\frac{w_8}{w_6}\right)^2\right] \qquad (6b)$$

Finally the positive equilibrium point $E_4=(s^*,i^*,z^*)$ exists uniquely in the interior of R_+^3 provided that there is a positive solution to the following algebraic system of equations.

$$\begin{aligned} &s(1-s^2-i)-w_1 z-w_2 si=0\\ &w_3 s^2-w_4 z-w_5=0 \\ &w_6 s+w_7 i-w_8=0\end{aligned} \quad (7)$$

Straightforward computation shows that system (7) has a unique positive solution given by

$$i^*=\frac{w_8-w_6 s^*}{w_7},\ z^*=\frac{w_3 s^{*2}-w_5}{w_4} \quad (8)$$

while s^* represents the unique positive root of the following third degree polynomial

$$w_4 s^3+\left(w_1 w_2-w_4(1+w_2)\frac{w_6}{w_7}\right)s^2 + w_4\left((1+w_2)\frac{w_8}{w_7}-1\right)s-w_1 w_5=0$$

provided that the following set of conditions are satisfied

$$w_1 w_2>w_4(1+w_2)\frac{w_6}{w_7} \ \text{ or } \ (1+w_2)\frac{w_8}{w_7}<1 \quad (9a)$$

$$\sqrt{\frac{w_5}{w_3}}<s^*<\frac{w_8}{w_6} \quad (9b)$$

Now since the general Jacobian matrix $DF=J(s,i,z)$ of system (3), where $F=(f_1,f_2,f_3)^T$, can be written as:

$$J(s,i,z)=(a_{ij})_{3\times 3} \quad (10)$$

where $a_{11}=1-3s^2-i-w_2 i$, $a_{12}=-s-w_2 s$, $a_{13}=-w_1$, $a_{21}=2w_3 si$, $a_{22}=w_3 s^2-w_4 z-w_5$, $a_{23}=-w_4 i$, $a_{31}=w_6 z$ $a_{32}=w_7 z$ and $a_{33}=w_6 s+w_7 i-w_8$. Then the Jacobian matrix at $E_0(0,0,0)$ is

$$J(E_0)=\begin{pmatrix}1 & 0 & -w_1\\ 0 & -w_5 & 0\\ 0 & 0 & -w_8\end{pmatrix} \quad (11a)$$

Clearly, the $J(E_0)$ has the following eigenvalues

$$\lambda_{0s}=1>0;\ \lambda_{0i}=-w_5<0;\ \lambda_{0z}=-w_8<0 \quad (11b)$$

where $\lambda_{0u};u=s,i,z$ represents the eigenvalue of $J(E_0)$ in the $u-$direction. Therefore the vanishing equilibrium point E_0 is a saddle point.

The Jacobian matrix of system (3) at the disease-predator free equilibrium point $E_1=(1,0,0)$ is given by

$$J(E_1)=\begin{pmatrix}-2 & -1-w_2 & -w_1\\ 0 & w_3-w_5 & 0\\ 0 & 0 & w_6-w_8\end{pmatrix}=(\tilde{a}_{ij}) \quad (12a)$$

Therefore the eigenvalues of $J(E_1)$ are

$$\lambda_{1s}=-2<0,\ \lambda_{1i}=w_3-w_5,\ \lambda_{1z}=w_6-w_8 \quad (12b)$$

It is easy to verify that all the above eigenvalues will be negative if they satisfy the following conditions

$$w_3<w_5 \quad (12c)$$

$$w_6<w_8 \quad (12d)$$

Hence E_1 is locally asymptotically stable in R_+^3 under the conditions (12c)-(12d) and it is unstable saddle point otherwise.

The Jacobian matrix of system (3) at the predator free equilibrium point $E_2=(\bar{s},\bar{i},0)$ can be written

$$J(E_2)=\begin{pmatrix}1-3\bar{s}^2-(1+w_2)\bar{i} & -(1+w_2)\bar{s} & -w_1\\ 2w_3\bar{s}\bar{i} & 0 & -w_4\bar{i}\\ 0 & 0 & w_6\bar{s}+w_7\bar{i}-w_8\end{pmatrix} =(\bar{a}_{ij}) \quad \text{……….. (13a)}$$

Then the characteristic equation is

$$(\lambda^2-T_2\lambda+D_2)(\bar{a}_{33}-\lambda)=0 \quad (13b)$$

here

$$T_2=\bar{a}_{11}=-2\frac{w_5}{w_3}<0 \ \text{ and } \ D_2=-\bar{a}_{12}\bar{a}_{21}=2w_5\left(1-\frac{w_5}{w_3}\right)>0$$

due to the existence condition (5a). Thus, the eigenvalues of $J(E_2)$ are:

$$\begin{aligned}&\lambda_{2s},\lambda_{2i}=\frac{1}{2}\left(T_2\pm\sqrt{T_2^2-4D_2}\right)\\ &\lambda_{2z}=w_6\bar{s}+w_7\bar{i}-w_8\end{aligned} \quad (13c)$$

Clearly, λ_{2s} and λ_{2i} have negative real parts, while λ_{2z} is negative provided that

$$w_6\bar{s}+w_7\bar{i}<w_8 \quad (13d)$$

Consequently, E_2 is locally asymptotically stable in R_+^3 under the condition (13d) and it is saddle point otherwise.

The Jacobian matrix of system (3) at the disease free equilibrium point $E_3=(\hat{s},0,\hat{z})$ can be written as follows:

$$J(E_3)=\begin{pmatrix}1-3\hat{s}^2 & -(1+w_2)\hat{s} & -w_1\\ 0 & w_3\hat{s}^2-w_4\hat{z}-w_5 & 0\\ w_6\hat{z} & w_7\hat{z} & 0\end{pmatrix}=\left(\hat{a}_{ij}\right) \quad (14a)$$

Hence the characteristic equation is

$$(\lambda^2-T_3\lambda+D_3)(\hat{a}_{22}-\lambda)=0 \quad (14b)$$

here $T_3=\hat{a}_{11}=1-3\left[\frac{w_8}{w_6}\right]^2$ and $D_3=w_1 w_6\hat{z}>0$.

Accordingly, the eigenvalues of $J(E_3)$ are written by:

$$\begin{aligned}&\lambda_{3s},\lambda_{3z}=\frac{1}{2}\left(T_3\pm\sqrt{T_3^2-4D_3}\right)\\ &\lambda_{2i}=w_3\hat{s}^2-w_4\hat{i}-w_5\end{aligned} \quad (14c)$$

Straightforward computation shows that the above eigenvalues have negative real parts provided that the following conditions hold.

$$1<3\left[\frac{w_8}{w_6}\right]^2 \quad (14d)$$

$$w_3\hat{s}^2 < w_4\hat{z} + w_5 \quad (14e)$$

Accordingly, the disease free equilibrium point E_3 is locally asymptotically stable provided that the conditions (14d)-(14e) are satisfied. However, if only the condition (14d) holds, E_3 will be unstable saddle point, while it's an unstable point when both the conditions (14d)-(14e) are violated.

Finally, The Jacobian matrix of system (3) at the positive equilibrium point $E_4 = (s^*, i^*, z^*)$ is given by

$$J(E_4) = \begin{pmatrix} 1-3s^{*2} - (1+w_2)i^* & -(1+w_2)s^* & -w_1 \\ 2w_3 s^* i^* & 0 & -w_4 i^* \\ w_6 z^* & w_7 z^* & 0 \end{pmatrix} = \left(a_{ij}^*\right) \quad \ldots\ldots\ldots\ldots\ldots(15a)$$

Thus the characteristic equation of $J(E_4)$ can be written as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (15b)$$

with $A_1 = -a_{11}^*$,

$A_2 = -a_{12}^* a_{21}^* - a_{13}^* a_{31}^* - a_{23}^* a_{32}^* > 0$,

$A_3 = a_{23}^*(a_{11}^* a_{32}^* - a_{12}^* a_{31}^*) - a_{13}^* a_{21}^* a_{32}^*$,

$$\Delta = A_1 A_2 - A_3 = a_{12}^*(a_{11}^* a_{21}^* + a_{23}^* a_{31}^*) + a_{13}^*(a_{11}^* a_{31}^* + a_{21}^* a_{32}^*).$$

Therefore, by using the Routh-Hurwitz criterion the following theorem can be proved directly.

Theorem (2): The positive equilibrium point E_4 is locally asymptotically stable in R_+^3 provided that the following condition is satisfied.

$$\max.\left\{\left(1 + 2\frac{w_3 w_7 s^* i^*}{w_6}\right), \left(1 + \frac{w_6(1+w_2)s^*}{w_7}\right)\right\} < 3s^* + (1+w_2)i^* \quad (16)$$

Proof. Straightforward computation shows that A_1, A_3 and Δ are positive due to the condition (16) and hence according to Routh-Hurwitz criterion all the eigenvalues of the $J(E_4)$ have negative real parts . Thus E_4 is locally asymptotically and the proof is complete. ■

4. Basin of attraction

In this section the region of global stability, that known as basin of attraction, of each equilibrium point is determined with the help of Lyapunov method as shown in the following theorems.

Theorem (3): Suppose that the disease-predator free equilibrium point E_1 is locally asymptotically stable, then it's a globally asymptotically stable in the region $\Omega_1 \subset R_+^3$, where $\Omega_1 = \{(s,i,z) \in R_+^3 : 1 < s < \gamma, i \geq 0, z \geq 0\}$ and γ is a constant given in the proof.

Proof: Define the function

$$V_1 = c_1\left[s - \tilde{s} - \tilde{s}\ln\left(\frac{s}{\tilde{s}}\right)\right] + c_2 i + c_3 z$$

where $c_j\,; j = 1,2,3$ are positive constants to be determined, V_1 is continuously differentiable positive definite real valued function with $V_1(\tilde{s},0,0) = V_1(1,0,0) = 0$ and $V_1(x,0,0) > 0$ for all $(x,0,0) \neq (1,0,0) \in R_+^3$. Now by differentiate V_1 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_1}{dt} = -c_1(s+1)(s-1)^2 - [c_1(1+w_2) - c_2 w_3 s]si - [c_2 w_5 - c_1(1+w_2)]i - [c_2 w_4 - c_3 w_7]iz - \left[c_1 w_1\left(1 - \frac{1}{s}\right) + c_3(w_8 - w_6 s)\right]z$$

So by choosing the constants as $c_1 = \frac{w_5}{1+w_2}, c_2 = 1$ and $c_3 = \frac{w_4}{w_7}$, we obtain that

$$\frac{dV_1}{dt} \leq -\left[\frac{w_5}{1+w_2}\right](s+1)(s-1)^2 - [w_5 - w_3 s]si - \left[\frac{w_1 w_5}{1+w_2}\left(1 - \frac{1}{s}\right) + \frac{w_4}{w_7}(w_8 - w_6 s)\right]z$$

Thus it is easy to verify that $\frac{dV_1}{dt}$ is negative definite under the sufficient condition

$$1 < s < \min.\left\{\frac{w_5}{w_3}, \frac{w_8}{w_6}\right\} = \gamma$$

Therefore, V_1 represents a Lyapunov function on Ω_1 and hence for any initial point in the region Ω_1 the solution of system (3) approaches asymptotically to E_1. Thus E_1 is globally asymptotically stable in Ω_1, which represents the basin of attraction of E_1. ■

Theorem (4): Suppose that the predator free equilibrium point $E_2 = (\bar{s}, \bar{i}, 0)$ is locally asymptotically stable, then it's a globally asymptotically stable in the region $\Omega_2 \subset R_+^3$, where $\Omega_2 = \{(s,i,z) \in R_+^3 : \bar{s} < s < \frac{w_8 - w_7\bar{i}}{w_6}, 0 \leq i < \bar{i}, z \geq 0\}$.

Proof: Consider the function

$$V_2 = \bar{c}_1\left[s - \bar{s} - \bar{s}\ln\left(\frac{s}{\bar{s}}\right)\right] + \bar{c}_2\left[i - \bar{i} - \bar{i}\ln\left(\frac{i}{\bar{i}}\right)\right] + \bar{c}_3 z$$

where $\bar{c}_j\,; j = 1,2,3$ are positive constants to be determined, V_2 is continuously differentiable positive definite real valued function with $V_2(\bar{s}, \bar{i}, 0) = 0$ and $V_2(s,i,0) > 0$ for all $(s,i,0) \neq (\bar{s}, \bar{i}, 0)$ in the R_+^3. Now by differentiate V_2 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_2}{dt} = -\bar{c}_1(s+\bar{s})(s-\bar{s})^2 - [\bar{c}_2 w_4 - \bar{c}_3 w_7]iz - [\bar{c}_1(1+w_2) - \bar{c}_2 w_3(s+\bar{s})](s-\bar{s})(i-\bar{i}) - \left[\bar{c}_3(w_8 - w_6 s) + \bar{c}_1 w_1\left(1 - \frac{\bar{s}}{s}\right) - \bar{c}_2 w_4 \bar{i}\right]z$$

So by choosing the constants as $\bar{c}_1 = \frac{w_3 \bar{s}}{1+w_2}, \bar{c}_2 = 1$ and $\bar{c}_3 = \frac{w_4}{w_7}$, we obtain, after some algebraic computation, that

$$\frac{dV_2}{dt} \le -\left[\frac{w_3 \bar{s}}{1+w_2}\right](s+\bar{s})(s-\bar{s})^2 + w_3 s(s-\bar{s})(i-\bar{i})$$
$$-\frac{z}{(1+w_2)w_7 s}\left[w_1 w_3 w_7 \bar{s}(s-\bar{s}) + w_4(1+w_2)s\left(w_8 - w_6 s - w_7 \bar{i}\right)\right]$$

Clearly, $\frac{dV_2}{dt}$ is negative definite under the following sufficient conditions

$$\bar{s} < s < \frac{w_8 - w_7 \bar{i}}{w_6};\ 0 \le i < \bar{i}$$

Therefore, V_2 represents a Lyapunov function on Ω_2 and hence for any initial point in the region Ω_2 the solution of system (3) approaches asymptotically to E_2. Thus E_2 is globally asymptotically stable in Ω_2, which represents the basin of attraction of E_2. ■

Theorem (5): Suppose that the disease free equilibrium point $E_3 = (\hat{s}, 0, \hat{z})$ is locally asymptotically stable, then it's a globally asymptotically stable in the region $\Omega_3 \subset R_+^3$, where $\Omega_3 = \{(s,i,z) \in R_+^3 : \gamma_1 < s < \gamma_2, i \ge 0, 0 \le z < \hat{z}\}$, where $\gamma_j;\ j = 1,2$ are constants given in the proof.

Proof: Define the function

$$V_3 = \hat{c}_1\left[s - \hat{s} - \hat{s}\ln\left(\frac{s}{\hat{s}}\right)\right] + \hat{c}_2 i + \hat{c}_3\left[z - \hat{z} - \hat{z}\ln\left(\frac{z}{\hat{z}}\right)\right]$$

where $\hat{c}_j;\ j = 1,2,3$ are positive constants to be determined, V_3 is continuously differentiable positive definite real valued function with $V_3(\hat{s}, 0, \hat{z}) = 0$ and $V_3(s, 0, z) > 0$ for all $(x, 0, z) \ne (\hat{x}, 0, \hat{z})$ in the R_+^3. Again by differentiate V_3 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_3}{dt} = -\hat{c}_1\left[s + \hat{s} - \frac{w_1 \hat{z}}{s\hat{s}}\right](s-\hat{s})^2 - \left[\hat{c}_2 w_4 - \hat{c}_3 w_7\right]iz$$
$$-\left[\hat{c}_1 \frac{w_1}{s} - \hat{c}_3 w_6\right](s-\hat{s})(z-\hat{z})$$
$$-\left[\hat{c}_1(1+w_2) - \hat{c}_2 w_3 s\right]si$$
$$-\left[\hat{c}_2 w_5 + \hat{c}_3 w_7 \hat{z} - \hat{c}_1(1+w_2)\hat{s}\right]i$$

So by choosing the constants as $\hat{c}_1 = \frac{w_5 + w_4 \hat{z}}{(1+w_2)\hat{s}}, \hat{c}_2 = 1$ and $\hat{c}_3 = \frac{w_4}{w_7}$, we obtain, after some algebraic computation, that

$$\frac{dV_3}{dt} = -\left(\frac{w_5 + w_4 \hat{z}}{(1+w_2)\hat{s}}\right)\left[s + \hat{s} - \frac{w_1 \hat{z}}{s\hat{s}}\right](s-\hat{s})^2$$
$$-\left[\frac{w_5 + w_4 \hat{z}}{\hat{s}} - w_3 s\right]si$$
$$-\left[\frac{w_1(w_5 + w_4 \hat{z})}{(1+w_2)\hat{s}s} - \frac{w_4 w_6}{w_7}\right](s-\hat{s})(z-\hat{z})$$

Obviously, it is easy to verify that $\frac{dV_3}{dt}$ is negative definite under the following sufficient conditions

$$\frac{w_1 \hat{z}}{\hat{s}^2} = \gamma_1 < s < \gamma_2 = \min.\left\{\hat{s}, \frac{w_1 w_7 (w_5 + w_4 \hat{z})}{(1+w_2) w_4 w_6 \hat{s}}\right\}$$

with $0 \le z < \hat{z}$

Therefore, V_3 represents a Lyapunov function on Ω_3 and hence for any initial point in the region Ω_3 the solution of system (3) approaches asymptotically to E_3. Thus E_3 is globally asymptotically stable in Ω_3, which represents the basin of attraction of E_3. ■

Theorem (6): Suppose that the disease free equilibrium point $E_4 = (s^*, i^*, z^*)$ is locally asymptotically stable, then it's a globally asymptotically stable in the region $\Omega_4 \subset R_+^3$, where $\Omega_4 = \{(s,i,z) \in R_+^3 : \gamma_1 < s < \gamma_2, i > i^*, 0 \le z < z^*\}$, where $\gamma_j;\ j = 1,2$ are constants given in the proof.

Proof: Define the function

$$V_4 = c_1^*\left[s - s^* - s^* \ln\left(\frac{s}{s^*}\right)\right] + c_2^*\left[i - i^* - i^* \ln\left(\frac{i}{i^*}\right)\right]$$
$$+ c_3^*\left[z - z^* - z^* \ln\left(\frac{z}{z^*}\right)\right]$$

where $c_j^*;\ j = 1,2,3$ are positive constants to be determined, V_4 is continuously differentiable positive definite real valued function with $V_4(s^*, i^*, z^*) = 0$ and $V_4(s,i,z) > 0$ for all $(s,i,z) \ne (s^*, i^*, z^*)$ in the R_+^3. So by differentiate V_4 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_4}{dt} = -c_1^*\left[s + s^* - \frac{w_1 z^*}{ss^*}\right](s-s^*)^2$$
$$-\left[c_1^*(1+w_2) - c_2^* w_3 (s+s^*)\right](s-s^*)(i-i^*)$$
$$-\left[\frac{c_1^* w_1}{s} - c_3^* w_6\right](s-s^*)(z-z^*)$$
$$-\left[c_2^* w_4 - c_3^* w_7\right](i-i^*)(z-z^*)$$

So by choosing the constants as $c_1^* = \frac{w_3 s^*}{(1+w_2)}, c_2^* = 1$ and $c_3^* = \frac{w_4}{w_7}$, we obtain, after some algebraic computation, that

$$\frac{dV_4}{dt} = -\left(\frac{w_3 s^*}{1+w_2}\right)\left[s + s^* - w_1 \frac{z^*}{ss^*}\right](s-s^*)^2 + w_3 s(s-s^*)(i-i^*) - \left[\frac{w_1 w_3 s^*}{(1+w_2)s} - \frac{w_4 w_6}{w_7}\right](s-s^*)(z-z^*)$$

It is easy to verify that $\frac{dV_4}{dt}$ is negative definite in the region Ω_4, where

$$\gamma_1 = \frac{w_1 z^*}{s^{*2}} \text{ and } \gamma_2 = \min.\left\{s^*, \frac{w_1 w_3 w_7 s^*}{(1+w_2) w_4 w_6 s^*}\right\}$$

Therefore, V_4 represents a Lyapunov function on Ω_4 and hence for any initial point in the region Ω_4 the solution of system (3) approaches asymptotically to E_4. Thus E_4 is globally asymptotically stable in Ω_4, which represents the basin of attraction of E_4. ■

5. Local bifurcation

It is well known that the bifurcation theory is interested by the change in the qualitative behavior of the solution of a system (3) as variations in the control parameter. Therefore in this section an application to the Sotomoyor's theorem [26] is performed to study the occurrence of local bifurcation near the equilibrium points of system (3). It is well known that the existence of a non hyperbolic equilibrium point is a necessary but not sufficient condition for occurrence of local bifurcation. Therefore the parameters, which change the equilibrium points from hyperbolic to non hyperbolic equilibrium point, are considered as a candidate bifurcation parameters of system (3) as shown in the next theorems.

Consider the Jacobian matrix of system (3) that is given in Eq.(10), then we can find that for any vector $V = (v_1, v_2, v_3)^T$

$$D^2 F(s,i,z)(V,V) = (u_{ij})_{3\times 1} \quad (17a)$$

$$D^3 F(s,i,z)(V,V,V) = (\omega_{ij})_{3\times 1} \quad (17b)$$

here $u_{11} = -6sv_1^2 - 2(1+w_2)v_1 v_2$,

$u_{21} = 2w_3 i v_1^2 + 4w_3 s v_1 v_2 - 2w_4 v_2 v_3$,

$u_{31} = 2w_6 v_1 v_3 + 2w_7 v_2 v_3$

$\omega_{11} = -6v_1^3$, $\omega_{21} = 6w_3 v_1^2 v_2$ and $\omega_{31} = 0$.

Theorem (7): Assume that condition (12c) holds and the parameter w_6 passes through the value $w_6^* = w_8$, then system (3) near the disease-predator free equilibrium point E_1 undergoes transcritical bifurcation but neither saddle-node nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_1 with $w_6^* = w_8$ is given by $\tilde{J}(E_1, w_6^*) = J(E_1)$ with $\tilde{a}_{33} = 0$, hence E_1 becomes non-hyperbolic equilibrium point when the parameter w_6 passes through the value $w_6^* = w_8$ with zero eigenvalue $\tilde{\lambda} = 0$. Let $\tilde{V} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T$ be the eigenvector corresponding to $\tilde{\lambda} = 0$ in $\tilde{J}$. Then we get that

$$\tilde{V} = \left(-\frac{w_1}{2}\tilde{v}_3, 0, \tilde{v}_3\right)^T \text{ with } \tilde{v}_3 \in R \text{ and } \tilde{v}_3 \neq 0$$

Let $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)^T$ be the eigenvector corresponding to $\tilde{\lambda} = 0$ in $\tilde{J}^T$. Then we get that

$$\tilde{\Psi} = (0,0,\tilde{\psi}_3)^T \text{ with } \tilde{\psi}_3 \in R \text{ and } \tilde{\psi}_3 \neq 0$$

Now, since $\tilde{\Psi}^T F_{w_6}(E_1, w_6^*) = 0$, then according to Sotomayor's theorem saddle-node bifurcation can't occur. Further, straightforward computation shows that

$$\tilde{\Psi}^T [DF_{w_6}(E_1, w_6^*)\tilde{V}] = \tilde{v}_3 \tilde{\psi}_3 \neq 0$$

Also due to Eq. (17a) we get that

$$\tilde{\Psi}^T [D^2 F(E_1, w_6^*)(\tilde{V}, \tilde{V})] = -w_1 w_6 \tilde{v}_3^2 \tilde{\psi}_3 \neq 0$$

Therefore transcritical bifurcation takes place but not pitchfork bifurcation and hence the proof is complete. ■

Theorem (8): As the parameter w_8 passes through the value $w_8^* = w_6 \bar{s} + w_7 \bar{i}$, then system (3) near the predator free equilibrium point E_2 undergoes transcritical bifurcation but neither saddle-node nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_2 with $w_8^* = w_6 \bar{s} + w_7 \bar{i}$ is given by $\bar{J}(E_2, w_8^*) = J(E_2)$ with $\bar{a}_{33} = 0$, hence E_2 becomes non-hyperbolic equilibrium point when the parameter w_8 passes through the value $w_8^* = w_6 \bar{s} + w_7 \bar{i}$ with zero eigenvalue $\bar{\lambda}_z = 0$. Let $\bar{V} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)^T$ be the eigenvector corresponding to $\bar{\lambda}_z = 0$ in $\bar{J}$. Then we get that

$$\bar{V} = (\bar{\alpha}_1 \bar{v}_3, \bar{\alpha}_2 \bar{v}_3, \bar{v}_3)^T \text{ with } \bar{v}_3 \in R \text{ and } \bar{v}_3 \neq 0$$

here $\bar{\alpha}_1 = -\frac{\bar{a}_{23}}{\bar{a}_{21}} > 0$ and $\bar{\alpha}_2 = \frac{\bar{a}_{11}\bar{a}_{23} - \bar{a}_{13}\bar{a}_{21}}{\bar{a}_{12}\bar{a}_{21}} < 0$

Let $\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)^T$ be the eigenvector corresponding to $\bar{\lambda}_z = 0$ in $\bar{J}^T$. Then we get that

$$\bar{\Psi} = (0,0,\bar{\psi}_3)^T \text{ with } \bar{\psi}_3 \in R \text{ and } \bar{\psi}_3 \neq 0$$

Now, since $\bar{\Psi}^T F_{w_8}(E_2, w_8^*) = 0$, then according to Sotomayor's theorem saddle-node bifurcation can't occur. Further, straightforward computation shows that

$$\bar{\Psi}^T [DF_{w_8}(E_2, w_8^*)\bar{V}] = -\bar{v}_3 \bar{\psi}_3 \neq 0$$

Also due to Eq. (17a) we obtain that

$$\bar{\Psi}^T [D^2 F(E_2, w_8^*)(\bar{V}, \bar{V})] = 2(w_6 \bar{\alpha}_1 + w_7 \bar{\alpha}_2)\bar{v}_3^2 \bar{\psi}_3 \neq 0$$

Therefore transcritical bifurcation takes place but not pitchfork bifurcation and hence the proof is complete. ■

Theorem (9): Assume that condition (14d) along with the following condition are satisfied

$$\frac{w_4}{w_1}\left(1 - \left(\frac{w_8}{w_6}\right)^2\right) < \frac{w_3 w_8}{w_6} \quad (18a)$$

Then as the parameter w_5 passes through the value $w_5^* = w_3\hat{s}^2 - w_4\hat{z}$, system (3) near the disease free equilibrium point E_3 has

1. No saddle node bifurcation.
2. Transcritical bifurcation provided the following condition holds, otherwise it has pitchfork bifurcation

$$\frac{w_4(1+w_2)}{w_1}\hat{s} \neq 2\frac{w_3 w_7}{w_6}\hat{s} + \frac{w_4 w_7}{w_1 w_6}(3\hat{s}^2 - 1) \qquad (18b)$$

Proof: Clearly the Jacobian matrix of system (3) at E_3 with $w_5^* = w_3\hat{s}^2 - w_4\hat{z}$, which is positive under condition (18a), is given by $\hat{J}(E_3, w_5^*) = J(E_3)$ with $\hat{a}_{22} = 0$, hence E_3 becomes non-hyperbolic equilibrium point when the parameter w_5 passes through the value $w_5^* = w_3\hat{s}^2 - w_4\hat{z}$ with zero eigenvalue $\hat{\lambda}_i = 0$. Let $\hat{V} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T$ be the eigenvector corresponding to $\hat{\lambda}_i = 0$ in $\hat{J}$. Then we get that

$\hat{V} = (\hat{\alpha}_1\hat{v}_2, \hat{v}_2, \hat{\alpha}_2\hat{v}_2)^T$ with $\hat{v}_2 \in R$ and $\hat{v}_2 \neq 0$

here $\hat{\alpha}_1 = -\frac{\hat{a}_{32}}{\hat{a}_{31}} < 0$ and $\hat{\alpha}_2 = \frac{\hat{a}_{11}\hat{a}_{32} - \hat{a}_{12}\hat{a}_{31}}{\hat{a}_{13}\hat{a}_{31}}$

Let $\hat{\Psi} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)^T$ be the eigenvector corresponding to $\hat{\lambda}_i = 0$ in $\hat{J}^T$. Then we obtain that

$\hat{\Psi} = (0, \hat{\psi}_2, 0)^T$ with $\hat{\psi}_2 \in R$ and $\hat{\psi}_2 \neq 0$

Now, since $\hat{\Psi}^T F_{w_5}(E_3, w_5^*) = 0$, then according to Sotomayor's theorem saddle-node bifurcation can't occur. Further, straightforward computation shows that

$$\hat{\Psi}^T[DF_{w_5}(E_3, w_5^*)\hat{V}] = -\hat{v}_2\hat{\psi}_2 \neq 0$$

Moreover according to Eqs. (17a)-(17b), we obtain that

$$\hat{\Psi}^T[D^2F(E_3, w_5^*)(\hat{V}, \hat{V})] = 2(2w_3\hat{s}\hat{\alpha}_1 - w_4\hat{\alpha}_2)\hat{v}_2^2\hat{\psi}_2$$

$$\hat{\Psi}^T[D^3F(E_3, w_5^*)(\hat{V}, \hat{V}, \hat{V})] = 6w_3\hat{\alpha}_1^2\hat{v}_2^2\hat{\psi}_2$$

Therefore it is easy to verify that if condition (18b) holds then $\hat{\Psi}^T[D^2F(E_3, w_5^*)(\hat{V}, \hat{V})] \neq 0$ and transcritical bifurcation takes place. Otherwise $\hat{\Psi}^T[D^2F(E_3, w_5^*)(\hat{V}, \hat{V})] = 0$ and $\hat{\Psi}^T[D^3F(E_3, w_5^*)(\hat{V}, \hat{V}, \hat{V})] \neq 0$, thus pitchfork bifurcation takes place and hence the proof is complete. ■

Theorem (10): Assume that

$$1 < 3s^* + (1+w_2)i^* < 1 + \frac{w_6(1+w_2)s^*}{w_7} \qquad (19a)$$

Then as the parameter w_4 passes through the value

$$w_4^* = \frac{2w_1 w_3 s^*}{1 - 3s^* - (1+w_2)i^* + \frac{w_6(1+w_2)}{w_7}s^*} \qquad (19b)$$

system (3) near the positive equilibrium point E_4 undergoes saddle-node bifurcation but neither transcritical nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_4 with $w_4 = w_4^*$, which is positive under condition (19a), can be written as $J^*(E_4, w_4^*) = J(E_4)$ with $a_{23}^* = -w_4^* i^*$. Now straightforward computation shows that A_3 in the characteristic equation (15b) vanishing ($A_3 = 0$) at $w_4 = w_4^*$ and hence the Jacobian matrix $J^*(E_4, w_4^*)$ has zero eigenvalue, say $\lambda^* = 0$. Thus E_4 is a non-hyperbolic point.

Let $V^* = (v_1^*, v_2^*, v_3^*)^T$ be the eigenvector corresponding to $\lambda^* = 0$ in J^*. Then we get that:

$V^* = (v_1^*, \alpha_1^* v_1^*, \alpha_2^* v_1^*)^T$ with $v_1^* \in R$ and $v_1^* \neq 0$

here $\alpha_1^* = -\frac{a_{31}^*}{a_{32}^*} < 0$ and $\alpha_2^* = -\frac{a_{21}}{a_{23}} > 0$.

Let $\Psi^* = (\psi_1^*, \psi_2^*, \psi_3^*)^T$ be the eigenvector corresponding to $\lambda^* = 0$ in J^{*T}. Then we obtain that:

$\Psi^* = (\psi_1^*, \sigma_1^*\psi_1^*, \sigma_2^*\psi_1^*)^T$ with $\psi_1^* \in R$ and $\psi_1^* \neq 0$

here $\sigma_1^* = -\frac{a_{13}^*}{a_{23}^*} < 0$ and $\sigma_2^* = -\frac{a_{12}^*}{a_{32}^*} > 0$. Now since

$$\Psi^{*T} F_{w_4}(E_4, w_4^*) = -\sigma_1^* i^* z^* \psi_1^* \neq 0$$

Then according to Sotomayor's theorem the first condition of saddle-node bifurcation is satisfied. Moreover, due to Eq. (17a), we have

$$\Psi^{*T}[D^2F(E_4, w^*{}_4)(V^*, V^*)]$$
$$= -2[3s^* - (1+w_2)\tfrac{w_6}{w_7} + \tfrac{w_1 w_3}{w_4}]v_1^{*2}\psi_1^* \neq 0$$

Thus saddle node bifurcation occurs but neither transcritical nor pitchfork bifurcation can occur. ■

6. Hopf bifurcation

It is well known that a hopf bifurcation (also known as a Poincaré−Andronov−Hopf bifurcation) is a bifurcation point at which an equilibrium point alters stability and a limit cycle (period orbit) is initiated. Therefore, the occurrence of hopf bifurcation condition in the interior of positive octant of system (3) is established in the following theorem.

Theorem (11): Assume that

$$1 + \frac{w_6(1+w_2)s^*}{w_7} < 3s^* + (1+w_2)i^* < 1 + 2\frac{w_3 w_7 s^* i^*}{w_6} \qquad (20a)$$

Then system (3) undergoes a hopf bifurcation around the positive equilibrium point E_4 in the $Int.R_+^3$, when the parameter w_1 passes through the value:

$$w_1^* = \frac{-(1+w_2)[2w_3(1-3s^* - (1+w_2)i^*)s^* - w_4 w_6 z^*]s^* i^*}{w_6(1 - 3s^* - (1+w_2)i^* + 2\frac{w_3 w_7}{w_6}s^* i^*)z^*}$$

………… (20b)

Proof. According to hopf bifurcation theorem, system (3) undergoes a hopf bifurcation around the positive equilibrium point E_4 if and only if the Jacobian matrix $J(E_4)$ has two complex conjugate eigenvalues, say

$$\lambda(w_1) = \xi_1(w_1) \pm i\,\xi_2(w_1)$$

with the third eigenvalue real and negative such that $\xi_1(w_1^*)=0$ and $\frac{d}{dw_1}\xi_1(w_1)\Big|_{w_1=w_1^*}\neq 0$, which is known transversality condition.

Clearly w_1^* is positive under condition (19a), also we obtained that Δ in Eq. (15b) at $w_1=w_1^*$ became $\Delta(w_1^*)=A_1(w_1^*)A_2(w_1^*)-A_3(w_1^*)=0$ and hence the roots of the characteristic equation can be written as $\lambda_1^*(w_1^*)=-A_1$, which is negative under condition (20a), while

$$\left.\begin{aligned}\lambda_2^*(w_1^*)&=\mathrm{i}\sqrt{A_2(w_1^*)}=\mathrm{i}\,\xi_2(w_1^*)\\ \lambda_3^*(w_1^*)&=-\mathrm{i}\sqrt{A_2(w_1^*)}=-\mathrm{i}\,\xi_2(w_1^*)\end{aligned}\right\},$$

which are pure imaginary.

Now since in the neighborhood of $w_1=w_1^*$ there is a range in which the Jacobian matrix $J(E_4)$ has two complex conjugate eigenvalues, say $\lambda(w_1)=\xi_1(w_1)\pm\mathrm{i}\,\xi_2(w_1)$. Thus by substituting these eigenvalues in the characteristic equation and determine the derivative with respect to the bifurcation parameter w_1 and then comparing the two sides of the resulting equation with equating their real and imaginary parts we obtain that

$$\xi_1'(w_1)=-\frac{H_1(w_1)H_3(w_1)+H_2(w_1)H_4(w_1)}{H_1^2(w_1)+H_2^2(w_1)}$$

$$\xi_2'(w_1)=\frac{-H_1(w_1)H_4(w_1)+H_2(w_1)H_3(w_1)}{H_1^2(w_1)+H_2^2(w_1)}$$

here

$$H_1(w_1)=3\xi_1^2(w_1)-3\xi_2^2(w_1)+A_2(w_1)+2\xi_1(w_1)A_1(w_1)$$

$$H_2(w_1)=6\xi_1(w_1)\xi_2(w_1)+2\xi_2(w_1)A_1(w_1)$$

$$\begin{aligned}H_3(w_1)&=\xi_1(w_1)A_2'(w_1)+\xi_1^2(w_1)A_1'(w_1)\\&\quad-\xi_2^2(w_1)A_1'(w_1)+(A_1(w_1)A_2(w_1))'\end{aligned}$$

$$H_4(w_1)=2\xi_1(w_1)\xi_2(w_1)A_1'(w_1)+\xi_2(w_1)A_2'(w_1)$$

Moreover, we have

$$\begin{aligned}&H_1(w_1^*)H_3(w_1^*)+H_2(w_1^*)H_4(w_1^*)=\\&-2w_6z^*\left[2\frac{w_3w_7}{w_6}s^*i^*+1-3s^{*2}-(1+w_2)i^*\right]A_2(w_1^*)\neq 0\end{aligned}$$

Thus the transversality condition holds, and hence the proof is complete. ■

7. Numerical simulation

In this section, the global dynamics of system (3) is investigated numerically and the effects of all the parameters on the system's dynamics are discussed. The purpose of this study is to confirm our obtained analytical results and specify the control set of parameters. Thus system (3) is solved numerically using the following set of biologically feasible hypothetical data and then the trajectories of the system are drawn as shown below in Fig. (1).

$$\begin{aligned}&w_1=0.2, w_2=0.15, w_3=2w_2, w_4=0.4\\&w_5=0.1,\ w_6=0.15,\ w_7=0.3, w_8=0.2\end{aligned}\qquad(21)$$

(a)

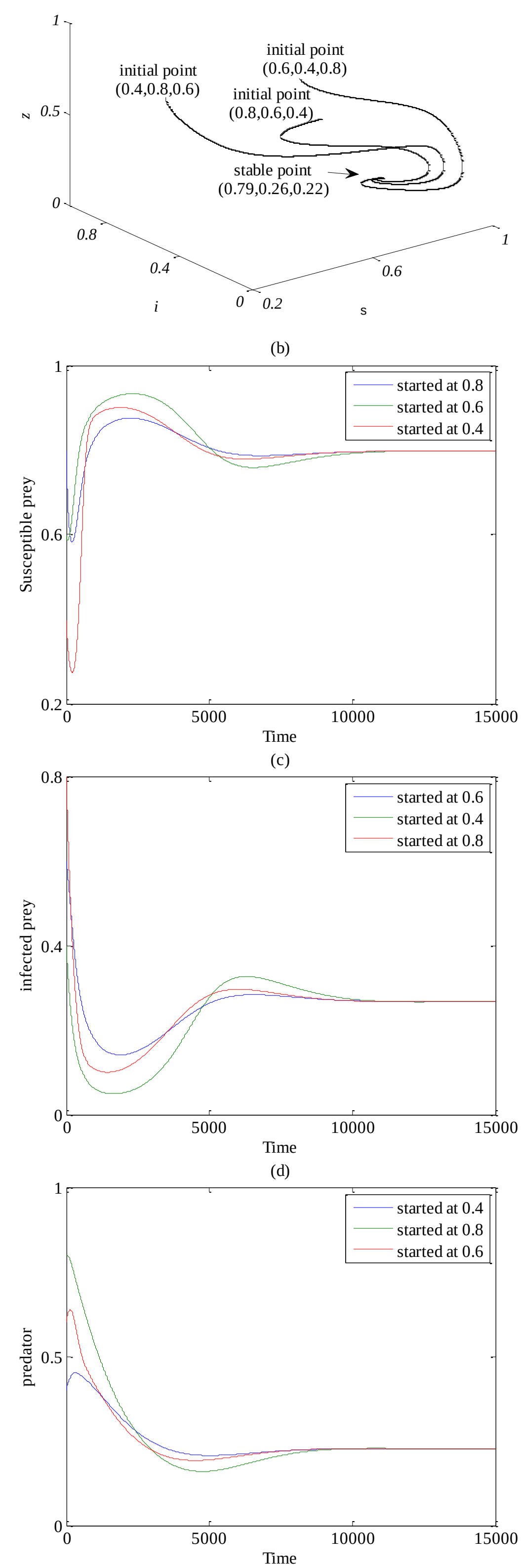

Fig.(1): The trajectories of system (3) approaches to $E_4=(0.79,0.26,0.22)$ asymptotically for the data (21) started from different initial points. (a) Phase portrait. (b) Time series

of phase portrait with respect to s. (c) Time series of phase portrait with respect to i. (d) Time series of phase portrait with respect to z

It is clear from Fig. (1) that the system (3) has a globally asymptotically stable positive equilibrium point, which confirm our obtained analytical result.

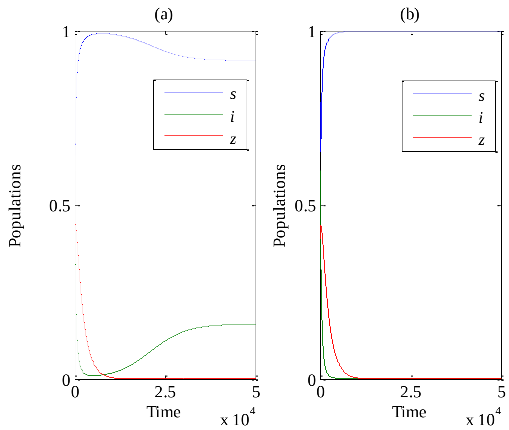

Fig. (2): The trajectory of system (3) as a function of time. (a) The solution approaches to $E_2 = (0.91, 0.15, 0)$ for the data (21) with $w_2 = 0.06$. (b) The solution approaches to $E_1 = (1, 0, 0)$ for the data (21) with $w_2 = 0.04$.

Now to investigate the behavior of the solution of system (3) when the parameters values are varying, the system is solved numerically for the set of data (21) with varying one parameter at a time. It is observed that, for the data given by (21) with $0.05 < w_2 \leq 0.067$ the trajectories of system (3) approach asymptotically to the predator free equilibrium point E_2 as shown in the typical figure, Fig. (2a), however for $0 < w_2 \leq 0.05$ the trajectories of system (3) approach asymptotically to the disease-predator free equilibrium point E_1 as explained in the typical figure given by Fig.(2b). Otherwise the solution of system (3) still approaches to the positive equilibrium point.

According to Fig.(2) the parameter w_2, which stands for the infection rate, pass through two bifurcation points; at the first point $w_2 = 0.067$ the solution approaches asymptotically to the predator free equilibrium point E_2 instead of the positive equilibrium point E_4, while the solution changes its stability again to the disease-predator free equilibrium point E_1 at the second bifurcation point $w_2 = 0.05$. Moreover the system (3) losses the persistence for $w_2 \leq 0.067$. Finally it is observed that the parameter $w_3 = 2w_2$ has similar behavior as that of w_2.

Now varying the parameter w_5, which stands for the infected species death rate, in the range $0.22 \leq w_5 < 0.3$ leads to destabilization of E_4 and the solution approaches to the predator free equilibrium point E_2 as shown in the typical figure, Fig.(3a). While for the range $0.3 \leq w_5$ the solution approaches to the disease-predator free equilibrium point E_1 as shown in Fig.(3b) below.

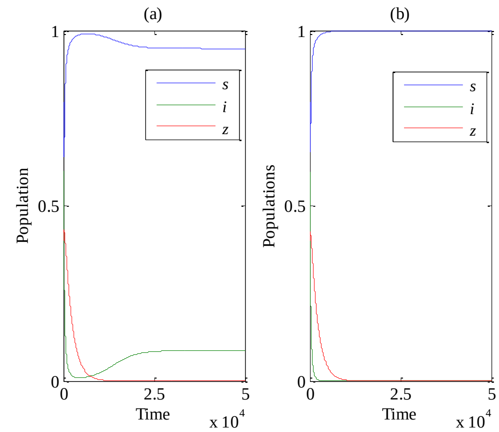

Fig. (3): The trajectory of system (3) as a function of time. (a) The solution approaches to $E_2 = (0.94, 0.08, 0)$ for the data (21) with $w_5 = 0.27$. (b) The solution approaches to $E_1 = (1, 0, 0)$ for the data (21) with $w_5 = 0.35$.

Again Fig.(3) shows that the parameter w_5 passes through two bifurcation points, at which the system change its stability behavior qualitatively and losing its persistence.

Now, varying the parameter w_6, which stands for conversion rate from susceptible species to predator, in the range $w_6 \leq 0.04$ leads to approaching to the predator free equilibrium point E_2 instead of positive equilibrium point E_4 as shown in Fig.(4a), while the solution approaches to the disease free equilibrium point E_3 when $0.21 \leq w_6 < 0.35$ as shown in Fig.(4b). Finally for $w_5 \geq 0.35$ the solution of system (3) approaches asymptotically to the periodic dynamics in the interior of $sz-$plane as explained in Fig.(5).

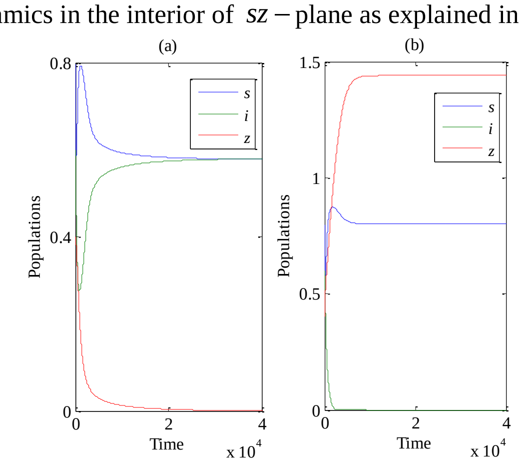

Fig. (4): The trajectory of system (3) as a function of time. (a) The solution approaches to $E_2 = (0.57, 0.57, 0)$ for the data (21) with $w_6 = 0.03$. (b) The solution approaches to $E_3 = (0.8, 0, 1.44)$ for the data (21) with $w_6 = 0.25$.

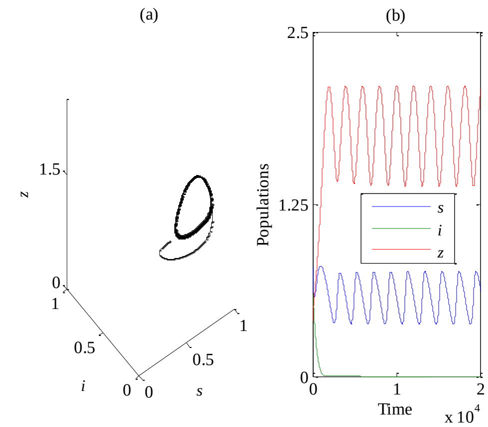

Fig. (5): The trajectory of system (3). (a) Periodic attractor in the interior of $sz-$plane for the data (21) with $w_6 = 0.35$. (b) The solution of system (3) as a function of time for the periodic attractor in (a).

Clearly, figures (4)-(5) explain that the system (3) undergoes qualitative change in their behavior when the parameter w_6 passes through three bifurcation points.

It is observed that the solution of system (3) approaches asymptotically to predator free equilibrium point E_2 when the parameter w_7, which stands for the conversion rate from the infected species to predator, varying in the range $w_7 \leq 0.19$ as shown in Fig.(6). While its still approaches to the positive equilibrium point otherwise.

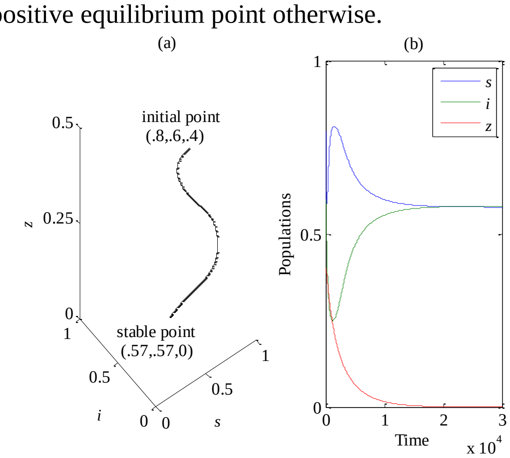

Fig. (6): The trajectory of system (3). (a) Asymptotically stable equilibrium point $E_2 = (0.57, 0.57, 0)$ in the interior of $sz-$plane for the data (21) with $w_7 = 0.15$. (b) The solution of system (3) as a function of time for the periodic attractor in (a).

Finally, for the data (21) with the parameter w_8, which stands for the death rate of predator species, in the range $w_8 \leq 0.14$ the solution of system (3) approaches asymptotically to the disease free equilibrium point E_3 in the interior of $sz-$plane as shown in Fig.(7a). However for the data (21) with $w_8 \geq 0.27$ the solution approaches asymptotically to the predator free equilibrium point E_2 in the interior of $si-$plane as shown in Fig.(7b) below. Moreover the system still persists at the positive equilibrium point otherwise.

Clearly figures (6)-(7) show the sensitivity of the behavior of solution of system (3) for varying in the parameters w_7 and w_8 respectively.

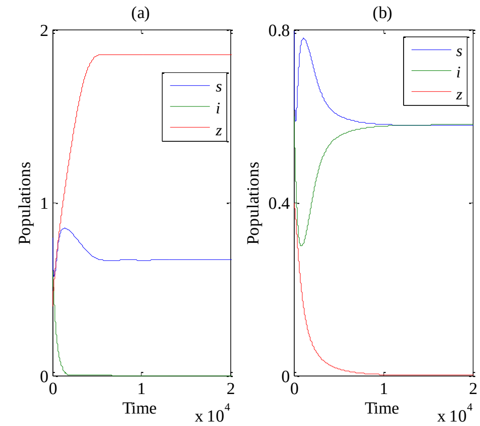

Fig. (7): The trajectory of system (3) as a function of time. (a) The solution approaches to $E_3 = (0.66, 0, 1.85)$ for the data (21) with $w_8 = 0.1$. (b) The solution approaches to $E_2 = (0.57, 0.57, 0)$ for the data (21) with $w_8 = 0.3$.

8. CONCLUSIONS AND DISCUSSION

In this paper the dynamics of prey-predator model with disease in prey and existence of herd behavior property is studied analytically as well as numerically. The model is formulated mathematically using system of first order nonlinear ordinary equations. The existence and uniqueness of solution of the proposed model are discussed. The boundedness of the solution is also studied. All possible equilibrium points with their local and global stability are investigated. The possibilities of occurrence of local bifurcations and hopf bifurcation are studied analytically. Finally, for the biologically feasible set of hypothetical data as given in Eq. (21), system (3) is solved numerically and the obtained results are explained in some typical figures and we will summarize them as follows.

1. Although, the linearized form of system (3) undergoes hopf bifurcation with respect to parameter w_1 as proved analytically under certain condition, the nonlinear system (3) has no periodic dynamics in the interior of R^3_+ for the data given in (21) rather than that it has periodic dynamics in the interior of $sz-$plane. But still there is possibility of having periodic dynamics for other set of data.
2. For the data given in Eq. (21), the trajectories of system (3) approached asymptotically to the global stable positive equilibrium point E_4 in the $Int.R^3_+$, which indicates to the persistence of all the species.
3. It is observed that decreasing the parameter related to infection rate w_2 destabilizing the positive equilibrium point and the system approaches asymptotically to predator free equilibrium point in the interior of $si-$plane first for a specific range and then approaches to disease-predator free equilibrium point on the $s-$axis for further decreasing in the parameter. Therefore decreasing this parameter prevents the persistence of the system. The

parameter w_3 has similar effects on the dynamics of system (3) as that shown with w_2.

4. Although, the linearized form of system (3) undergoes saddle node bifurcation with respect to parameter w_4 as proved analytically under certain condition, varying the parameter w_4 has no effect on the dynamical behavior of the nonlinear system (3) for the data given in (21). But still there is possibility of having bifurcation at this parameter for other set of data.
5. Increasing the parameter related to death rate of infected species w_5 destabilizing the positive equilibrium point and the system approaches asymptotically to predator free equilibrium point in the interior of $si-$plane first for a specific range and then approaches to the disease-predator free equilibrium point on the $s-$axis for further increasing in the parameter. Therefore increasing this parameter prevents the persistence of the system.
6. Decreasing the parameter related with conversion rate from the susceptible species w_6 causes destabilizing of the positive point and the solution approaches asymptotically to the predator free equilibrium point in the interior of $si-$plane. However increasing this parameters makes the solution approaches asymptotically to the disease free equilibrium point in the interior of $sz-$plane first and then to periodic dynamics in the same plane. Therefore the persistence of system (3) is sensitive to the varying in the parameter w_6.
7. Decreasing the parameter related with conversion rate from the infected species w_7 causes destabilizing of the positive equilibrium point and the solution approaches asymptotically to the predator free equilibrium point in the interior of $si-$ plane, which means the system will loss the persistence too.
8. Decreasing the parameter related with death rate of the predator w_8 causes destabilizing of the positive point and the solution approaches asymptotically to the disease free equilibrium point in the interior of $sz-$plane. However increasing this parameters makes the solution approaches asymptotically to the predator free equilibrium point in the interior of $si-$plane. Therefore the persistence of system (3) is sensitive to the varying the parameter w_8.

According to the above discussion, it's observed that the prey-predator system with the disease in prey and existence of herd prey's behavior property is very sensitive to any varying in the values of parameters especially those in predator equation.

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