SCALING FUZZY TOPOLOGY BY THE UNDERLYING CRISP TOPOLOGY

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ABSTRACT: *This paper is the first part devoted to the study of the papers of Burgin M.([1],[3] and [4]), in the case where our topologies are the fuzzy topologies and the scales are some particular fuzzy sets. The paper [1] introduces many important and interesting concepts. In order to elaborate our study we follow the same approach as in the references cited above and improve or complete any concept which seems ambiguous. In this paper, we consider fuzzy topological spaces are enrich them by additional structures in order to give them more realistic meaning and so make them useful by considering powerful technique developed in topology. As we have said, the suggested approach is based on the [1] and [3]. In our approach the initial topology will be the fuzzy topology on given set X and the scale will be the topology defined by some family of characteristic functions. We explore the (f, g)-continuity, weak (f, g)-continuity, and g-continuity. Different properties of scales, f -open, f -closed sets in the case of fuzzy topology are investigate. For the introduction of the notion of fuzzy topology we can see [3]. The references [4]-[7] can provide the reader with a complement on the fuzzy topology theory.*

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1 PRELIMINARIES

1.1 Scaled Spaces.

We start with conventional topological spaces. Let X be a topological space with a topology τ and τ' be a subset of τ, i.e. τ' consists of some open sets of X.

Definition 1. [1] *A scale or discontinuity structure f on> X is a mapping*: $f: X \longrightarrow 2^{\tau'}$ that satisfies the following conditions:

1. (SC1) For all points x from X, if O is an element of $f(x)$, then $x \in O$.
2. (SC2) $\forall O \in \tau', \exists x \in X$ such that $O \in f(x)$.

Remark 1.

1. From the axiom (SC1), we have $\bigcap_{o\in f(x)} O \neq \emptyset, \exists x \in X$
2. Sometimes, we can assume that the scale f on X satisfy the additional condition $(F): X$ is an element of $f(x)$ for all x in X. In **[1]** it is said that this condition implies the axiom (SC2), but this is not true to see this, it suffices to consider the following example.
Let $X = \{a, b, c\}$, τ be the discrete topology on X $\tau' = \{\{a\}, \{a, b\}, X\}$ and $f(a) = f(b) = f(c) = \{X\}$. It is clear that the last condition is satisfied but (SC2) is not.
3. It is easy to see from the axioms (SC1) and (SC2) that. $\emptyset \notin \tau'$ and $X \in \tau$
4. f is called a discontinuity structure because it determines admissible discontinuity, that is, it determines to what extent a mapping from one topological space into another may be discontinuous. Informally and since the elements of $f(x)$ are open sets and any $O \in f(x)$ is such $x \in O$ then O is a neighborhood of x, so f relates each point x of X to some set of neighborhoods of x. We call f −neighborhoods of x and denote by f the set $f(X)$.
5. As mappings are just special kinds of binary relations, it is possible to define for them all set theoretical relations like inclusion or intersection. With this identification, the inclusion of such mappings is just inclusion of images for all points of X. For example, if f and g are two discontinuity structures on a topological space X, $f \subset g$ means that $f(x) \subset g(x)$ for all x in X.

Definition 2. *A scaled or distructured topological space is a triplet (X, τ, f) where τ is a topology and f is a scale or a discontinuity structure on X.*

As it is possible to define different topologies on a given set, in a similar way, it is possible to define different scales in a given topological space.

Among the following examples we have extracted some of them from those given in [2], [3]. They are just an illustration of the introduced concepts.

Example 1. *Let X be a set and τ be the discrete topology on X. If $\tau = \{\{a\} \mid a \in X\}$ and $f: X \to 2^{\tau'}, f(x) = \{\{x\}\}$, then f is a scale on the topological space (X, τ).*

Example 2. *Let X be the n-dimensional Euclidean space $\mathbb{R}^n, a > 0$ and $\tau' = \{O \in \tau \mid \exists x \in X \; s.t \; \overline{B(x,a)} \subset O\}$. If $f_a(x)$ consists of all neighborhoods of a point x from $\mathbb{R}^n$, that contain a closed n −dimensional ball with the radius a and center x, f_a is a discontinuity structure on $\mathbb{R}^n$.*

Example 3. *Let X be a metric space with the metric d and $a > 0$. If we define $f(x)$ as the set of all connected neighborhoods of a point x containing an open ball with the radius a and center x.* If

$\tau' = \{connected\ neighborhods\ V \mid \exists x \in X \; s.t \; B(x,a) \subset V\}$

we get a discontinuity structure on X. We denote such scale f by f_a^d.

Example 4. *(P-structure). Let us1 fix for each point x in X some neighborhood O_x. Then we define $f(x)$ as the set of all open sets in X containing O_x. Such scale is called a P-structure. For example, the scales of examples 4 and 5 are P-structure on the space $\mathbb{R}^n$.*

Definition 3. *The scaled topological space(X,τ,f) is called a (F-structure) if for all x, the set $f(x)$ is a filter in τ of neighborhoods of x, that is, $f(x)$ is closed with respect to finite intersections and supersets of its elements. Scales from the example 4 is a F-structure. Moreover, it is principal F-structure or $P-$structure because all sets $f(x)$ are principal filters. On the other hand the scales of the examples 3 and 5 are not F-structures.*

Any discontinuity P-structure is a discontinuity F-structure.

Definition 4. *Let (X,τ,f) be a scaled topological space. We introduce the following axioms:*

1. (X,τ,f) is called U-structure if for all $x\in X, O\in f(x)$ and for any $O\in\tau'$; $O\cup O'\in f(x)$
2. (X,τ,f) is called a weak U-structure if for all $x\in X, O\in f(x)$ and $O\in\tau$; $O\cup O'\in f(x)$

Any discontinuity F-structure is a discontinuity U-structure.

Definition 5.

1. The scaled topological space (X,τ,f) is called a L-structure if for all x, $f(x)$ is a lattice of neighborhoods of x.
2. The scaled topological space (X,τ,f) is called a weak L-structure if $f(X)$ is a lattice.

1.2 Fuzzy Topology.

Definition 6. *Let X be a set called in the sequel the universe. A fuzzy subset is a mapping $\mu: X\to[0,1]$. If α is a constant in $[0,1]$ and μ is a fuzzy subset such that $\forall x\in X$, $\mu(x)=\alpha$, we denote this fuzzy subset by $\underline{\alpha}$.*

In the sequel a fuzzy set means a fuzzy subset in the unverse X.

Definition 7. *A family $\tau=\{\mu_i, i\in I\}$ of fuzzy sets of X is called a fuzzy topology for X if it satisfies the following three axioms:*

1. the fuzzy constant sets $\underline{0},\underline{1}\in\tau$,
2. if $\mu_1,\mu_2\in\mathrm{T}$ then $\mu_1\bigcap\mu_2\in\tau$,
3. if μ_i is a family (finite or infinite) of elements of τ then $\bigcup_i\mu_i\in\tau$.

It is clear that the fuzzy sets $\mu\cap\nu$ and $\mu\cup\nu$ are defined by

$\mu\cap\nu(x)=min\{\mu(x),\nu(x)\}$ and
$\mu\cup\nu(x)=max\{\mu(x),\nu(x)\}$

The pair (X,τ) is called a fuzzy topological space or FTS, for short. The elements of τ are called fuzzy open sets. A fuzzy set K is called fuzzy closed if $K^c\in\tau$. We denote by τ^c the collection of all fuzzy closed sets in this fuzzy topological space.

2 MAIN RESULTS

In the case of fuzzy sets theory, it is well known that if X is our universe and A is a subset of X, the expression $x\in A$ is replaced by $\mu_A(x)>0$, where μ is a mapping from X to the real interval $[0,1]$. In the case of crisp sets the above notion (when μ_A takes its values in the pair $\{0,1\}$), the mapping μ_A is replaced by the well known function; the characteristic function χ_A. So with this remark it is easy to adapt the above definitions with the concept of the fuzzy notions.

Definition 8. *The support of a fuzzy set μ is the set* $supp\mu=\{x\in X \mid \mu(x)\neq 0\}$.

Definition 9. *Let (X,τ) be a fuzzy topological space. A fuzzy set $\mu: X\to[0,1]$ is called a neighborhood of a point $x\in X$ if there exists an element $\alpha\in\tau$ such that $x\in supp\alpha$ and $\alpha\le\mu$*

Now introduce the concept of direct image and inverse image in term of fuzzy sets.

Definition 10. *Let $\phi:(X,\tau)\to(Y,\tau')$ be a mapping between two fuzzy topological spaces. If $\mu\in\tau$ and $\nu\in\tau'$. We define the sets:*

$$\phi(\mu)=\quad\alpha:\ Y\ \to[0,1]$$
$$y\ \to\alpha(y)=\begin{cases}\mu(x) & if\ \ y=\phi(x)\\ 0 & otherwise\end{cases}$$

and

$$\phi^{-1}(\nu)=\quad\gamma:\ X\ \to[0,1]$$
$$x\ \to\gamma(x)=\ \nu(\phi(x))$$

In the sequel we suppose that the supports of the elements of τ is a cover of the set X, that is $\bigcup_{\mu\in\mathrm{T}}supp\mu=X$ and this is insured by the fact that $1\in\tau$.

In the case of fuzzy topology the Definition 1 can be formulated as follows.

Definition 11. *Let X be a fuzzy topological space with a topology τ_f and τ' be a subset of τ_f, i.e., τ' consists of some open sets of X. A scale or discontinuity structure f on X is a mapping*

$f: X\to 2^{\tau'}$ that satisfies the following conditions:

1. (SC'1) $\forall x \in X$, if μ is an element of $f(x)$, then $\mu(x) = 1$.
2. (SC'2) $\forall \mu \in \tau'$, $\exists x \in X$ such that $\mu \in f(x)$.

Proposition 12. *Let τ be a fuzzy topology on a set X such that for all $x \in X$, $\exists A \subset X, x \in A$ and $\chi_A \in \tau$ and $\tau = \{\chi_A \in \tau \mid A \neq \emptyset\}$ be the set of all non identically null characteristic functions with X as universe and belonging to τ. If $f : X \to 2^{\tau'}$ is defined by $f(x) = \{\chi_A \mid x \in A \subset supp\mu, \mu \in \tau\}$. The triplet (X, τ, f) is a scaled topology on X.*

Proof. It is clear that $\forall x \in X = supp\underline{1} \in \tau$ so $\chi_X \in f(x)$ and $f(x) \neq \emptyset$ so f is well defined.

1. It is easy to see that $\tau' \subset \tau$ since any element of τ is in fact a particular fuzzy set.
2. Let $x \in X$ and $\mu \in f(x)$ then there exists $\emptyset \neq A \subset X$ such that $x \in A, \mu = \chi_A$ and so $\chi_A(x) = 1$.
3. Let $\chi_A \in \tau'$ then $A \neq \emptyset$ so there exists at least one element a of X in A and then $\chi_A \in f(a)$.

In addition this scaled topology verifies the condition F, that is $\forall x \in X$, $\chi_X \equiv 1 \in f(x)$.

Proposition 13. *The scaled topology of the above Proposition is a $F-$structure.*

Proof.

1. Let χ_A, χ_B be two elements of $f(x)$, then there exist two fuzzy sets $\mu_1, \mu_2 \in \tau$ such that $x \in A \subset supp\mu_1, x \in B \subset supp\mu_2$. As $x \in A \cap B \subset supp\mu_1 \cap supp\mu_2 \subset supp(\mu_A \cap \mu_B)$ and $\mu_A \cap \mu_B \in \tau$, then $\chi_{A\cap B} \in f(x)$, so $\chi_A \cap \chi_B \in f(x)$.
2. Let $x \in X$, $\chi_A \in f(x)$ and $\chi_A \subset \chi_B$. As $1 = \chi_A(x) \leq \chi_B(x)$, then $x \in B$ but $\chi_B \in \tau$, necessary $\chi_B \in f(x)$. Consequently the family $f(x)$ is a filter for any $x \in X$.

If $X \neq \emptyset$, the family τ' cannot be a base of the topology τ. That is there exists an element of τ, the constant fuzzy set $\mu(x) = \frac{1}{2}$ and an element $x \in supp(\mu)$ such that for all $X \supset A \neq \emptyset$ if $x \in supp(\chi_A), \chi_A$ is not less than μ.

Example 5. *Let $X = \mathbb{N}$ and $\tau = \{\mu_i : X \to [0,1]\} \cup \{\chi_A \mid \emptyset \neq A \subset X\}$, where the fuzzy sets μ_i are defined by*

$$\forall x \in X, \mu_i(x) = \begin{cases} 1 & if\ x \leq i-1 \\ \frac{i}{x} & if\ x \geq i \end{cases}$$

It is easy to prove that $\forall i \in X, \underline{0} \leq \mu_i \leq \mu_{i+1} \leq \underline{1}$ so the family τ is a fuzzy topology on X.

Now if we set $\tau' = \{\chi_A \mid \emptyset \neq A \subset X\}$ and $\forall x \in X$, $f(x) = \{\chi_A \in \tau \mid x \in A\}$, the triplet (X, τ, f) is a scale of the fuzzy topological space (X, τ). Indeed

1. $\forall x \in X$, if $\mu \in f(x)$ then there exists a nonempty subset A of X such that $\mu = \chi_A$ and $x \in A$. It follows that $\mu(x) = 1$.
2. In another hand if $\mu \in \tau'$ then there exists $\emptyset \neq A \subset X$ such that $\mu = \chi_A$. And since A is not empty then there exists at least one element $a \in A$. Consequently $\mu(a) = \chi_A(a) = 1$ and $\mu \in f(a)$.

Example 6. *Let $X = \mathbb{N}$ and $\tau = \{\mu_i : X \to [0,1]\} \cup \{\chi_A \mid \emptyset \neq A \subset X\}$, where the fuzzy sets μ_i are defined by*

$$\forall x \in X, \mu_i(x) = \begin{cases} 1 & if\ x \leq i-1 \\ \frac{i}{x} & if\ x \geq i \end{cases}$$

It is easy to prove that $\forall i \in X, \underline{0} \leq \mu_i \leq \mu_{i+1} \leq \underline{1}$ so the family τ is a fuzzy topology on X.

Let for $a \in X, |[a, \infty|[$ denotes the set of all naturals great or equal to a. If we set $\tau' = \{\chi_A \mid \exists a \in X \text{ and } A = |[a, \infty|[\}$ and

$\forall x \in X$, $f(x) = \{\chi_A \in \tau \mid |[x, \infty|[\subset A\}$, the triplet (X, τ, f) is also a scale of the fuzzy topological space (X, τ). Indeed

1. $\forall x \in X$, if $\mu \in f(x)$ then there exists a nonempty subset A of X such that $\mu = \chi_A$ and $[x, \infty|[\subset A$. It follows that $\mu(x) = 1$.
2. In another hand if $\mu \in \tau'$ then there exists a set $A = |[x, \infty|[\subset X$ such that $\mu = \chi_A$. Then $\mu \in f(x)$.

Definition 14. *Let f be a scale of a fuzzy topological space (X, τ) associated to τ'. If $g : X \to 2^{\tau'}$ is a scale on (X, τ) such that $\forall x \in X$, one has $g(x) \subset f(x)$ we say that g is a subscale of the scale f.*

Definition 15. *Let f be a scale of a fuzzy topological space*

(X,τ) associated to τ' and $g: X \to 2^{\tau'}$ be a subscale of the scale f. We say that f has a nonempty trace on g if for all $x \in X$ and $A \in f(x)$ there exists $A' \in g(x), A'' \in \tau'$ such that $A = A' \cup A''$.

Proposition 16. *Let f be a scale of a fuzzy topological space (X,τ) associated to τ' and $g: X \to 2^{\tau'}$ be a subscale of the scale f such that f has a nonempty trace on g. If the scale g is a $U-$structure then so is f.*

Proof.

Let $A \in f(x)$ and A_i be an arbitrary family of elements of τ'. As $A \in f(x)$ and f has a nonempty trace on g, there exists $A' \in g(x), A'' \in \tau'$ such that $A = A' \cup A''$. But $\bigcup_i A_i \cup A = (\bigcup_i A_i \cup A'') \cup A'$ which is in $g(x)$ by hypothesis and so it is in $f(x)$ and f is a $U-$structure.

Definition 17. *Let f be a scale on a fuzzy topological space (X,τ) associated to τ'. Any element of $f(x)$ is called $f-$open and its complement in X is called a $f-$closed set.*

Definition 18. *Let f be a scale on a fuzzy topological space (X,T) associated to T'. Then*

1. f is a $I-$ structure if
$$\forall x \in X, \forall A \in f(x), \forall O \in \mathrm{T}',\ A \cap O \in f(x).$$
2. f is a weak $I-$ structure if
$$\forall x \in X, \forall A, B \in f(x),\ A \cap B \in f(x).$$

Proposition 19.

1. A scale f is a weak $U-$structure if and only if the intersection of $f-$closed sets is a $f-$closed set.
2. A scale f is a weak I-structure if and only if any finite union of $f-$closed sets is a $f-$closed set.

Proof.

1. Let $\{F_i,\ i \in I\}$ be a family of $f-$closed sets, then there exists a family $\{O_i,\ i \in I\}$ of $f-$open sets such for all $\forall i \in I, F_i = \complement^{O_i}$. So $\cap_i F_i = \cap_i \complement^{O_i} = \complement^{(\cup_i O_i)}$
By definition of the weak $U-$structure $\bigcup_{i \in I} O_i \in f(X)$ and its complement is then a $f-$closed set. The converse is also easy to establish.

2.Let $x \in X$ and $\{F_i, i \in I\}$ be a finite family of closed sets of $f(x)$. For all $i \in I$, $\complement^{F_i}$ is an open set of $f(x)$ so since f is a weak $I-$structure the finite intersection of the family $\{\complement^{F_i}\}$ is still in $f(x)$ but $\cap_i \complement^{F_i} = \complement^{\cup_i F_i}$. And so $\bigcup_i F_i$ is a closed set.

It is easy to prove the following corollary.

Corollary 20. *A scale f is a weak $L-$structure if and only if the intersection of two $f-$closed sets is a f-closed set and the union of two f-closed sets is a f-closed set.*

Let us assume that (X,τ_1,f) and (Y,τ_2,g) are scaled topological spaces. In the following definition, a neighborhood of x (resp. y) relatively to $f(x)$ (resp. $g(y)$) means a subset A (resp. B) of X (resp. Y) such that there exists an open set O in $f(x)$ (resp. $O' \in g(y)$) such that $O \subset A$ (resp. $O' \subset B$).

Definition 21. *A mapping $\phi: X \to Y$ is called:*

1. (f,g)-continuous at a point $x \in X$ if for $y = \phi(x)$ and any element neighborhood O_y of y relatively to $g(y)$, the set $\phi^{-1}(O_y)$ is a neighborhood of x relatively to $f(x)$.
2. locally (f,g)-continuous if it is (f,g)-continuous at all points of X.
3. (f,g)-continuous if for any g-open set V, the set $\phi^{-1}(V)$ is f-open.
4. g-continuous at a point $x \in X$ if for all $W \in g(\phi(x)),\ (g \circ \phi)^{-1}(W)$ is a neighborhood of x in (X,τ_1).
5. Locally g-continuous if it is g-continuous at all points of X.
6. g-continuous if for any $g-$open set O, $\phi^{-1}(O) \in \tau_1$.
7. f-continuous at a point $x \in X$ if for any neighborhood $W \in \tau_2'$ of $\phi(x), (\phi)^{-1}(W)$ is in $f(x)$.
8. f-continuous if ϕ is f-continuous at any point $x \in X$.
9. Partially f-continuous if for any $O \in \tau_2'$, $\phi^{-1}(O) \in f(x)$ for some $x \in X$.

As one can remark the above definition is not exactly the Definition 9 in [Burgin]. We have introduced some change on the first point, otherwise the Example 12 is not true as we will give Why.

In [Burgin], the author gives a lemma (Lemma 1) without a proof. We think that if the mapping is continuous at x, it need not be (f,g)-continuous at x since the inverse image of an element of $g(y)$ need not be in $f(x)$. But in our case of fuzzy topologies as chosen above, the following propositions are the case where continuity at a point implies the (f,g)-continuity at this point.

Proposition 22. *Let $\phi: X \to Y$ be a mapping, (X,τ_1,f) and (Y,τ_2,g) are the scaled topological spaces as defined in Proposition 16. If a mapping ϕ is continuous at $x \in X$, then it is (f,g)-continuous at x.*

Proof. Suppose that $y = \phi(x)$ and $\chi_B \in g(y)$ then there exist $B \neq \emptyset$ and $\chi_B \in \tau_2$. As ϕ is continuous at x and $\chi_B \in \tau_2$. then $\phi^{-1}(\chi_B) \in \tau_1$. Let us prove that $\chi_{\phi^{-1}(B)} = \phi^{-1}(\chi_B)$.

As introduced in Definition 14,

$$\phi^{-1}(\chi_B) = \quad \gamma: \quad X \to [0,1]$$
$$z \to \gamma(z) = \chi_B(\phi(z))$$

So as

$z \in \phi^{-1}(B) \Leftrightarrow \phi(z) \in B$,then

$\chi_{\phi^{-1}(B)}(z) = 1 \Leftrightarrow \phi^{-1}(\chi_B)(z) = 1.$

$\phi^{-1}(\chi_B) \in \tau_1 \Rightarrow \chi_{\phi^{-1}(B)} \in \tau_1$ and then $\chi_{\phi^{-1}(B)} \in \tau_{1'}$.

The result follows.

Proposition 23. *Let $\phi: X \to Y$ be a mapping, (X,τ_1,f) and (Y,τ_2,g) are the scaled topological spaces as defined in Proposition 16. If a mapping ϕ is (f,g) continuous at $x \in X$, then it is continuous at x.*

Proof.

Let μ be a neighborhood of $y = \phi(x)$, then there exists $\beta \in \tau_2$ such that $x \in supp\beta$ and $\beta \leq \mu$. We want to prove that $\phi^{-1}(mu)$ is a neighborhood of x, that is there exists $\gamma \in \tau_1$ such that $x \in supp\gamma$ and $\gamma \leq \phi^{-1}(mu)$.

Let $\chi_B \in g(y)$ then there exists $A \subset X$ such that $x \in A$ and $\phi^{-1}(\chi_B) = \chi_A$. If we take $\gamma = \mu \cap \chi_B$, then necessary $\gamma = \chi_C$ and C is not empty since it contains y. It is clear that $x \in supp\gamma$. On the other hand since γ is the intersection of μ with another fuzzy set its value at any point is less than the value of μ at this point so $\gamma(\phi(z)) \leq \mu(\phi(z))$ and this expression is equivalent to

$$\phi^{-1}(\gamma)(z) \leq \phi^{-1}(\mu)(z), \forall\, z \in X \;\Leftrightarrow\; \phi^{-1}(\gamma) \leq \phi^{-1}(\mu).$$

Corollary 24. *Let $\phi: X \to Y$ be a mapping, (X,τ_1,f) and (Y,τ_2,g) are the scaled topological spaces as defined in Proposition 16. Then ϕ* is continous if and only if it is locally $(f,g)-$ continous.

Proposition 24. *If X, Y are as in the above propositions and $\phi: X \to Y$ is a surjective locally (f,g)-continuous mapping, then ϕ is (f,g)-continuous.*

Proof.

Indeed, let ϕ be a locally (f,g)-continuous surjection and O be an g-open set. Then there exists $y \in Y$ such that $O \in g(y)$. As ϕ is a surjection, there is an element $x \in X$ such that $\phi(x) = y$. By the hypothesis on $\phi, \phi^{-1}(O) \in f(x)$ and $\phi^{-1}(O)$ is $f-$open.

As said above, the function given in Example 12 of [Burgin] is not locally (f,g)-continuous in the sense of Definition 9 of the same paper. To see this it suffices to take $x = 0.51$, then there exists a neighborhood O_y =]0.31,0.71[in $g(y)$,but

$\phi^{-1}(O_y)$ =]0.31,0.5[$\cup$]0.5,0.71[$\notin f(x)$. On the other hand and in the sense of Definition 27 above, the same function becomes locally (f,g)-continuous.

Inspired by the example 12 in [3], we show that also in the case of our fuzzy scaling, the surjectivity of ϕ is necessary.

Example 7. *Let $X = [0,\frac{1}{2}) \cup (\frac{1}{2},1], Y = [0,1]$ and the mapping $\phi: X \to Y$, the identity on X, i.e., $\phi(x) = x$ for all $x \in X$. For an arbitrary point z either from X or from Y, both scales $f(z)$ and $g(z)$ consist of all characteristic functions of the form $\chi_{(z-\varepsilon,z+\varepsilon)}, \varepsilon > 0$ with domain respectively in X or in Y.*

ϕ is locally (f,g)-continuous, indeed for any $x \in X$ and any neighborhood $\chi_{(x-\varepsilon,x+\varepsilon)}$ of $y = x$ relatively to $g(y)$ there exists an element χ_A of $g(y)$, such that

$$A = \begin{cases} (x-\varepsilon, x+\varepsilon) & if \;\; x-\varepsilon > \frac{1}{2} \;\; or \;\; x+\varepsilon < \frac{1}{2} \\ (x-\varepsilon, \frac{1}{2}) & if \;\; x-\varepsilon < \frac{1}{2} \;\; and \;\; x+\varepsilon \geq \frac{1}{2} \\ (\frac{1}{2}, x+\varepsilon) & if \;\; x-\varepsilon \leq \frac{1}{2} \;\; and \;\; x+\varepsilon > \frac{1}{2} \end{cases}$$

and then

$$\phi^{-1}(\chi_A) = \begin{cases} \chi_{(x-\varepsilon,x+\varepsilon)} & if \;\; x-\varepsilon > 1/2 \;\; or \;\; x+\varepsilon < 1/2 \\ \chi_{(x-\varepsilon,1/2)} & if \;\; x-\varepsilon < 1/2 \;\; and \;\; x+\varepsilon \geq 1/2 \\ \chi_{(1/2,x+\varepsilon)} & if \;\; x-\varepsilon \leq 1/2 \;\; and \;\; x+\varepsilon > 1/2 \end{cases}$$

It is clear that $\phi^{-1}(\chi_A)$ is a neighborhood of x since there exists $\chi_B \in f(x)$ and $\chi_B \subset \phi^{-1}(\chi_A)$ with

$$B=\begin{cases}(x-\varepsilon,x+\varepsilon) & if\ \ x-\varepsilon>\frac{1}{2}\ \ or\ \ x+\varepsilon<\frac{1}{2}\\ (x-\delta,x+\delta),\delta\le\min\{\varepsilon,\frac{1}{2}-x\} & if\ \ x-\varepsilon<\frac{1}{2}\ \ and\ \ x+\varepsilon\ge\frac{1}{2}\\ (x-\delta,x+\delta),\delta\le\min\{\varepsilon,x-\frac{1}{2}\} & if\ \ x-\varepsilon\le\frac{1}{2}\ \ and\ \ x+\varepsilon>\frac{1}{2}\end{cases}$$

On the other hand the fuzzy set $\chi_{(0,1)}$ is an element of $g\left(\frac{1}{2}\right)$ but its inverse image by ϕ can not be a $f-$open set, since $\phi^{-1}(\chi_{(0,1)})=\vee\{\chi_{(0,\frac{1}{2})},\chi_{(\frac{1}{2},1)}\}$. Consequently ϕ is not (f,g)-continuous.

If we suppose that the scale is such any element of τ_2' is a neighborhood of its elements, the condition on the surjectivity of ϕ in Proposition 31 can be weakened as follows.

Proposition 25. *If X,Y are such that for any $y\in Y$, for any $O\in g(y)$ there exist $z\in O, x\in X$ satisfying $z=\phi(z)$. If $\phi:X\to Y$ locally (f,g)-continuous mapping, then ϕ is (f,g)-continuous.*

Proof.

Indeed, let ϕ be a locally (f,g)-continuous. If $y\in Y$, $O\in g(y)$ there exist $z\in O, x\in X$ such that $z=\phi(x)$. As O is also a neighborhood of z relatively to $g(z)$ and ϕ be a locally (f,g)-continuous, then it is (f,g)-continuous at x. This implies that $\phi^{-1}(O)\in f(x)$ and the proposition follows.

A natural question rises from the above results. What happens when only the arrival topological space or the departure space is endowed with a scale?

In [3], the author explored the relationship between the different types of the $g-$continuity, in the following we establish some relationship between the different types of the f-continuity.

3 f-continuity

To begin with this section let us give an example where the mapping is partially $f-$continuous but not continuous. To show this let us return to Example 32 and where the topology on Y is the usual topology.

Example 8. *The mapping ϕ is partially f-continuous but not $f-$continuous at the point 0.51 for example. To see this, let $\chi_{]0.3,0.6[}$ be an open fuzzy set of (Y,τ). It is clear that there exists $z=0.45$ such that $\phi^{-1}(\chi_{]0.3,0.6[})\in f(z)$ but $\phi^{-1}(\chi_{]0.3,0.6[})\notin f(x)$. So ϕ can not be $f-$continue at x and then not $f-$continue.*

In the sequel we suppose that the scale f verifies the condition (SSC1), if

$\forall x,y\in\tau_1$ and $O\in f(x)$, then O is also an element of $f(y)$

Proposition 26. *Let $\phi:X\to Y$ be a mapping (X,τ_1,f), (Y,τ_2,g) be a scaled topological space and (Y,τ_2) a topological space. If a mapping ϕ is partially (f)-continuous on X, then it is f-continuous on X.*

Proof. Let $x\in X$ and $O\in\tau_2'$ a neighborhood of $\phi(x)$. As $\phi(x)$ is partially f-continue there exists $z\in X$ such that $\phi^{-1}(O)\in f(z)$. But the condition (SC1), we get $z\in\phi^{-1}(O)\in f(z)$ and since $x\in\phi^{-1}(O)$, by (SSC1), we get $\phi^{-1}(O)\in f(x)$ and the result follows.

Proposition 27. *Let $\phi:X\to Y$ be (f,g)-continue at $x\in X$ and $\psi:Y\to Z$ be $g-$continue at $\phi(x)$. The mapping $\psi\circ\phi$ is $f-$continue at x.*

Proof. If $x\in X$ then $\phi(x)\in Y$ and as ψ is $g-$continue at $\phi(x)$, for any neighborhood $W\in\tau_3$ of $\psi(\phi(x)),\psi^{-1}(W)\in g(\phi(x))$. On the other hand as ϕ is $(f,g)-$continue at x and $\psi^{-1}(W)\in g(\phi(x))$ so a neighborhood of $\phi(x)$, then $\phi^{-1}(\psi^{-1}(W))\in f(x)$. From the equality $\phi^{-1}(\psi^{-1}(W))=(\psi\circ\phi)^{-1}(W)$ we deduce the result.

4 CONCLUSION.

In this paper, it is demonstrated how conventional fuzzy topological spaces can be scaled by crisp sets. The goal is to study the situations when it is reasonable to disregard relatively small gaps or when there is no information about them. Such situations are frequent topology. This study is done in discontinuous topology, which is a new field, in which methods and constructions of the classical topology are utilized for investigation of the notion of (f-g)-continuity. The association of the discontinuous structure and topological methods gives birth to the name discontinuous topology. It is sure that the results obtained in this topic make possible to obtain classical results for continuous mappings as direct corollaries. They show that discontinuous topology is a natural extension of the classical topology, which illustrate real-life situations such as computation or measurement of topological characteristics. Many other properties that can be developed in these topics. As topology is developed in the context of categories and functors (cf., for example Johnstone [6]). It might be interesting to build a discontinuous topology in a categorical setting. Among our objectives in future works, we will be concerned by the completion of this study and the investigation on some questions posed by Burgin. in [1] and [3].

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