THE DYNAMICAL OF A HARVESTED PREY-PREDATOR MODEL WITH THE SAME DISEASE IN THE PREY AND PREDATOR

Dina Sultan Al-Jaf
Department of Mathematics, College of Since, University of Baghdad, Baghdad, Iraq
dinasialjaf@gmail.com

ABSTRACT: *In this paper the dynamism of a harvested prey-predator model with the same disease inthe prey and predator is projected and analyze. We will say that each population, which are divided in to two classes, which are susceptible population and infected population because of exposure to the infectious disease. We will say that the susceptible predator which is only attack the prey (susceptible and infected) and the susceptible predator which are only feed on the susceptible prey and infected prey. we will say the prey (susceptible and infected) are the only one that will be affected by harvest. All analyzed local and global stability. All equilibria biological points accepted for the system we studied have dynamic behavior. with be of assistance of appropriate Lyapunov functions The global ynamics of the model are explore. Conditions Persists of the model are recognized.*

"Keywords": harvest, prey-predator model, infectious disease, stability Lyapunov Function, persistence.

1. INTRODUCTION

There has been going up concern in the revision of diseases in prey-predator models. It is well made out so as to, in usual world species does not stay alive alone. In truth, any firm habitat might contain dozens or hundreds of species, from time to time thousands. in view of the fact that any species has smallest amount the impending to act together species in its habitat, the possibility of partition of the disease in a society hurriedly becomes enormous as the number of infected species in the habitation increases. It is more of biological consequence to schoolwork the achieve of disease on the dynamicalactions of interacting species. In the last decades, some prey-predator models by wayof infectious disease have been painstaking [1,2,4,5,6,10,11,12,13]. Every one of these studies, reached by the winding up that disease may reason vital changes in the dynamics of an ecological unit. On the other offer, harvesting has in general a strappingcontact on the population dynamics of a harvested class. The severity of this contactdepends on the character of the implemented harvesting tactic, which in revolve mayseries from the express reduction to the complete perpetuation of a population.The study of inhabitants dynamics with harvesting is a issue of mathematical bio- economics, and it is related to the most favorable administration of renewablepossessions [8]. The effect of even charge of harvesting on the dynamical performance of interacting species has been measured by countless researchers [3,8,9]. these studies reached to conclusions of canister be alive summarize as follows:

Harvesting may well be worn as a biological manage pro the coexistence of the species, but unfettered harvesting strength go ahead to destruction in one or extra species. Keeping the above in sight, the outcome of disease on the dynamical conduct of the harvested prey-predator models, is important from economical position. Little thought has been salaried so future in this path.\ Recently,"Chattopadhyay et al"[7], projected and analyze a model in mathematical on the a harvested prey-predator model with disease in prey population. They unsaid that, the predator feeds on top of the susceptible prey inhabitants according to "Holling type-II" functional response, at the same time as it feeds on infected prey inhabitants according to "Lotka-Voltera" predation type. They reached to the subsequent answer, harvesting of infected prey may well be used as biological organize for the persistence in an infected prey-predator model.In this paper, " Chattopadhyay et al"[7] modified by assuming that only the susceptible predator feeds on the prey (susceptible and infected) according to Lotka-Voltera functional response type. In this paper, a prey-predator model relating both a harvesting and infectious disease is projected and analyzed. The effects of the harvest and infectious disease on the dynamical behavior of prey-predator model are well thought-out analytically as well as numerically

2. "Mathematical model"

In this paper, an eco-epidemiological model is proposal used for study. The model onsists of a prey, whose entirety population density is denote by $X(t)$, interacting with predator whose entirety population density is denoted by $Y(t)$.It is assumed that the prey population are subjected to infection by a disease and a harvested while the predator population are subjected to infection by a disease only. Further, the following assumptions of the dynamics of a harvested prey-predator model with the same disease in the prey and predator are made in formulating the basic eco epidemiological model:

1. epidemic disease of form SI survive in prey population divides the population prey in to two classes that is $S_1(T)$ which represents at time T the density of susceptible prey species and $I_1(T)$ which represents the density of infected prey species at time T. then at any T ,we have $X(T) = S_1(T) + I_1(T)$**.** additional susceptible prey becomes infected by a disease at a specific infection rate of $c_1(c_1 > 0)$.

2. epidemic disease of form SI survive in predator population divides the population predator in to two classes that is $S_2(T)$ which represents the density of susceptible Predator species at time T and $I_2(T)$ which represents the density of infected predator species at time T.Therefore at any T,we have $Y(t) = S_2(T) + I_2(T)$. added susceptible prey becomes infected by a disease at a specific infection rate of $c_2(c_2 > 0)$ **.3.** It is unsaid that the susceptible prey are alone able of reproducing logistically by carrying capacity $K(K > 0)$ and an inherent birth pace constant $r(r > 0)$, and that the infected prey population dies, by specific death pace constant $d_1(d_1 > 0)$, before having the chance to reproduce. However, the infected prey population still contributes along with susceptible prey population to population growth towards the carrying capacity .Then the evolution equations of the prey can represented as:

$$\frac{dS_1}{dT} = rS_1\left(1-\frac{S_1+I_1}{k}\right) - c_1S_1I_1 - h_1S_1; S_1(0) \geq o$$
$$\frac{dI_1}{dT} = c_1S_1I_1 - d_1I_1 - h_2I_1; I_1(0) \geq 0 \qquad (1)$$

4. It is unsaid that as the population of predator attack the population of prey the same disease which exist in the prey population separate in predator population.

5. In the nonexistence of the prey, the predator (susceptible and infected) decay exponentially with natural death rates $d_2(d_2 > 0)$ and $d_3(d_3 > 0)$ correspondingly.

6. we will study the effect of a harvesting only on the prey with its susceptible and infected, non-negative constants h_1 and h_2 are represent the harvesting efforts for the susceptible and infected prey correspondingly.

7. To finish, it is unsaid that only the predator susceptible consume the prey (without make different between infected I_1 and well S_1 prey) according to Lotka Voltera form of functional response at constant consumption rates $a_1 > 0$ (from susceptible prey) and $a_2 > 0$ (from infected prey).

Consequently, the dynamics of a harvested prey-predator model with the same disease in the prey and predator be able to represented in the next set of equations:

$$\frac{dS_1}{dT} = rS_1\left(1-\frac{S_1+I_1}{k}\right) - c_1S_1I_1 - h_1S_1 - a_1S_1S_2 = S_1 f_1(S_1, I_1, S_2, I_2)$$
$$\frac{dI_1}{dT} = c_1S_1I_1 - a_2I_1S_2 - (d_1+h_2)I_1 = I_1 f_2(S_1, I_1, S_2, I_2)$$
$$\frac{dS_2}{dT} = e_1a_1S_1S_2 + e_2a_2I_1S_2 - c_2S_2I_2 - d_2S_2 = S_2 f_3(S_1, I_1, S_2, I_2)$$
$$\frac{dI_2}{dT} = c_2S_2I_2 - d_3I_2 = I_2 f_4(S_1, I_1, S_2, I_2) \qquad (2)$$

in the region

$$R_+^4 = \{(S_1, I_1, S_2, I_2) \in R^4 : S_1 \geq 0, I_1 \geq 0, S_2 \geq 0, I_2 \geq o\}$$

we will tacked any point to used the initial condition for system (2) .clearly, the relations functions in the right hand part of system (2) are continuously differentiable functions on R_+^4, and so they are Lipschitizian. as a result the solution of system (2) exists and is unique. Further, all the solutions of system (2) with non- negative initial condition are uniformly bounded as exposed in the next theorem.

Theorem 1. All solutions for the system (2), which start in R_+^4 are uniformly bounded.

Proof: assume that $(S_1(T), I_1(T), S_2(T), I_2(T))$ be any solution of the system (2) through non negative initial condition $(S_{10}, I_{10}, S_{20}, I_{20})$ .given that we have

$$\frac{dS_1}{dT} \leq r\, S_1\,(1-\frac{S_1}{k})$$

next according to the theory of differential inequality ["Birkhoof and Rota "(1982)], we have

$$Sup\ S_1(T) \leq M,\ \ \forall T \geq 0,$$

where $M = \max\{S_{10}, k\}$

Define the function:

$$W(T) = S_1(T) + I_1(T) + S_2(T) + I_2(T)$$

The derivative time of $W(T)$ along the solution of the system (2) is:

$$\frac{dW}{dT} \leq rS_1 - h_1S_1 - (d_1 + h_2)I_1 - d_2S_2 - d_3I_2.$$

therefore we get:

$$\frac{dW}{dT} \leq rM - mW(T)$$

where $m = \min\{h_1, (d_1 + h_2), d_2, d_3\}$.

Again, suitable to the theory of differential inequalities we get

$$W(T) \leq \frac{rM}{m} - \frac{rM}{me^{mT}} + \frac{W_0}{e^{mT}},$$ where $W_0 = (S_1(0), I_1(0), S_2(0), I_2(0))$.

Thus, $\forall T \geq 0$ we have that $0 \leq W(T) \leq \frac{rM}{m}$

So every part of solutions of system (2) are uniformly bounded, hence the proof is

3. "Stability analysis of 2D predator liberated subsystem"

It is famous that according to the prey-predator relations, the prey species can stay alive in the nonexistence of predator, and as the prey population is separated into two classes that is susceptible prey population S_1 and the infected prey population. I_1 Hence, the follow 2D predator liberated subsystem is obtained:

$$\frac{dS_1}{dT} = S_1\left[r\left(1-\frac{S_1+I_1}{K}\right) - C_1I_1 - h_1\right] = L_1(S_1, I_1)$$
$$\frac{dI_1}{dT} = I_1[c_1S_1 - d_1 - h_2] = L_2(S_1, I_1)$$

The study of local and global subsystem (3) is discuss and give the following results:

1. The vanishing equilibrium point $P_0 = (0,0)$ all the time exist and its locally asymptotically stable under the following condition:

$$r < h_1 \qquad (4)$$

2. The axial equilibrium point on the $S_1 -$ axis is given by $P_1 = (\hat{S}_1, 0)$, where

$$\hat{S}_1 = \frac{(r-h_1)K}{r} \qquad (5)$$

Which exists under the next condition:

$$r > h_1 \qquad (6a)$$

And it is locally asymptotically stable in the $R^2_{+(S_1I_1)}$ if and only if the following condition hold:

$$(r - h_1) < \frac{r(d_1+h_2)}{c_1K} \qquad (6b)$$

3. The interior stability point $P_2 = (\tilde{S}_1, \tilde{I}_1)$ exists in the $Int.R^2_{+(S_1I_1)}$ under the following condition :

$$K > \frac{(d_1 + h_2)r}{c_1(r - h_1)} \quad (7)$$

Where

$$\tilde{S}_1 = \frac{d_1 + h_2}{c_1} \; ; \; \tilde{I}_1 = \frac{r(c_1K - (d_1 + h_2)) - c_1h_1K}{c_1(r + c_1K)} \quad (8)$$

the equilibrium point P_2 is an unstable saddle point.

4. "Stability analysis of 2D prey - predator disease librated subsystem"

In this subsystem the disease is not exist which makes that the population (prey and predator) contains only susceptible individuals respectively and the next 2D disease librated subsystem of system (2) come into view:

$$\frac{dS_1}{dT} = S_1\left[r\left(1 - \frac{S_1}{K}\right) - h_1S_1 - a_1S_1S_2\right] = S_1J_1(S_1, S_2)$$

$$\frac{dS_2}{dT} = S_2[e_1a_1S_1 - d_2] = S_2J_2(S_2, S_2) \quad (9)$$

The analysis of subsystem (9) gave the following result:

1. The vanishing equilibrium point $F_0 = (0,0)$ always exist and it is locally asymptotically stable under condition (4) holds, while it is unstable saddle point under condition (6a) holds.

2. The axial equilibrium point $F_1 = (\hat{S}_1, 0)$, exist under condition (6a) and is given in equation (5) which it is locally asymptotically stable under following condition

$$(r - h_1) < \frac{d_2 r}{e_1 a_1 K} \quad (10)$$

3. The interior equilibrium point $F_2 = (\ddot{S}_1, \ddot{I}_1)$, where

$$\ddot{S}_1 = \frac{d_2}{e_1a_1} \; , \quad \ddot{S}_2 = \frac{e_1a_1K(r - h_1) - rd_2}{e_1a_1^2K} \quad (11)$$

Which exist in the $Int.R^2_{+(S_1S_2)}$ under the following condition :

$$r > \frac{rd_2}{e_1a_1K} + h_1 \quad (12)$$

the equilibrium point F_2 is locally asymptotically stable at any time it exists, additional the global dynamics of F_2 is approved t in the subsequently theorem.

Theorem 3. The interior equilibrium point $F_2 = (\ddot{S}_1, \ddot{I}_1)$ is globally asymptotically stable in the $Int.R^2_{+(S_1S_2)} = \{(S_1, S_2) \in R^2; S_1 > 0, S_2 > 0\}$.

Proof: Assume that $H(S_1, S_2) = \frac{1}{S_1S_2}$; clearly, $H(S_1, S_2)$ is C^1 function in the $Int.R^2_{+(S_1S_2)}$ and it is positive for all. $(S_1, S_2) \in Int.R^2_{+(S_1S_2)}$ Note that

$$\Delta(S_1, S_2) = \frac{\partial(HJ_1)}{\partial S_1} + \frac{\partial(HJ_2)}{\partial S_2} = \frac{-r}{KS_2} < 0$$

make a note of that $\Delta(S_1, S_2)$ does not change sign plus it is not identically zero in the $Int.R^2_{+(S_1S_2)}$ of S_1S_2_ plane . So there is no periodic solution in the $Int.R^2_{+(S_1S_2)}$ of S_1S_2 _ plane, according to Bendixon-Dulic criteria.

Now, since all the solutions of the subsystem (9) are uniformly bounded and F_2 is unique positive equilibrium point in the $Int.R^2_{+(S_1S_2)}$.Hence by Poincare- Bendixon theorem F_2 is a globally asymptotically stable and hence the proof is complete. ∎

5. "Stability analysis of 3D prey-predator with disease subsystem"

In this part the disease exist only in the prey population which divided in to two classes Susceptible S_1 and infected I_1 and the predator attacked only the susceptible prey which makes the predator only susceptible ,therefore the following 3D subsystem of system (2) appears:

$$\frac{dS_1}{dT} = S_1\left[r(1 - \frac{S_1 + I_1}{K}) - c_1I_1 - h_1 - a_1S_2\right] = S_1g_1(S_1, I_1, S_2)$$

$$\frac{dI_1}{dT} = I_1[c_1S_1 - a_2S_2 - (d_1 + h_2)] = I_1g_2(S_1, I_1, S_2)$$

$$\frac{dS_2}{dT} = S_2[e_1a_1S_1 + e_2a_2I_1 - d_2] = S_2g_3(S_1, I_1, S_2) \quad (13)$$

The analysis of subsystem (13) show in the following results:

1. The vanishing equilibrium point $Q_1 = (0,0,0)$ always exist and it is locally asymptotically stable under condition (4) holds , while it is unstable saddle point under condition (6a) holds. **2.** The axial equilibrium point $Q_1 = (\hat{S}_1, 0, 0)$, exist under condition (6a) and $\hat{S}_1$ isgiven in equation (5) which it is locally asymptotically stable under given conditions in (6b) and (10).

3. The interior equilibrium point $Q_2 = (\bar{S}_1, \bar{I}_1, \bar{S}_2)$, where

$$\bar{S}_1 = \frac{a_1K(d_1 + h_2) + (r - h_1)a_2K - a_2(r + c_1K)\bar{I}_1}{ra_2 + a_1c_1K},$$

$$\bar{I}_1 = \frac{e_1a_1^2K(d_1 + h_2) + (r - h_1)e_1a_1a_2K - d_2(ra_2 + a_1c_1K)}{e_1a_1a_2(r + c_1K) - e_2a_2(ra_2 + a_1c_1K)},$$

$$\bar{S}_2 = \frac{c_1\bar{S}_1 - d_1 - h_2}{a_2} \quad (14)$$

Which exist in the $Int.R^3_{+(S_1I_1S_2)} = \{(S_1, I_1, S_2) \in R^3 : S_1 > 0, I_1 > 0, S_2 > 0\}$ under condition (6a) and the following conditions :

$$c_1\bar{S}_1 > d_1 + h_2 \quad (15a)$$

$$ra_2 + a_1c_1K < \min\left\{\frac{e_1a_1(r + c_1K)}{e_2}, \frac{e_1a_1^2K(d_1 + h_2) + (r - h_1)e_1a_1a_2K}{d_2}\right\} \quad (15b)$$

$$a_1K(d_1 + h_2) + (r - h_1)a_2K > a_2(r + c_1K)\bar{I}_1 \quad (15c)$$

In adding together it is locally asymptotically stable in the $Int.R^3_{+(S_1I_1S_2)}$ if the following conditions hold:

$$r - \frac{2r}{K}\bar{S}_1 - (\frac{r}{K} + c_1)\bar{I}_1 - a_1\bar{S}_2 < h_1 < h_2 \quad (16a)$$

$$\frac{(r + c_1K)e_1}{c_1K} < e_2 < \frac{d_2}{a_2\bar{I}_1} - e_1a_1\bar{S}_1 \quad (16b)$$

also, the global stability of Q_2 is investigated in the next theorem .

Theorem 4. Assume that $Q_2 = (\bar{S}_1, \bar{I}_1, \bar{S}_2)$ is locally asymptotically stable with

$$e_1 > e_2 \quad (17)$$

Then Q_2 is globally asymptotically stable in the sub region of R^3_+ which can define as: $\omega = \{(S_1, I_1, S_2): S_1 > \bar{S}_1, I_1 > \bar{I}_1, S_2 > \bar{S}_2\}$

Proof:

regard as the following function:

$$U(S_1,I_1,S_2) = b_1\left[S_1-\bar{S}_1-\bar{S}_1\ln\frac{S_1}{\bar{S}_1}\right] + b_2\left[I_1-\bar{I}_1-\bar{I}_1\ln\frac{I_1}{\bar{I}_1}\right] + b_3\left[S_2-\bar{S}_2-\bar{S}_2\ln\frac{S_2}{\bar{S}_2}\right]$$

Note that $U: R^3_{+(S_1I_1S_2)} \to R$, and is a C^1 positive definite function, where b_i, ($i =$ 1,2,3) are nonnegative constants to be firm. Now, the derivative of U along the trajectory of subsystem (13) be able to written as:

$$\frac{dU}{dT} = \frac{-rb_1}{K}(S_1-\bar{S}_1)^2 - \frac{rb_1}{K}(S_1-\bar{S}_1)(I_1-\bar{I}_1) - b_1c_1(S_1-\bar{S}_1)(I_1-\bar{I}_1) - b_1a_1(S_1-\bar{S}_1)(S_2-\bar{S}_2) + b_2c_1(S_1-\bar{S}_1)(I_1-\bar{I}_1) - b_2a_2(I_1-\bar{I}_1)(S_2-\bar{S}_2) + b_3e_1a_1(S_1-\bar{S}_1)(S_2-\bar{S}_2) + b_3e_2a_2(I_1-\bar{I}_1)(S_2-\bar{S}_2)$$

Now by choosing a suitable positive constants $b_1 = b_2 = 1$ and $b_3 = 1/e_1$, yield that $\frac{dU}{dT} = \frac{-r}{K}(S_1-\bar{S}_1)^2 - \frac{r}{K}(S_1-\bar{S}_1)(I_1-\bar{I}_1) - a_2\left(1-\frac{e_2}{e_1}\right)(I_1-\bar{I}_1)(S_2-\bar{S}_2)$

Clearly $\frac{dU}{dT} < 0$ in ω under condition (17), hence U is a strictly Lyapunov function .as a result Q_2 is globally asymptotically stable in the region ω .

6**." Local stability analysis of system (2)"**

In this section , we studied the existence and the local stability analysis of all equilibrium points of system (2) and obtained the following results:

1. the vanishing equilibrium point $E_0 = (0,0,0,0)$ always exist.

2. the axial equilibrium point $E_1 = (\hat{S}_1, 0,0,0,)$ exist under condition (6a) ;where $\hat{S}_1$given in equation (5).

3.The predator librated equilibrium point $E_2 = (\tilde{S}_1, \tilde{I}_1, 0,0)$ exist under condition (7); where $\tilde{S}_1$ and $\tilde{I}_1$ are given in equation (8).

4. The disease librated equilibrium point $E_3 = (\ddot{S}_1, 0, \ddot{S}_2, 0)$ exist under condition (12) where $\ddot{S}_1$ and $\ddot{S}_2$ are given in equation (11).

5. The equilibrium point $E_4 = (\bar{S}_1, \bar{I}_1, \bar{S}_2, 0)$ exist under conditions (6a) and (15a-15c); where $\bar{S}_1, \bar{I}_1$ and $\bar{S}_2$ are given in equation (14).

6. The positive equilibrium point $E_5 = (S_1^*, I_1^*, S_2^*, I_2^*)$ exist uniquely in the Int. R^4_+ under the following conditions :

$$r > h_1 + \frac{a_1d_3}{c_2} + \frac{r(a_2d_3+c_2(d_1+h_2))}{c_1c_2K} \quad (18a)$$

$$e_2a_2I_1^* > d_2 \quad (18b)$$

; where

$$S_1^* = \frac{a_2d_3+c_2(d_1+h_2)}{c_1c_2}, I_1^* = \frac{(r-h_1)c_1c_2K - r(a_2d_3+c_2(d_1+h_2)) - c_1a_1d_3K}{c_1c_2(r+c_1K)},$$

$$S_2^* = \frac{d_3}{c_2},$$

$$I_2^* = \frac{e_1a_1(a_2d_3+c_2(d_1+h_2))+c_1c_2(e_2a_2I_1^*-d_2)}{c_1c_2} \quad (19)$$

In adding together, it is observed that, the eigenvalues of the Jacobian matrix of system (2) at E_0 , say $J(E_0)$, are:

$$\lambda_{0S_1} = r-h_1, \quad \lambda_{0I_1} = -(d_1+h_2), \quad \lambda_{0S_2} = -d_2, \quad \lambda_{0I_2} = -d_3 \quad (20)$$

hence , E_0 is unstable saddle point with locally stable manifold in the $R^3_{+(I_1S_2I_2)}$ (i.e.dim ω^s =3) and with locally unstable manifold in the S_1 −direction (i.e.dim $\omega^u = 1$)provided that condition (6a) holds. on the other hand, it is locally asymptotically stable provided that condition (4) holds.the eigenvalues of the Jacobian matrix of system (2) at E_1 , say $J(E_1)$, satisfy the following relations:

$$\lambda_{1S_1} + \lambda_{1I_1} = -(r-h_1) + \frac{c_1K(r-h_1)}{r} - (d_1+h_2) \quad (21a)$$

$$\lambda_{1S_1}.\lambda_{1I_1} = \frac{-c_1K_1(r-h_1)^2}{r} + (r-h_1)(d_1+h_2) \quad (21b)$$

$$\lambda_{1S_2} + \lambda_{1I_2} = \frac{e_1a_1K(r-h_1)}{r} - (d_2+d_3) \quad (21c)$$

$$\lambda_{1I_2} = \frac{-e_1a_1d_3K(r-h_1)}{r} + d_2d_3 \quad (21d)$$

make a note of , all the eigenvalues of $J(E_1)$ have negative real parts according to Eqs. (21a-21d) and so E_1 is locally asymptotically stable in the R^4_+ if and only if conditions (6b) and (10) hold. However, E_1 is an unstable saddle point in the R^4_+ with locally unstable manifold of dimension less than or equal two (i.e. dim$\omega^u \le 2$) and with locally stable manifold of dimension greater than or equal two (i.e. dim$\omega^s \ge 2$) if the one of the conditions (6b) or (10) hold.

Further, the eigenvalues of Jacobian matrix at the predator librated equilibrium point $E_2 = (\tilde{S}_1, \tilde{I}_1, 0,0)$ satisfy the following relations:

$$\lambda_{2S_1} + \lambda_{2I_1} = -\frac{(d_1+h_2)}{c_1K} + 1 \quad (22a)$$

$$\lambda_{2S_1}.\lambda_{2I_1} = -\frac{(d_1+h_2)}{c_1K} < 0 \quad (22b)$$

$$\lambda_{2S_2} + \lambda_{2I_2} = e_1a_1\tilde{S}_1 + e_2a_2\tilde{I}_1 - (d_2+d_3) \quad (22c)$$

$$\lambda_{2S_2}.\lambda_{2I_2} = d_2d_3 - d_3(e_1a_1\tilde{S}_1 + e_2a_2\tilde{I}_1) \quad (22d)$$

Note that, it is simple to confirm that, according to Eq. (22b) the equilibrium point E_2 is an unstable saddle point in the R^4_+ with locally unstable manifold of dimension greater than or equal two (i.e. dim$\omega^u \ge 2$) and with locally stable manifold of dimension less than or equal two (i.e. dim$\omega^s \le 2$) .

Now, we can write the Jacobian matrix at the equilibrium point $E_3 = (\ddot{S}_1, 0, \ddot{S}_2, 0)$ as the following:

$$J(E_3) = [a_{ij}]_{4\times4} \; ; i,j = 1,2,3,4 \quad (23)$$

Where $a_{11} = -\frac{rd_2}{e_1a_1K} < 0$, $a_{12} = -\frac{d_2(r+c_1K)}{e_1a_1K} < 0$, $a_{13} = -\frac{d_2}{e_1} < 0$, $a_{14} = 0$ $a_{21} = 0$, $a_{22} = c_1\ddot{S}_1 - a_2\ddot{S}_2 - (d_1+h_2)$, $a_{23} = 0$, $a_{24} = 0$, $a_{31} = e_1a_1\ddot{S}_2 > 0$
$a_{32} = e_2a_2\ddot{S}_2 > 0$, $a_{33} = 0$, $a_{34} = 0, a_{41} = 0, a_{42} = 0, a_{43} = 0,$
$a_{44} = c_2\ddot{S}_2 - d_3$

So we can write the characteristic equation of the Jacobian matrix $J(E_3)$ as thefollowing form

$$(a_{44}-\lambda)(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3) = 0 \quad (24a)$$ So, either

$(a_{44}-\lambda)=0$
gives
$\lambda=a_{44}=c_2\ddot{S}_2-d_3$ (24b) Or
$(\lambda^3+A_1\lambda^2+A_2\lambda+A_3)=0$ (24c) Where
$A_1=-(a_{11}+a_{22})$, $A_2=a_{11}a_{22}-a_{13}a_{31}>0$, $A_3=a_{13}a_{31}a_{22}>$
$\Delta=A_1A_2-A_3=-a_{11}^2a_{22}+a_{11}a_{13}a_{31}-a_{11}a_{22}^2>0$ (24d)

It is observed that eq. (24b) represent the eigenvalue of $J(E_3)$ that describe the dynamics in the I_2 direction which is negative under the following condition holds:

$$\ddot{S}_2<\frac{d_3}{c_2} \quad (25a)$$

However, the roots of eq.(24c), which represent the eigenvalues of $J(E_3)$ that describe the dynamics in the $R^3_{(S_1I_1S_2)}$, which have negative real parts if and only if the following condition holds:

$$\ddot{S}_2>\frac{c_1\ddot{S}_1-(d_1+h_2)}{a_2} \quad (25b)$$

As a result E_3 is locally asymptotically stable in the R^4_+ provided that conditions (25a) And (25b) are hold otherwise it is unstable saddle point.

However, we can write the Jacobian matrix at the equilibrium point $E_4=(\bar{S}_1,\bar{I}_1,\bar{S}_2,0)$as the following:

$$J(E_4)=\left[b_{ij}\right]_{4\times4} \ ; i,j=1,2,3,4 \quad (26)$$

Where $b_{11}=r-\frac{2r\bar{S}_1}{K}-\left(\frac{r}{K}+c_1\right)\bar{I}_1-h_1-a_1\bar{S}_2$, $b_{12}=-\left(\frac{r}{K}+c_1\right)\bar{S}_1<0$, $b_{13}=-a_1\bar{S}_1<0$, $b_{14}=0$, $b_{21}=c_1\bar{I}_1>0$, $b_{22}=h_1-h_2$, $b_{23}=-a_2\bar{I}_1<0$, $b_{31}=e_1a_1\bar{S}_2>0$, $b_{32}=e_2a_2\bar{S}_2>0$, $b_{33}=e_1a_1\bar{S}_1+e_2a_2\bar{I}_1-d_2$, $b_{34}=-C_2\bar{S}_2<0$, $b_{41}=0$, $b_{42}=0$, $b_{43}=0$, $b_{44}=c_2\bar{S}_2-d_3$

So we can write characteristic equation of the Jacobian matrix $J(E_4)$ as the following
Form

$$(b_{44}-\lambda)(\lambda^3+B_1\lambda^2+B_2\lambda+B_3)=0 \quad (27a)$$

So, either
$(b_{44}-\lambda)=0$
We get :

$$\lambda=c_2\bar{S}_2-d_3 \quad (27b)$$

Or

$$\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0 \quad (27c)$$

Where $B_1=-(b_{11}+b_{22}+b_{33})$, $B_2=b_{11}b_{33}+b_{22}b_{33}+b_{11}b_{22}-b_{13}b_{31}-b_{23}b_{32}-b_{12}b_{21}$, $B_3=-b_{11}b_{22}b_{33}-b_{12}b_{23}b_{31}-b_{13}b_{21}b_{32}+a_{13}a_{31}a_{22}+b_{11}b_{23}b_{32}+b_{12}b_{21}b_{33}$

$$\bar{\Delta}=B_1B_2-B_3 = -b_{11}^2a_{33}-2b_{11}b_{22}b_{33}-b_{11}^2b_{22}+b_{11}b_{13}b_{31}+b_{11}b_{12}b_{21}-b_{22}^2b_{33}-b_{11}b_{22}^2+b_{22}b_{23}b_{32}+b_{22}b_{12}b_{21}-b_{11}b_{33}^2-b_{22}b_{33}^2+b_{13}b_{31}b_{33}+b_{23}b_{32}b_{33}+b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32} \quad (27d)$$

It is observed that eq.(27b) represent eigenvalue of $J(E_4)$ that describe the dynamics in the I_2 direction which is negative under the following condition holds:

$$\ddot{S}_2<\frac{d_3}{c_2} \quad (28a)$$

on the other hand, the roots of eq.(27c), which represent the eigenvalues of $J(E_4)$ that describe the dynamics in the $R^3_{(S_1I_1S_2)}$, which have negative real parts if and only if the following condition holds:

$$r-\frac{2r}{K}\bar{S}_1-(\frac{r}{K}+c_1)\bar{I}_1<h_1<h_2 \quad (28b)$$

$$\frac{(r+c_1K)e_1}{c_1K}<e_2<\frac{d_2}{a_2\bar{I}_1}-e_1a_1\bar{S}_1 \quad (28c)$$

hence E_4 is locally asymptotically stable in the R^4_+ provided that conditions (28a)- (28c)are hold otherwise it is unstable saddle point.

Finally, we can write the Jacobian matrix of system (2) at the positive equilibrium point $E_5=(S_1^*,I_1^*,S_2^*,I_2^*)$ as the following :

$$J(E_5)=\left[c_{ij}\right]_{4\times4} \ ; i,j=1,2,3,4, \quad (29)$$

Where $c_{11}=r-\frac{2rS_1^*}{K}-\left(\frac{r}{K}+c_1\right)I_1^*-h_1-a_1S_2^*$, $c_{12}=-\left(\frac{r}{K}+c_1\right)S_1^*<0$, $c_{13}=-a_1S_1^*<0$, $c_{14}=0$, $c_{21}=c_1I_1^*>0$, $c_{22}=c_1S_1^*-a_2S_2^*-(d_1+h_2)$, $c_{23}=-a_2I_1^*<0$, $c_{24}=0$, $c_{31}=e_1a_1S_2^*>0$, $c_{32}=e_2a_2S_2^*>0$, $c_{33}=e_1a_1S_2^*+e_2a_2I_1^*-c_2I_2^*-d_2$, $c_{34}=-c_2S_2^*<0$, $c_{41}=0$, $c_{42}=0$, $c_{43}=c_2I_2^*>0$, $c_{44}=c_2S_2^*-d_3$.

According to the characteristic equation of $J(E_5)$ is given by:

$$\lambda^4+\Gamma_1\lambda^3+\Gamma_2\lambda^2+\Gamma_3\lambda+\Gamma_4=0 \quad (30)$$

Where $\Gamma_1=-(\gamma_1+\gamma_2)$, $\Gamma_2=c_{44}\gamma_1+\gamma_3+\gamma_4+\gamma_5+\gamma_6$ $\Gamma_3=-\gamma_1\gamma_6-\gamma_2\gamma_3+c_{44}\gamma_7-\gamma_8+\gamma_9$, $\Gamma_4=\gamma_3\gamma_6+c_{44}\gamma_8-c_{44}\gamma_9$,
With
$\gamma_1=c_{11}+c_{22}$, $\gamma_2=c_{33}+c_{44}$, $\gamma_3=c_{11}c_{22}-c_{12}c_{21}>0$, $\gamma_5=c_{22}c_{33}-c_{23}c_{32}>0$, $\gamma_6=c_{33}c_{44}-c_{34}c_{43}>0$, $\gamma_7=c_{13}c_{31}+c_{23}c_{32}<0$, $\gamma_8=c_{12}c_{23}c_{31}+c_{21}c_{32}c_{13}$, $\gamma_9=c_{11}c_{23}c_{32}+c_{22}c_{13}c_{31}>0$,
And

$$\Delta^*=\Gamma_1\Gamma_2\Gamma_3-\Gamma_3^2-\Gamma_1^2\Gamma_4$$
$$=\gamma_1\gamma_6[a_{44}\gamma_1^2+(\gamma_1+\gamma_2)(\gamma_4+\gamma_5)+\gamma_2\gamma_6+a_{44}\gamma_1\gamma_2-\gamma_8+a_{44}\gamma_7+\gamma_9]$$
$$+\gamma_1\gamma_2\gamma_3[a_{44}(\gamma_1+\gamma_2)+\gamma_3+\gamma_4+\gamma_5-2\gamma_6]+\gamma_2\gamma_3[\gamma_2(\gamma_4+\gamma_5)+a_{44}\gamma_7+\gamma_9]$$
$$-a_{44}\gamma_1\gamma_7[a_{44}(\gamma_1+\gamma_2)+\gamma_3+\gamma_4+\gamma_5]+a_{44}\gamma_7[-\gamma_2(\gamma_4+\gamma_5+\gamma_6)-a_{44}\gamma_7+2\gamma_8+2\gamma_9]+\gamma_8[(\gamma_1+\gamma_2)(\gamma_4+\gamma_5)+\gamma_3(\gamma_1-\gamma_2)+\gamma_2\gamma_6-\gamma_8+2\gamma_9-a_{44}\gamma_2(\gamma_1+\gamma_2)]+\gamma_9[-(\gamma_1+\gamma_2)(\gamma_4+\gamma_5)-\gamma_1\gamma_3-\gamma_9+a_{44}\gamma_2(\gamma_1+\gamma_2)]$$

Therefore, the local stability conditions of the positive equilibrium point E_5 is conventional in the following theorem :

Theorem 5. The positive equilibrium point $E_5=(S_1^*,I_1^*,S_2^*,I_2^*)$ of the system (2) is locally asymptotically stable in the Int. R^4_+ under the following conditions:

$$e_1<\frac{c_1e_2K}{r+c_1K} \quad (31a)$$

$$S_1^*<\min\left\{\frac{a_2S_2^*+d_1+h_2}{c_1},\frac{c_2I_2^*+d_2-e_2a_2I_1^*}{e_1a_1}\right\} \quad (31b)$$

$$\frac{r-\frac{2r}{K}S_1^*-\left(\frac{r}{K}+c_1\right)I_1^*-h_1}{a_1}<S_2^*<\frac{d_3}{c_2} \quad (31c)$$

$$\Delta^*>0 \quad (31d)$$

Proof:

According to the Routh-Hurwtiz criterion for dimension four, the proof is follow, if and only if $\Gamma_i(i=1,3,4)>0$ and $\Delta^*>0$. Now, straight onward computations and elements of $J(E_4)$ due to the coefficients of equation (30) we get that $A_i(i=1,3)>0$ under conditions (31a)-(31c), and visibly as of the sign of coefficients of $J(E_4)$ we have $A_4>0$ (always).

Thus, all the eigenvalues of the $J(E_4)$ contain negative real parts.as a result E_4 is locally asymptotically stable in the $Int.R_+^4$ so as the proof is whole .

7."Global dynamical behavior of system (2)"

In this part we l using Lyapunov method to prove the global stability for the equilibrium
points of system (2) as shown in the next theorems.

Theorem 6. Assume that the vanishing equilibrium point E_0 of system (2) is locally
asymptotically stable in the R_+^4 . then E_0 is globally asymptotically stable in the R_+^4, and let the following condition holds:

$$e_2 \le e_1 \qquad (32)$$

Proof:

Consider the following positive definite function

$$V_0(S_1,I_1,S_2,I_2) = b_1S_1 + b_2I_1 + b_3S_2 + b_4I_2$$

Clearly $V_0: R_+^4 \to R$ is C^1 , where $b_i,(i = 1,2,3,4,)$nonnegative constants to be gritty are. Now, as the derivative of V_0 along the path of system (2) canister be written as:

$$\frac{dV_0}{dT} = (b_1r - b_1h_1)S_1 - \frac{b_1r}{K}S_1^2 - \left(\frac{b_1r}{K} + b_1c_1 - b_2c_1\right)S_1I_1 - (b_1a_1 - b_3e_1a_1)S_1S_2 - (b_2 - b_3e_2)a_2I_1S_2 - b_2d_1I_1 - b_2h_2I_1 - (b_3 - b_4)c_2S_2I_2 - b_3d_2S_2 - b_4d_3I_2$$

By choosing the positive constants as: $b_1 = b_2 = 1, b_3 = b_4 = \frac{1}{e_1}$, yield that

$$\frac{dV_0}{dT} < (r - h_1)S_1 - \frac{r}{K}S_1^2 - \frac{r}{K}S_1I_1 - \left(1 - \frac{e_2}{e_1}\right)a_2I_1S_2 - (d_1 + h_2)I_1 - \frac{d_2}{e_1}S_2 - \frac{d_3}{e_1}I_2$$

Clearly $\frac{dV_0}{dT} < 0$ under the local stability condition (4) and given condition (32), hence
V_0 is strictly Lyapunov function. Therefore E_0 is globally asymptotically stable in the R_+^4.

Theorem 7. Assume that the axial equilibrium point E_1 of system (2) is locally
asymptotically stable in the R_+^4 . then E_1 is globally asymptotically stable in the R_+^4, and let the following condition holds:

$$r - h_1 \le \frac{r(d_1+h_2)}{r+c_1K} \qquad (33)$$

Proof:

Consider the following positive definite function

$$V_1(S_1,I_1,S_2,I_2) = b_1[S_1 - \hat{S}_1 - \hat{S}_1\ln\frac{S_1}{\hat{S}_1}] + b_2I_1 + b_3S_2 + b_4I_2$$

Clearly $V_1: R_+^4 \to R$ is C^1 , where $b_i,(i = 1,2,3,4,)$nonnegative constants to be gritty are. Now, as the derivative of V_1 along the path of system (2) canister be written as:

$$\frac{dV_1}{dT} = -(S_1 - \hat{S}_1)^2 - (1 + b_1c_1 - b_2c_1)S_1I_1 + (\hat{S}_1 + b_1c_1\hat{S}_1 - b_2d_1 - b_2h_2)I_1 - (b_1a_1 - b_3e_1a_1)S_1S_2 + (b_1a_1\hat{S}_1 - b_3d_2)S_2 - (b_2a_2 - b_3e_2a_2)I_1S_2 - (b_3c_2 - b_4c_2)S_2I_2$$

By choosing the positive constants as: $b_1 = b_2 = \frac{K}{r}, b_3 = b_4 = \frac{K}{re_1}$, yield that

$$\frac{dV_1}{dT} \le -(S_1 - \hat{S}_1)^2 - S_1I_1 + \left(\left(1 + \frac{c_1K}{r}\right)\hat{S}_1 - \frac{K}{r}(d_1 + h_2)\right)I_1 + \left(\frac{a_1K^2(r - h_1)}{r^2} - \frac{Kd_2}{re_1}\right)S_2 - \left(1 - \frac{e_2}{e_1}\right)\frac{Ka_2}{r}I_1S_2$$

Clearly $\frac{dV_1}{dT} < 0$ under the local stability condition (6b) and conditions (32) ,(33).

Hence V_1 is strictly Lyapunov function. Therefore E_1 is globally asymptotically stable in
the R_+^4.

Theorem 8. Assume that the equilibrium point E_3 of system (2) is locally asymptotically stable in the R_+^4 , with the following condition holds

$$\left(\frac{r}{K} + c_1\right)\ddot{S}_1 < d_1 + h_2 + \frac{e_2}{e_1}a_2\ddot{S}_2 \qquad (34)$$

Then E_3 is globally asymptotically stable in the R_+^4

Proof:

Consider the following positive definite function

$$V_3(S_1,I_1,S_2,I_2) = \left[S_1 - \ddot{S}_1 - \ddot{S}_1\ln\frac{S_1}{\ddot{S}_1}\right] + I_1 + \frac{1}{e_1}[S_2 - \ddot{S}_2 - \ddot{S}_2\ln\frac{S_2}{\ddot{S}_2}] + \frac{1}{e_1}I_2$$

Clearly $V_3: R_+^4 \to R$ is C^1 . Now we have

$$\frac{dV_3}{dT} \le -\frac{r}{K}(S_1 - \ddot{S}_1)^2 - \frac{r}{K}S_1I_1 + (\frac{r}{K}\ddot{S}_1 + c_1\ddot{S}_1 - (d_1 + h_2) - \frac{e_2}{e_1}a_2\ddot{S}_2)I_1 - \left(1 - \frac{e_2}{e_1}\right)a_2I_1S_2 + (c_2\ddot{S}_2 - d_3)\frac{1}{e_1}I_2$$

Clearly $\frac{dV_2}{dT} < 0$ under the local stability condition (24b) and conditions (32),(34)

Hence V_3 is strictly Lyapunov function. Therefore E_3 is globally asymptotically stable in
the R_+^4.

Theorem 9. Assume that the equilibrium point E_4 of system (2) is locally asymptotically stable in the R_+^4 , Then E_4 is globally asymptotically stable in the sub region of R_+^4 which can define as the following

$$\varphi = \{(S_1,I_1,S_2,I_2): S_1 > \bar{S}_1, I_1 > \bar{I}_1, S_2 > \bar{S}_2, I_2 > 0\}$$

Proof: Consider the following positive definite function

$$V_4(S_1,I_1,S_2,I_2) = \left[S_1 - \bar{S}_1 - \bar{S}_1\ln\frac{S_1}{\bar{S}_1}\right] + \left[I_1 - \bar{I}_1 - \bar{I}_1\ln\frac{I_1}{\bar{I}_1}\right] + \frac{1}{e_1}\left[S_2 - \bar{S}_2 - \bar{S}_2\ln\frac{S_2}{\bar{S}_2}\right] + \frac{1}{e_1}I_2$$

Clearly $V_3: R_+^4 \to R$ is C^1 . Now we have

$$\frac{dV_4}{dT} \le -\frac{r}{K}(S_1 - \bar{S}_1)^2 - \frac{r}{K}(S_1 - \bar{S}_1)(I_1 - \bar{I}_1) - \left(1 - \frac{e_2}{e_1}\right)a_2(I_1 - \bar{I}_1)(S_2 - \bar{S}_2) + (c_2\bar{S}_2 - d_3)\frac{1}{e_1}I_2$$

Clearly $\frac{dV_4}{dT} < 0$ under the local stability condition (28a) and condition (32)

Hence V_3 is strictly Lyapunov function. Therefore E_4 is globally asymptotically stable in
the region φ .

Theorem 10. Assume that the equilibrium point E_5 of system (2) is locally asymptotically stable in the $Int.R_+^4$, Then E_5 is globally asymptotically stable in the sub region of region of $Int.R_+^4$ which can define as the following

$$\Phi = \{(S_1,I_1,S_2,I_2): S_1 > S_1^*, I_1 > I_1^*, S_2 > S_2^*, I_2 > 0\}$$

Proof: Consider the following positive definite function

$$V_5(S_1,I_1,S_2,I_2)=\left[S_1-S_1^*-S_1^*\ln\frac{S_1}{S_1^*}\right]+\left[I_1-I_1^*-I_1^*\ln\frac{I_1}{I_1^*}\right]$$
$$+\frac{1}{e_1}\left[S_2-S_2^*-S_2^*\ln\frac{S_2}{S_2^*}\right]+\frac{1}{e_1}\left[I_2-I_2^*-I_2^*\ln\frac{I_2}{I_2^*}\right]$$

Clearly $V_3: R_+^4 \to R$ is C^1. Now since

$\frac{dV_5}{dT}=-\frac{r}{K}(S_1-S_1^*)^2-\frac{r}{K}(S_1-S_1^*)(I_1-I_1^*)-a_1(S_1-S_1^*)(S_2-S_2^*)$ $-c_1(S_1-S_1^*)(I_1-I_1^*)+c_1(I_1-I_1^*)(S_1-S_1^*)-a_2(I_1-I_1^*)(S_2-S_2^*)$ $+a_1(S_1-S_1^*)(S_2-S_2^*)+\frac{e_2}{e_1}a_2(S_2-S_2^*)(I_1-I_1^*)-\frac{c_2}{e_1}(S_2-S_2^*)(I_2-I_2^*)+\frac{c_2}{e_1}(S_2-S_2^*)(I_2-I_2^*)$

Hence, we can easy to verify that:

$\frac{dV_5}{dT}\le-\frac{r}{K}(S_1-S_1^*)^2-\frac{r}{K}(S_1-S_1^*)(I_1-I_1^*)-a_2\left(1-\frac{e_2}{e_1}\right)(I_1-I_1^*)(S_2-S_2^*)$Clearly $\frac{dV_5}{dT}<0$ under the condition (32) , hence V_5 is strictly Lyapunov function Therefore E_5 is globally asymptotically stable in The sub region Φ of $\text{Int.} R_+^4$.

In the following part, we will study the persistence condition of system (2).

8."Persistence Analysis of system (2)"

In this part we studied the persistence of system (2). It is famous that ,we say that the system is persists if and only if each of species is persists. Mathematically, that means, we say that system (2) is persists if the solution of the system with positive initial condition does not have omega limit set on the boundary of its domain [see Xiao and Chen (2003)]. Now, we can establish the persistence condition of system (2) in the next theorem.

Theorem 11. Suppose that the equilibrium point E_3 and E_4 of system (2) are globally asymptotically stable in the Interior of $R^2_{+(S_1S_2)}$ and $R^3_{+(S_1I_1S_2)}$ respectively. adding together , if condition (6a) hold with the following set of conditions :

$$r-h_1>\frac{r(d_1+h_2)}{c_1K} \quad (35a)$$
$$r-h_1>\frac{rd_2}{e_1a_1K} \quad (35b)$$
$$c_2\ddot{S}_2>d_3 \quad (35c)$$
$$c_2\bar{S}_2>d_3 \quad (35d)$$

Then system (2) is persists

Proof: Assume that x is a position in the $\text{Int.} R_+^4$ and $o(x)$ is the path during x. Let $\Omega(x)$ is the omega limit set of the $o(x)$ Note that $\Omega(x)$ is bounded, due to the boundedness of the system (2).

We first claim that $E_0 \notin \Omega(x)$. Assume the contrary.

Since E_0 is a saddle point due to condition (6a), E_0 cannot be the only point in $\Omega(x)$, and hence by Butler-McGhee lemma there is at least one other point p such that $p \in \omega^s(E_0)\cap\Omega(x)$, where $\omega^s(E_0)$ is the stable manifold of E_0 .

Now, since $\omega^s(E_0)$ is the $R^3_{+(I_1S_2I_2)}$ and the entire orbit through p , say $O(p)$, is contained in $\Omega(x)$.

Then, if p is on either boundary axis of $R^3_{+(I_1S_2I_2)}$, we obtain that the positive specific axis (that containing p) is contained in $\Omega(x)$. contradicting its boundedness.

Now, let $p\in \text{Int.} R^3_{+(I_1S_2I_2)}$. Since there is no equilibrium point in the $\text{Int.} R^3_{+(I_1S_2I_2)}$, the orbit through p which is contained in $\Omega(x)$ must be unbounded. Giving a contradiction too, this shows that$E_0 \notin \Omega(x)$.

Now, we show that $E_1=(r-h_1,0,0,0)$ cannot be in $\Omega(x)$. Since E_1 is saddle point under condition (35a) or (35b) holds . Then again by Butler-McGhee lemma there exist $q\in\omega^s(E_1)\cap\Omega(x)$, Also, since $\omega^s(E_1)$ could be either($R^2_{+(S_1I_2)}, R^2_{+(I_1S_2)}, R^2_{+(I_1I_2)}, R^3_{+(S_1I_1S_2)}, R^3_{+(S_1I_1I_2)} R^3_{+(S_1S_2I_2)}, R^3_{+(I_1S_2I_2)}$)

Suppose that $\omega^s(E_1)$ is the $R^2_{+(S_1I_2)}$ (similar proof for others).

Not that, if q is on either boundary axis of $R^2_{+(S_1I_2)}$, then we get contradiction as in the first part of proof.

Let now, $q\in \text{Int.} R^2_{+(S_1I_2)}$. Since there is no equilibrium point in the $\text{Int.} R^2_{+(S_1I_2)}$, then the $O(q)\subset\Omega(x)$, is unbounded, which gives a contradiction to the boundedness of $\Omega(x)$,.Thus $E_1\notin\Omega(x)$.

For the points E_2, E_3 and E_4, by argument completely analogous to the above we have E_2, E_3 and E_4 cannot contained in $\Omega(x)$. Thus $\Omega(x)$ must be $\text{Int.} R_+^4$, which proves persistence. ■

REFERENCES

1. " Anderson, R.M. and May, R.M.", (1986). "The invasion and spread of infectious disease with in animal and plant communities", Philos. Trans. R. Soc. Lond. B Biol. Sci. 314, p.533-570.
2. " Beltrami E. and Carroll T.O.", (1994). "Modeling the role of viral disease in recurrent phytoplankton blooms", J. of Mathematical Biology, 32, p.857-863.
3. "Brauer F. and Soudack A.C.",(1981)."Coexistence properties of some predator prey systems under constant rate harvesting and stocking", J. of Mathematical Biology, 12, p. 101-114,.
4. " Chattopadhyay J. and Arino O"., (1999). "A predator-prey model with disease in the Prey", Nonlinear Analysis, 36, p.747-766.
5. " Chattopadhyay, J., Ghosal G. and Chaudhuri K.S.",(1999)."Non selective harvesting of a prey-predator community with infected prey", Korean J. Computation and applied Mathematics, 6, p. 601-616.
6. "Chattopadhyay J.and Pal, S.", (2002). "Viral infection on phytoplankton zooplankton system-a mathematical model", Ecological Modeling, 151, p.15-28.
7. " Chattopadhyay, Sarkar R.R. and Ghosal G.",(2002). " Removal of infected prey prevent limit cycle oscillation in an infected prey-predator system-a mathematical study", Ecological modeling, 156, p. 113-121.
8. " Clark C.W.",(1990). " Mathematical Bioeconomics: The Optimal Management of Renewable Resources", Wiely, New York, USA.
9. "Dai G. and Tang M. ",(1998)."Coexistence region and global dynamics of a harvested predator prey system ",Siam. J. of applied Mathematical, 58,p.193-210.

10. " Freedman, H.I.", (1990). "A model of predator-prey dynamics as modified by the action of parasite", Mathematical Biosciences, 99, p.143-155.
11." Mukherjee, D.", (2003). "Persistence in a prey-predator system with disease in the Prey", J. of Biological systems, 11, p.101-112.
12." Venturino, E.", (1995). "Epidemics in predator-prey models: disease in the prey", In: O. Arino, D. Axelrod, M. kimmel and M. Langlasis (Eds), Mathematical population Dynamics: Analysis of Heterogeneity, Theory of Epidemics, Vol. l. S. Wuerz, Winnipeg, p.381-393.
13." Xiao, Y.N., chen, L.", (2001). "Modeling and analysis of predator-prey model with disease in the prey", Mathematical Biosciences, 171, p.59-82.

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