THE DYNAMICS OF SI_CIR MODEL WITH NONLINEAR INCIDENCE RATE AND SATURATED TREATMENT FUNCTION

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ABSTRACT: *In this paper, an epidemic model consisting of susceptible, carriers, infected and removal is formulated mathematically. It's assumed that the disease is transmitted between the individuals according to the nonlinear incidence rate represented by Beddington-DeAngelis type of function. However the treatment that given to the infected individuals follows the saturated treatment function. The proposed model has at most two equilibrium points disease-free equilibrium and endemic equilibrium. The local and global stability conditions of all possible equilibrium points are established. The local bifurcation analysis is carried out. Finally, numerical simulations are given to illustrate our obtained analytical results. It's observed that the system has only one type of attractors that represented by point attractor and undergoes backward bifurcation due to existence of saturated treatment function.*

Keywords*:* Epidemic model; Saturated treatment function; Stability; Local bifurcation; Backward bifurcation.

1. INTRODUCTION

Epidemiological models have been known as valuable tools in analyzing the spread and control of infectious diseases. In epidemiological models, for some infectious diseases, there are individuals who are able to transmit their disease but do not exhibit any symptoms. These individuals are called carriers, and they play an important role in the transmission of the disease. The incidence rate type as well as the treatment rate type play an important role too on the dynamics of the transmission of diseases. There are different types of incidence rate, which represented the number of individuals who become infected per unit of time in epidemiology. The authors in [1-9] have been used classical linear type of incidence rate in their models that given by the form βSI where β is stand for infection rate. On the other hand several authors [10-18] suggested different types of nonlinear incidence rates. The saturated incidence rate $\frac{\alpha SI}{1+\beta S}$, where β is stand for saturation factor, was first introduced by [10] and later on used by many authors [11, 12, 19, 20] and the references therein. However, Li et al [12,13] suggested an SIR epidemic model with nonlinear incidence rate given by $\frac{\alpha SI}{1+\gamma I}$. They assumed in this type of an incidence rate that the number of effective contacts between infected and susceptible individuals may saturate at high infected levels due to crowding of infected individuals or due to the protection measures by the susceptible individuals. On the other hand Beddington and DeAngelisz [21-22] introduced independently nonlinear incidence rate known after that as Beddington-DeAngelis type incidence rate $\frac{\alpha SI}{1+\beta S+\gamma I}$, which adopted both the saturation factor and the effect of crowding of infected individuals. Later on, some authors [23-27] used this incidence rate to describe the transmission of disease in their epidemiological models.

It is well known that the treatment is an important method to control the spread of diseases. In classical epidemic models, the treatment rate of infected individuals is assumed to be either constant or proportional to the number of the infected individuals. Therefore, it is very important to choose a proper treatment rate of a disease. [28] considered an SIR epidemic model with constant treatment rate as given $h(I) = \begin{cases} r, & I > 0 \\ 0, & I = 0 \end{cases}$ where r is a positive constant and I is the number of infected individuals. They studied stability analysis and showed that this model exhibits various types of local bifurcations. Further, [29] modified the treatment rate to Holling type II

$$h(I) = \frac{\beta I}{1+\gamma I}\ ,\ I \geq 0\ ,\beta > 0\ ,\gamma > 0$$

They have shown that, with varying amount of medical resources and their supply efficiency, then the epidemic model may have both backward bifurcation and Hopf bifurcation. Finally [30] have also used Holling type II, III and IV as treatment rates to study their model.

Keeping the above in view, in this paper we proposed and analyzed a mathematical model describing epidemiological system having SI_CIR (susceptible, carriers, infected and removal) model with Beddington-DeAngelis type of incidence rate and saturated treatment rate simultaneously.

2. PROBLEM FORMULATION

In this section an epidemic problem involving carrier individuals, which defined as those individuals who harbor the specific organisms of a disease without manifest symptoms and is capable of transmitting the infection, is proposed for study. To formulate the problem mathematically , its assumed that the entire population is divided into four classes: susceptible class, carrier class ,symptomatically infectious class or simply infectious class and removal or recovered class, where the number of individuals at time t in each class represented by $S(t)$, $I_C(t)$, $I(t)$ and $R(t)$ respectively.Furthermore the following two main hypotheses are adopted in formulating the dynamical equations of the model:

1. A susceptible individual can be contracts the disease through direct contact with an infectious individual or a carrier individual with Beddington-DeAngelis type of incidence rate and transfers to carrier class or an infectious class with probabilities p and $(1-p)$ respectively.
2. The infected individuals may be recovered and transfer to removal class in two methodology: auto recovery due to the immune response of the body or by treatment using the saturated treatment rate.

Accordingly the dynamical behavior of the model can can be described by the following set of nonlinear first order differential equations

$$\begin{aligned}
\frac{dS}{dt} &= A - d_0 S - SH - \theta S \\
\frac{dI_C}{dt} &= pS\,H - d_1 I_C - \beta_1 I_C \\
\frac{dI}{dt} &= (1-p)S\,H - d_2 I - \beta_2 I - \frac{aI}{1+bI} + \beta_1 I_C \\
\frac{dR}{dt} &= \theta S + \beta_2 I + \frac{aI}{1+bI} - d_3 R
\end{aligned} \tag{1}$$

Where $H = \left(\frac{\alpha_1 I_C + \alpha_2 I}{1+\gamma_1 S+\gamma_2 I_C+\gamma_3 I}\right)$

Here $S(0) > 0\,, I_C(0) \geq 0\,, I(0) \geq 0$ and $R(0) > 0,$ while the other parameters can be describe in the following table.

Table 1: Parameters description in the system (1)

The parameter	***Description***
$A > 0$	Represents the susceptible be influx at a constant rate
$d_i > 0; i = 0,3$	Represent natural death rate of the susceptible and removal respectively
$d_i > 0; i = 1,2$	Represent death rates for I_C and I individuals, respectively including both natural and because disease
$\beta_1 > 0$	Represents transmission rate for carrier individual into infected individual.
$\beta_2 > 0$	Represents natural recovery rate
$\theta > 0$	Represents vaccination rate
$\alpha_i > 0; i = 1,2$	Represent contact rate between the individuals of S with I_C and I respectively
$\gamma_i > 0; i = 1,2,3$	Represent a measure of inhibition effect due to contact between the individuals of the same class for S, I_C and I respectively
$a > 0$	Represents the maximum treatment rate
$b \geq 0$	Represents the measure of inhibition recovered rate due to the long time responded of infected individual to the treatment

According to the form of system (1) it's clear that once the dynamic of (S, I_C, I) are understood, then the dynamics of R can be then determined from the equation

$$\frac{dR}{dt} = \theta S + \beta_2 I + \frac{aI}{1+bI} - d_3 R \tag{2}$$

Therefore from now onward we will consider the following reduced system

$$\begin{aligned}
\frac{dS}{dt} &= A - d_0 S - S\,H - \theta S \\
\frac{dI_C}{dt} &= pS\,H - d_1 I_C - \beta_1 I_C \\
\frac{dI}{dt} &= (1-p)S\,H - d_2 I - \beta_2 I - \frac{aI}{1+bI} + \beta_1 I_C
\end{aligned} \tag{3}$$

It is easy to verify that the solution of system (3) is bounded in the sub region of the first octant as shown in the following theorem

Theorem (1): All solutions of system (3), which initiate in the first octant are uniformly bounded in the region $\Gamma = \left\{(S, I_C, I) \in R^3 : S(t) \leq \frac{A}{d_0+\theta}\ \ , S + I_C + I \leq \frac{A}{\bar{d}}\right\}$.

Proof. From the first equation of (3) we have

$$\frac{dS}{dt} = A - d_0 S - SH - \theta S \leq A - S(d_0 + \theta)$$

Which gives that

$$S \leq S_0 e^{-(d_0+\theta)t} + \frac{A}{d_0+\theta}\left(1 - e^{-(d_0+\theta)t}\right)$$

Hence as $t \to \infty$ it is observed that

$$S(t) \leq \frac{A}{d_0+\theta}$$

Now let $N = S + I_C + I$, then

$$\frac{dN}{dt} = A - (d_0 + \theta)S - d_1 I_C - (d_2 + \beta_2)I - \frac{aI}{1+bI}$$

Thus for $\bar{d} = min\{d_0 + \theta, d_1, d_2 + \beta_2\}$ we obtain that

$$\frac{dN}{dt} \leq A - \bar{d}N$$

So solving this linear differential inequality gives for $t \to \infty$

$$N(t) \leq \frac{A}{\bar{d}}$$

Hence all the solutions of system (3) are uniformly bounded in the region Γ and the proof is complete. ■

3.Equilibrium points and their stability

It is observed that, system (3) has at most two biologically feasible equilibrium points the existence conditions for each of these equilibrium points are discussed in the following:

1. The disease free equilibrium point, which is denoted by $E_0 = \left(\frac{A}{d_0+\theta}, 0,0\right)$always exists
2. The endemic equilibrium point, say $E_1 = (S^*, {I_C}^*, I^*)$, of the system (3) exists if there is a positive solution that denoted by $(S^*, {I_C}^*, I^*)$ to the following set of equations

$$A - d_0 S - SH - \theta S = 0 \quad (4a)$$

$$pSH - d_1 I_C - \beta_1 I_C = 0 \quad (4b)$$

$$(1-P)SH - d_2 I - \beta_2 I - \frac{aI}{1+bI} + \beta_1 I_C = 0 \quad (4c)$$

Obviously form equation (4a) we get that

$$D_1 S^2 + D_2 S + D_3 = 0$$

where $D_1 = \gamma_1(d_0 + \theta) > 0$

$$D_2 = (d_0 + \theta)(1 + \gamma_2 I_C + \gamma_3 I) + (\alpha_1 I_C + \alpha_2 I) - A\gamma_1$$

$$D_3 = -A(1 + \gamma_2 I_C + \gamma_3 I) < 0$$

Therefore the unique positive real root is given by

$$S = \frac{-D_2 + \sqrt{{D_2}^2 - 4D_1 D_3}}{2D_1} \quad (4d)$$

Now, substituting the value of Sinto Eq. (4b)-(4c) give us the following two isoclines equations. Keeping the above in view the positive intersection point of these two isoclines (if such point exist), which denoted by$({I_C}^*, I^*)$, together with S^* that results from $(4d)$ after substituting $({I_C}^*, I^*)$ in it represents an endemic equilibrium point.

$$f(I_C, I) = \frac{p(\alpha_1 I_C + \alpha_2 I)\, B}{2D_1(1+\gamma_2 I_C + \gamma_3 I) + \gamma_1 B} - I_C(d_1 + \beta_1) = 0 \quad (4e)$$

$$g(I_C, \mathrm{I}) = \frac{(1-P)(\alpha_1 I_C + \alpha_2 I)B}{2D_1(1+\gamma_2 I_C + \gamma_3 I) + \gamma_1 B} - I\left(d_2 + \beta_2 + \frac{a}{1+bI}\right) + \beta_1 I_C = 0 \quad (4f)$$

Clearly as $I \to 0$ the first isoclines (4e) intersects the horizontal $I_C - axis$ at a unique point, say z_1. While as $I \to 0$ in the second isoclines (4f), it will intersect the horizontal $I_C - axis$ at a unique point, say z_2.

Moreover, by using implicit differentiation we obtain that

$$\frac{dI}{dI_C} = -\frac{\partial f}{\partial I_C} \Big/ \frac{\partial f}{\partial I} \quad and \quad \frac{dI}{dI_C} = -\frac{\partial g}{\partial I_C} \Big/ \frac{\partial g}{\partial I}$$

Thus it is easy to verify that the two isoclines (4e) and (4f) intersect at a unique positive intersection point, namely $({I_C}^*, I^*)$, provided that

$$0 < z_1 < z_2 \quad (4g)$$

$$\frac{\partial f}{\partial I_C} > 0 \ ; \ \frac{\partial f}{\partial I} < 0 \ \text{ OR } \ \frac{\partial f}{\partial I_C} < 0; \frac{\partial f}{\partial I} > 0 \quad (4h)$$

$$\frac{\partial g}{\partial I_C} > 0 \ ; \ \frac{\partial g}{\partial I} > 0 \ \text{ OR } \ \frac{\partial g}{\partial I_C} < 0; \frac{\partial g}{\partial I} < 0 \quad (4i)$$

Consequently system (3) will have a unique endemic equilibrium point $E_1 = (S^*, {I_C}^*, I^*)$ provided that the above sufficient conditions (4g)-(4i) are satisfied. ■

It is well known that the basic reproduction number that is denoted by R_0, represents the mean number of secondary infections caused by a single infective introduced into a susceptible population, can be determined using the next generation matrix method [31]

Therefore in order to compute R_0, we set $X = (I_C, I, S)^T$, then system (3) can be rewritten as

$$\frac{dX}{dt} = \mathcal{F}(x) - \nu(x)$$

Here $\mathcal{F}(x)$is the matrix of new infection terms, while $\nu(x)$ is the matrix of transfer terms into compartment and out of compartment, which can be written as

$$\mathcal{F}(x) = \begin{bmatrix} pSH \\ (1-P)SH \\ 0 \end{bmatrix};$$

$$\nu(x) = \begin{bmatrix} d_1 I_C + \beta_1 I_C \\ d_2 I + \beta_2 I + \dfrac{aI}{1+bI} - \beta_1 I_C \\ -A + d_0 S + SH + \theta S \end{bmatrix}$$

Therefore the Jacobian of matrices $\mathcal{F}(x)$ and $\nu(x)$ at the disease free equilibrium E_0 can be written as:

$$D\mathcal{F}(E_0) = \begin{bmatrix} \dfrac{pA\alpha_1}{C_1} & \dfrac{pA\alpha_2}{C_1} & 0 \\ \dfrac{(1-p)A\alpha_1}{C_1} & \dfrac{(1-p)A\alpha_2}{C_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D\nu(E_0) = \begin{bmatrix} C_2 & 0 & 0 \\ -\beta_1 & C_3 & 0 \\ \dfrac{A\alpha_1}{C_1} & \dfrac{A\alpha_2}{C_1} & d_0 + \theta \end{bmatrix}$$

Where $C_1 = d_0 + \theta + \gamma_1 A$, $C_2 = d_1 + \beta_1$

$$C_3 = d_2 + \beta_2 + a$$

Thus we obtain that

$$F = \frac{A}{C_1}\begin{bmatrix} p\alpha_1 & p\alpha_2 \\ (1-p)\alpha_1 & (1-p)\alpha_2 \end{bmatrix}$$

$$V = \begin{bmatrix} C_2 & 0 \\ -\beta_1 & C_3 \end{bmatrix}$$

According to the next generation matrix method, the basic reproduction number is the maximum eigenvalues of the matrix $F\,V^{-1}$. Now since

$$F\,V^{-1} = \frac{A}{C_1C_2C_3}\cdot \begin{bmatrix} p\alpha_1C_3 + p\alpha_2\beta_1 & p\alpha_2C_2 \\ (1-p)\alpha_1C_3 + (1-p)\alpha_2\beta_1 & (1-p)\alpha_2C_2 \end{bmatrix}$$

Therefore the eigenvalues of $F\,V^{-1}$ are given by

$$\lambda_1 = 0$$

$$\lambda_2 = \frac{A(p[\alpha_1C_3 + \alpha_2\beta_1] + (1-p)\alpha_2C_2)}{C_1C_2C_3}$$

From which it is clear that the reproduction number

$$R_0 = \lambda_2 \qquad (5)$$

In the following the local stability analysis of each equilibrium points is studied with the help of basic reproduction number as shown in the next two theorems. First the Jacobian matrices at the equilibrium points E_0 and E_1 are given below respectively.

$$J(E_0) = \begin{bmatrix} -(d_0+\theta) & \frac{-A\alpha_1}{C_1} & \frac{-A\alpha_2}{C_1} \\ 0 & \frac{pA\alpha_1}{C_1} - C_2 & \frac{pA\alpha_2}{C_1} \\ 0 & \frac{(1-p)A\alpha_1}{C_1} + \beta_1 & \frac{(1-p)A\alpha_2}{C_1} - C_3 \end{bmatrix} = [b_{ij}]_{3\times3} \qquad (6)$$

While

$$J(E_1) = [c_{ij}]_{3\times3} \qquad (7)$$

where

$$c_{11} = -d_0 - \frac{H_0H_5}{{H_3}^2} - \theta;\quad c_{12} = -S^*\left(\frac{H_1\alpha_1 - \gamma_2\alpha_2I^*}{{H_3}^2}\right)$$

$$c_{13} = -S^*\left(\frac{H_2\alpha_2 - \gamma_3\alpha_1{I_C}^*}{{H_3}^2}\right),\qquad c_{21} = p\left(\frac{H_0H_5}{{H_3}^2}\right)$$

$$c_{22} = p\,S^*\left(\frac{H_1\alpha_1 - \gamma_2\alpha_2I^*}{{H_3}^2}\right) - C_2$$

$$c_{23} = pS^*\left(\frac{H_2\alpha_2 - \gamma_3\alpha_1{I_C}^*}{{H_3}^2}\right), c_{31} = (1-p)\left(\frac{H_0H_5}{{H_3}^2}\right)$$

$$c_{32} = (1-p)S^*\left(\frac{H_1\alpha_1 - \gamma_2\alpha_2I^*}{{H_3}^2}\right) + \beta_1$$

$$c_{33} = (1-p)S^*\left(\frac{H_2\alpha_2 - \gamma_3\alpha_1{I_C}^*}{{H_3}^2}\right) - H_4$$

Here $H_0 = (1 + \gamma_2{I_C}^* + \gamma_3I^*)$;

$H_1 = (1 + \gamma_1S^* + \gamma_3I^*)$;

$H_2 = (1 + \gamma_1S^* + \gamma_2{I_C}^*)$;

$H_3 = (1 + \gamma_1S^* + \gamma_2{I_C}^* + \gamma_3I^*)$;

$H_4 = d_2 + \beta_2 + \frac{a}{(1+bI^*)}$

and $H_5 = (\alpha_1{I_C}^* + \alpha_2I^*)$

Thus the local stability results near the above equilibrium points can be presented in the following theorems.

Theorem (2):The disease-free equilibrium $E_0 = \left(\frac{A}{d_0+\theta}, 0, 0\right)$ is locally asymptotically stable when $R_0 < 1$ and unstable for $R_0 > 1$.

Proof. Accordingly to the Jacobian matrix $J(E_0)$, given by Eq. (6), the characteristic equation can be written as:

$$(b_{11} - \lambda)(\lambda^2 - T\lambda + D) = 0 \qquad (8)$$

where $T = b_{22} + b_{33}$ and $D = b_{22}b_{33} - b_{32}b_{23}$. Clearly the first eigenvalue that given by $\lambda_1 = b_{11} = -(d_0 + \theta) < 0$, while the other two eigenvalues, which represent the roots of second order polynomial part of (8), have negative real parts if and only if $T < 0$ and $D > 0$.

Straightforward computation shows that $R_0 < 1$ guarantees that $b_{22} < 0$ and $b_{33} < 0$ and hence $T < 0$. On the other hand, direct calculation shows that

$$D = -\frac{pA\alpha_1C_3 + pA\alpha_2\beta_1 + (1-p)A\alpha_2C_2}{C_1} + C_2C_3 \qquad (9)$$

It is easy to verify that $D > 0$ provided that $R_0 < 1$ and $D < 0$ provided that $R_0 > 1$. Thus E_0 is locally asymptotically stable under the condition $R_0 < 1$ and unstable saddle point for $R_0 > 1$. Hence the proof is complete. ■

Theorem (3): The endemic equilibrium point $E_1$of system (3) is locally asymptotically stable if the following sufficient conditions hold

$$0 < p\,S^*(H_1\alpha_1 - \gamma_2\alpha_2I^*) < C_2{H_3}^2 \qquad (10a)$$

$$0 < (1-p)S^*(H_2\alpha_2 - \gamma_3\alpha_1 {I_C}^*) < {H_3}^2 H_4 \quad (10b)$$

$$H_4\left[C_2{H_3}^2 - p\,S^*(H_1\alpha_1 - \gamma_2\alpha_2 I^*)\right] > S^*(H_2\alpha_2 - \gamma_3\alpha_1 {I_C}^*)\left((1-p)d_1 + \beta_1\right) \quad (10c)$$

$$c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} > 2c_{11}c_{22}c_{33} \quad (10d)$$

Proof. According to the Jacobian matrix given in (7), the characteristic equation can be written as:

$$\lambda^3 + W_1\lambda^2 + W_2\lambda + W_3 = 0 \quad (11)$$

where $W_1 = -(c_{11} + c_{22} + c_{33})$

$$W_2 = c_{11}c_{22} - c_{12}c_{21} + c_{11}c_{33} - c_{13}c_{31} + c_{22}c_{33} - c_{23}c_{32}$$

$$W_3 = -[c_{11}(c_{22}c_{33} - c_{23}c_{32}) + c_{12}(c_{23}c_{31} - c_{21}c_{33}) + c_{13}(c_{21}c_{32} - c_{22}c_{31})]$$

while

$$\Delta = W_1W_2 - W_3$$
$$= -(c_{11} + c_{22})(c_{11}c_{22} - c_{12}c_{21}) - (c_{11} + c_{33})(c_{11}c_{33} - c_{13}c_{31}) - (c_{22} + c_{33})(c_{22}c_{33} - c_{23}c_{32}) - 2c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32}$$

Now according to Routh-Hurwitz criterion [32] the endemic equilibrium point E_1 is locally asymptotically stable, provided that $W_1 > 0, W_3 > 0$ and $\Delta = W_1W_2 - W_3 > 0$. Straightforward computation shows that conditions (10a)-(10b) guarantee that $c_{12} < 0$, $c_{13} < 0$, $c_{22} < 0$, $c_{23} > 0$, $c_{32} > 0$ and $c_{33} < 0$. Hence $W_1 > 0$, further the conditions (10a)-(10c) lead to $W_3 > 0$. Finally its easy to verify that the given conditions guarantee that $\Delta > 0$. Thus the proof is complete.

Now the global stability analysis of each equilibrium points of system (3) is studied analytically with the help of Lyapunov method [33]as shown in the following theorems.

Theorem (4): Assume that the equilibrium point E_0 is a locally asymptotically stable in R_+^3, then it is a globally asymptotically stable provided that

$$d_1(d_0 + \theta)^2 > \alpha_1 A(A + d_0 + \theta) \quad (12a)$$

$$(d_2 + \beta_2)(d_0 + \theta)^2 > \alpha_2 A(A + d_0 + \theta) \quad (12b)$$

Proof. Consider the following positive definite real valued function

$$L_1 = \frac{(S - S_0)^2}{2} + I_C + I$$

Clearly, $L_1: R^3 \to R$ that is continuously differentiable function so that $L_1(S_0, 0, 0) = 0$ and $L_1(S, I_C, I) > 0\ \forall (S, I_C, I) \neq (S_0, 0, 0)$. Therefore by differentiating this function with respect to the time, we get:

$$\frac{dL_1}{dt} = \frac{\partial L_1}{\partial S}.\frac{dS}{dt} + \frac{\partial L_1}{\partial I_C}.\frac{dI_C}{dt} + \frac{\partial L_1}{\partial I}.\frac{dI}{dt}$$

Substituting the value of $\frac{dS}{dt}$, $\frac{dI_C}{dt}$ and $\frac{dI}{dt}$ from system (3) in the above equation gives

$$\frac{dL_1}{dt} = (S - S_0)[A - d_0S - SH - \theta S] + [pSH - d_1I_C - \beta_1I_C] + \left[(1-P)SH - d_2I - \beta_2I - \frac{aI}{1+bI} + \beta_1I_C\right]$$

Now, by doing some algebraic manipulation

$$\frac{dL_1}{dt} = -(d_0 + \theta)(S - S_0)^2 + S[-S + S_0 + 1]H - d_1I_C - I\left[(d_2 + \beta_2) + \frac{a}{1+bI}\right]$$

Consequently, we obtain that

$$\frac{dL_1}{dt} \leq -(d_0 + \theta)(S - S_0)^2 + (S_0 + 1)S(\alpha_1I_C + \alpha_2I) - d_1I_C - \left(d_2 + \beta_2 + \frac{a}{1+bI}\right)I$$

Now, since $S(t) \leq \frac{A}{d_0+\theta}$ and $S_0 = \frac{A}{d_0+\theta}$ then we obtain that

$$\frac{dL_1}{dt} \leq -(d_0 + \theta)(S - S_0)^2 - \left[d_1 - \alpha_1\frac{A(A+d_0+\theta)}{(d_0+\theta)^2}\right]I_C - \left[d_2 + \beta_2 + \frac{a}{1+bI} - \alpha_2\frac{A(A+d_0+\theta)}{(d_0+\theta)^2}\right]I$$

Therefore due to the conditions (12a)-(12b) we obtain that $\frac{dL_1}{dt} < 0$. Hence E_0 is a locally asymptotically stable and the proof is complete ■

Theorem (5): Assume that the endemic equilibrium point $E_1$is a locally asymptotically stable, then it is a globally asymptotically stable in the region that satisfy the following conditions:

$$C_2H_3\bar{H}_3 > pS^*B_1 \quad (13a)$$

$$(d_2 + \beta_2)H_3\bar{H}_3 + B_3 > (1-p)S^*B_2 \quad (13b)$$

$${d_{12}}^2 < d_{11}d_{22} \quad (13c)$$

$${d_{13}}^2 < d_{11}d_{33} \quad (13d)$$

$${d_{23}}^2 < d_{22}d_{33} \quad (13e)$$

where d_{ij} will be given in the proof.

Proof. Consider the following positive definite real valued function

$$L_2 = \frac{(S-S^*)^2}{2} + \frac{(I_C - {I_C}^*)^2}{2} + \frac{(I-I^*)^2}{2}$$

Clearly $L_2 : R^3 \to R$ be a continuously differentiable function so that $L_2(S^*, {I_C}^*, I^*) = 0$ and $L_2(S, I_C, I) > 0 \ , \forall (S, I_C, I) \neq (S^*, {I_C}^*, I^*)$. Therefore by differentiating this function with respect to the time, we get:

$$\frac{dL_2}{dt} = \frac{\partial L_2}{\partial S}.\frac{dS}{dt} + \frac{\partial L_2}{\partial I_C}.\frac{dI_C}{dt} + \frac{\partial L_2}{\partial I}.\frac{dI}{dt}$$

Substituting the value of $\frac{dS}{dt}, \frac{dI_C}{dt}$ and $\frac{dI}{dt}$ from system (3) in the above equation and doing some algebraic manipulation gives that

$$\begin{aligned}
\frac{dL_2}{dt} &= -\left[\frac{(d_0+\theta)H_3\bar{H}_3 + H_0\bar{H}_5}{H_3\bar{H}_3}\right](S-S^*)^2 \\
&-\left[\frac{C_2H_3\bar{H}_3 - pS^*B_1}{H_3\bar{H}_3}\right](I_C - {I_C}^*)^2 \\
&-\left[\frac{(d_2+\beta_2)H_3\bar{H}_3 + B_3 - (1-p)S^*B_2}{H_3\bar{H}_3}\right](I-I^*)^2 \\
&+\left[\frac{pH_0\bar{H}_5 - S^*B_1}{H_3\bar{H}_3}\right](S-S^*)(I_C - {I_C}^*) \\
&+\left[\frac{(1-p)H_0\bar{H}_5 - S^*B_2}{H_3\bar{H}_3}\right](S-S^*)(I-I^*) \\
&+\left[\frac{pS^*B_2 + (1-p)S^*B_1 + \beta_1H_3\bar{H}_3}{H_3\bar{H}_3}\right](I_C - {I_C}^*)(I-I^*)
\end{aligned}$$

Here $B_1 = \alpha_1(1+\gamma_1S+\gamma_3I^*) - \alpha_2\gamma_2I^*$;

$B_2 = \alpha_2(1+\gamma_1S+\gamma_2{I_C}^*) - \alpha_1\gamma_3{I_C}^*$;

$$B_3 = \frac{aH_3\bar{H}_3}{(1+bI^*)(1+bI)}$$

$\bar{H}_3 = 1+\gamma_1S+\gamma_2I_C+\gamma_3I$;

$\bar{H}_5 = \alpha_1I_C + \alpha_2I$;

Clearly the coefficient of $(I_C - {I_C}^*)^2$ and $(I-I^*)^2$ are negative due to the conditions (13a) and (13b) respectively. However the other conditions guarantee that

$$\begin{aligned}
\frac{dL_2}{dt} \leq &-\frac{1}{H_3\bar{H}_3}\left[\sqrt{\frac{d_{11}}{2}}(S-S^*) - \sqrt{\frac{d_{22}}{2}}(I_C - {I_C}^*)\right]^2 \\
&-\frac{1}{H_3\bar{H}_3}\left[\sqrt{\frac{d_{11}}{2}}(S-S^*) - \sqrt{\frac{d_{33}}{2}}(I-I^*)\right]^2 \\
&-\frac{1}{H_3\bar{H}_3}\left[\sqrt{\frac{d_{22}}{2}}(I_C - {I_C}^*) - \sqrt{\frac{d_{33}}{2}}(I-I^*)\right]^2
\end{aligned}$$

Here

$$\begin{aligned}
d_{11} &= (d_0+\theta)H_3\bar{H}_3 + H_0\bar{H}_5 \\
d_{22} &= C_1H_3\bar{H}_3 - pS^*B_1 \\
d_{33} &= (d_2+\beta_2)H_3\bar{H}_3 + B_3 - (1-p)S^*B_2 \\
d_{12} &= pH_0\bar{H}_5 - S^*B_1 \\
d_{13} &= (1-p)H_0\bar{H}_5 - S^*B_2 \\
d_{23} &= pS^*B_2 + (1-p)S^*B_1 + \beta_1H_3\bar{H}_3
\end{aligned}$$

Obviously due to the given condition the value of $\frac{dL_2}{dt}$ is negative definite and hence the endemic equilibrium point $E_1$is a globallyasymptotically stable.

■

4. BIFURCATION ANALYSIS:

In this section, the possibility of occurrence of local bifurcation (such as transcritical, pitchfork and saddle-node) around equilibrium points is studied by applying the Sotomayor's theorem [34]. Consider system (3), which can be rewritten as

$$\frac{dY}{dt} = H(Y) \qquad (14)$$

where $Y = (S, I_C, I)^T$ and $H(Y) = (h_1, h_2, h_3)^T$ represents the vector of interaction functions in system (3). Recall that, the general Jacobian matrix of system (3) can be represented by

$$DH = \frac{\partial H}{\partial Y} = J(S, I_C, I) \qquad (15)$$

Then straightforward computation shows that for any non-zero vector $V = (v_1, v_2, v_3)^T$ the second derivative of the vector $H(Y)$ with respect to Y can be written as the following:

$$D^2H(\mathrm{V},\mathrm{V}) = \left(u_{ij}\right)_{3\times1} \qquad (16)$$

where

$$\begin{aligned}
u_{11} &= \mathrm{e}_1{v_1}^2 - (\mathrm{e}_2+\mathrm{e}_4)v_1v_2 - (\mathrm{e}_3+\mathrm{e}_7)v_1v_3 + \mathrm{e}_5{v_2}^2 \\
&\quad - (\mathrm{e}_6+\mathrm{e}_8)v_2v_3 + \mathrm{e}_9{v_3}^2
\end{aligned}$$

$$\begin{aligned}
u_{21} &= p[-\mathrm{e}_1{v_1}^2 + (\mathrm{e}_2+\mathrm{e}_4)v_1v_2 + (\mathrm{e}_3+\mathrm{e}_7)v_1v_3 - \mathrm{e}_5{v_2}^2 \\
&\quad + (\mathrm{e}_6+\mathrm{e}_8)v_2v_3 - \mathrm{e}_9{v_3}^2]
\end{aligned}$$

$$\begin{aligned}
u_{31} &= (1-p)\Big[-\mathrm{e}_1{v_1}^2 + (\mathrm{e}_2+\mathrm{e}_4)v_1v_2 + (\mathrm{e}_3+e_7)v_1v_3 \\
&\quad - \mathrm{e}_5{v_2}^2 + (\mathrm{e}_6+\mathrm{e}_8)v_2v_3 - (\mathrm{e}_9 \\
&\quad + \tfrac{ab}{(1+bI)^2})v_3{}^2\Big]
\end{aligned}$$

Here

$$\mathrm{e}_1 = \frac{2\gamma_1\bar{H}_0\bar{H}_5}{{\bar{H}_3}^3}$$

$$\mathrm{e}_2 = \frac{\bar{H}_3[\alpha_1\bar{H}_6 - \alpha_2\gamma_2I] - 2\gamma_1S(\alpha_1\bar{H}_1 - \alpha_2\gamma_2I)}{{\bar{H}_3}^3}$$

$$e_3 = \frac{\bar{H}_3[\alpha_2\bar{H}_7 - \alpha_1\gamma_3 I_C] - 2\gamma_1 S(\alpha_2\bar{H}_2 - \alpha_1\gamma_3 I_C)}{\bar{H}_3{}^3}$$

$$e_4 = \frac{\bar{H}_3[\alpha_1\bar{H}_8 + \alpha_2\gamma_2 I\,] - 2\gamma_2\bar{H}_0\bar{H}_5}{\bar{H}_3{}^3}$$

$$e_5 = 2\gamma_2 S\left(\frac{\alpha_1\bar{H}_1 - \alpha_2\gamma_2 I}{\bar{H}_3{}^3}\right)$$

$$e_6 = \frac{S(\gamma_2\alpha_2 - \alpha_1\gamma_3)\bar{H}_3 - 2S\gamma_2(\alpha_2\bar{H}_2 - \alpha_1\gamma_3 I_C)}{\bar{H}_3{}^3}$$

$$e_7 = \frac{\bar{H}_3[\alpha_1 I_C\gamma_3 + \alpha_2\bar{H}_9] - 2\gamma_3(\bar{H}_0\bar{H}_5)}{\bar{H}_3{}^3}$$

$$e_8 = S\left(\frac{(\gamma_3\alpha_1 - \alpha_2\gamma_2)\bar{H}_3 - 2\gamma_3(\alpha_1\bar{H}_1 - \alpha_2\gamma_2 I)}{\bar{H}_3{}^3}\right)$$

$$e_9 = 2\gamma_3 S\left(\frac{(\alpha_2\bar{H}_2 - \alpha_1\gamma_3 I_C)}{\bar{H}_3{}^3}\right)$$

With $\bar{H}_0 = (1 + \gamma_2 I_C + \gamma_3 I)$;$\bar{H}_1 = (1 + \gamma_1 S + \gamma_3 I)$; $\bar{H}_2 = (1 + \gamma_1 S + \gamma_2 I_C)$,$\bar{H}_3 = (1 + \gamma_1 S + \gamma_2 I_C + \gamma_3 I)$

$\bar{H}_4 = d_2 + \beta_2 + \frac{a}{(1+bI)}$, $\bar{H}_5 = (\alpha_1 I_C + \alpha_2 I)$

$\bar{H}_6 = (1 + 2\gamma_1 S + \gamma_3 I)$,$\bar{H}_7 = (1 + 2\gamma_1 S + \gamma_2 I_C)$, $\bar{H}_8 = (1 + 2\gamma_2 I_C + \gamma_3 I)$, $\bar{H}_9 = (1 + \gamma_2 I_C + 2\gamma_3 I)$

Theorem (6): If the following condition holds

$$R_0 = 1 \qquad (17)$$

Then system (3) near the free disease equilibrium point E_0 has

1. No saddle node bifurcation
2. A transcritical bifurcation

Proof. It is easy to verify that the Jacobain matrix of system (3) at E_0 that given by matrix$[b_{ij}]_{3\times3}$ in Eq. (6) has zero eigenvalue (say $\lambda_1^0 = 0$)provided that $R_0 = 1$ or equivalently at

$$\widetilde{\alpha_2} \equiv \alpha_2 = C_3\frac{C_1C_2 - pA\alpha_1}{A\ (p\beta_1 + (1-p)C_2)} \qquad (18)$$

Clearly α_2 is positive by construction and it considered as a candidate parameter bifurcation (similarly $R_0 = 1$ is equivalent to any other parameter).

Now by substituting the value of $(E_0\,, \widetilde{\alpha_2})$in the Eq. (6), its obtain that:

$$\check{J} = J(E_0, \widetilde{\alpha_2}) = \begin{bmatrix} -(d_0+\theta) & \frac{-A\alpha_1}{C_1} & \frac{-A\widetilde{\alpha_2}}{C_1} \\ 0 & \frac{pA\alpha_1}{C_1} - C_2 & \frac{pA\widetilde{\alpha_2}}{C_1} \\ 0 & \frac{(1-p)A\alpha_1}{C_1} + \beta_1 & \frac{(1-p)A\widetilde{\alpha_2}}{C_1} - C_3 \end{bmatrix}$$

Let $K = (K_1, K_2, K_3)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_1^0 = 0$ of $\check{J}$. Then straightforward computation to solve $\check{J}K = \mathbf{0}$, gives that

$$K = (k_{11}K_3\,, k_{21}K_3\,,\ K_3)^T \qquad (19a)$$

Where

$$k_{11} = \frac{-C_2C_3}{(d_0+\theta)(p\beta_1 + (1-p)C_2)} < 0$$

$$k_{21} = \frac{pC_3}{p\beta_1 + (1-p)C_2} > 0$$

and K_3 be any non-zero real number. On the other hand let $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_1^0 = 0$ of the matrix $\check{J}^T$. Then straightforward computation to solve $\check{J}^T\Psi = \mathbf{0}$ gives that

$$\Psi = (0, \omega_{21}\Psi_3\,, \Psi_3)^T \qquad (19b)$$

where

$$\omega_{21} = \frac{-\big((1-p)A\alpha_1 + \beta_1C_1\big)}{pA\alpha_1 - C_2C_1}$$

and $\Psi_3$be any non-zero real number, also its clear that $pA\alpha_1 - B_2B_1 < 0$ due to the condition (17) and hence $\omega_{21} > 0$.

Now since

$$\frac{\partial H}{\partial\alpha_2} = H_{\alpha_2}(Y, \alpha_2) = \left(\frac{-SI}{\bar{H}_3}, \frac{pSI}{\bar{H}_3}, \frac{(1-p)SI}{\bar{H}_3}\right)^T$$

Then $H_{\alpha_2}(E_0\,, \widetilde{\alpha_2}) = (0,0,0)^T$

Hence $\Psi^T H_{\alpha_2}(E_0\,, \widetilde{\alpha_2}) = 0$

Thus, according to Sotomayor's theorem [34] the saddle-node bifurcation cannot occur. Furthermore the derivative of $H_{\alpha_2}(Y, \alpha_2)$ with respect to Y can be written

$$DH_{\alpha_2}(Y, \alpha_2) = \frac{1}{\bar{H}_3{}^2}\begin{bmatrix} -I\bar{H}_0 & SI\gamma_2 & -S\bar{H}_2 \\ p\,I\bar{H}_0 & -p\,SI\gamma_2 & pS\bar{H}_2 \\ (1-p)I\bar{H}_0 & -(1-p)SI\gamma_2 & (1-p)S\bar{H}_2 \end{bmatrix}$$

Hence

$$DH_{\alpha_2}(E_0\,, \widetilde{\alpha_2}) = \begin{bmatrix} 0 & 0 & \frac{-A}{C_1} \\ 0 & 0 & p\frac{A}{C_1} \\ 0 & 0 & (1-p)\frac{A}{C_1} \end{bmatrix}$$

Therefore, it's easy to get that

$$\Psi^T\left[DH_{\alpha_2}(Y,\alpha_2)K\right] = -AK_3\Psi_3(p\beta_1+(1-p)C_2) \neq 0$$

Moreover, by substituting the values of $(E_0, \widetilde{\alpha_2})$ and the eigenvector K in Eq. (16) it's obtain that

$$u_{11}(E_0, \widetilde{\alpha_2}) = \left[-2m_1k_{11}k_{21} - 2m_2k_{11} + m_3{k_{21}}^2 + 2m_4k_{21} + 2m_5\right]{K_3}^2$$

$$u_{21}(E_0, \widetilde{\alpha_2}) = p\left[2m_1k_{11}k_{21} + 2m_2k_{11} - 2m_3{k_{21}}^2 - 2m_4k_{21} - 2m_5\right]{K_3}^2$$

$$u_{31}(E_0, \widetilde{\alpha_2}) = (1-p)[2m_1k_{11}k_{21} + 2m_2k_{11} - 2m_3{k_{21}}^2 - 2m_4k_{21} - (2m_5+ab)]{K_3}^2$$

Where

$$m_1 = \frac{\alpha_1}{{\bar{\bar{H}}_3}^2}, \qquad m_2 = \frac{\alpha_2}{{\bar{\bar{H}}_3}^2}, \qquad m_3 = \frac{\alpha_1\gamma_2 S_0}{{\bar{\bar{H}}_3}^2},$$

$$m_4 = S_0\left(\frac{\alpha_1\gamma_3+\alpha_2\gamma_2}{{\bar{\bar{H}}_3}^2}\right), \qquad m_5 = \gamma_3 S_0\left(\frac{\alpha_2}{{\bar{\bar{H}}_3}^2}\right)$$

Here $\bar{\bar{H}}_3 \equiv \bar{H}_3(E_0) = (1+\gamma_1 S_0)$. Consequently,

$$\Psi^T\left[D^2H(E_0, \widetilde{\alpha_2})(K,K)\right] = 2\Psi_3{K_3}^2\left[(p\omega_{21}+(1-p))\left(m_1k_{11}k_{21} + m_2k_{11} - m_3{k_{21}}^2 - m_4k_{21}\right) - (1-p)\frac{ab}{2}\right]$$

Clearly, due to the sign of k_{11}, k_{21} and ω_{21}, the value of $\Psi^T\left[D^2H(E_0, \widetilde{\alpha_2})(K,K)\right] \neq 0$. Indeed it has negative sign and hence according to Sotomayor's theorem system (3) has a transcritical bifurcation at E_0 provided that $R_0 = 1 \quad or \quad \alpha_2 = \widetilde{\alpha_2}$. ■

Theorem (7): Assume that conditions (10a)-(10b) hold, then system (3) undergoes a saddle node bifurcation near the endemic equilibrium point, as the parameter β_2 passing the following value

$$\tilde{\beta}_2 = \frac{(d_0+\theta)S^*B_5[p\beta_1+(1-p)C_2]}{C_2H_0H_5+(d_0+\theta)\bar{B}} - \left(d_2 + \frac{a}{(1+bI^*)}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Provided that the following conditions are hold.

$$H_4 < \frac{(d_0+\theta)S^*B_5[p\beta_1+(1-p)C_2]}{C_2H_0H_5+(d_0+\theta)\bar{B}} \qquad (21a)$$

$${H_3}^2 > max\left\{\frac{(\gamma_1S^*B_4+\gamma_2H_0H_5)}{\alpha_1}, \frac{(\gamma_1S^*B_5+\gamma_3H_0H_5)}{\alpha_2}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12b)$$

Here $B_4 = (H_1\alpha_1 - \gamma_2\alpha_2I^*)$; $B_5 = (H_2\alpha_2 - \gamma_3\alpha_1{I_C}^*)$

and $\bar{B} = B_2{H_3}^2 - p\,S^*B_4$

Proof. According to the parameter value $\tilde{\beta}_2$ given by Eq. (20), which is positive under the condition (21a), it is easy to verify that the value of the determinant of $J(E_1)$, that given by W_3 in the Eq. (11), is $W_3(\tilde{\beta}_2) = 0$.

Therefore the Jacobian matrix $\bar{\bar{J}} = J\left(E_1, \tilde{\beta}_2\right) = \left(\bar{\bar{c}}_{ij}\right)_{3\times3}$ has zero eigenvalue, say $\bar{\bar{\lambda}} = 0$, where $\bar{\bar{c}}_{ij} = c_{ij};\ \forall\, i,j = 1{,}23$ that given in Eq. (7) while $\bar{\bar{c}}_{33} = c_{33}(\tilde{\beta}_2)$. Hence E_1 is a non-hyperbolic point.

Recall that conditions (10a)-(10b) specify the sign of c_{ij} as mentioned in theorem (3). While condition (21a)-(21b) guarantees the positivity of $\tilde{\beta}_2$ in addition to possibility of having zero determinant of $J(E_1)$.

Let $\mathcal{Z} = (z_1, z_2, z_3)^T$ be the eigenvector that corresponding to the eigenvalue $\bar{\bar{\lambda}} = 0$ of the matrix $\bar{\bar{J}}$. Thus $\bar{\bar{J}}\mathcal{Z} = \mathbf{0}$, which gives:

$$\mathcal{Z} = (\xi_1z_3, \xi_2z_3, z_3)^T \qquad (22a)$$

Here

$$\xi_1 = \frac{-S^*B_5\left(C_2H_0H_5 + (d_0+\theta)C_2{H_3}^2\right)}{\left(H_0H_5+(d_0+\theta){H_3}^2\right)\left(C_2H_0H_5+(d_0+\theta)\bar{B}\right)}$$

$$\xi_2 = \frac{p(d_0+\theta)S^*B_5}{C_2H_0H_5+(d_0+\theta)\bar{B}}$$

While z_3 be any nonzero real number. Clearly due to condition (10a) then $\xi_1 < 0$ and $\xi_2 > 0$.

Let $U = (u_1, u_2, u_3)^T$ be the eigenvector associated with the eigenvalue $\bar{\bar{\lambda}} = 0$ of the matrix $\bar{\bar{J}}^T$. Then solving $\bar{\bar{J}}^TU = 0$, gives

$$U = (\varpi_1u_3, \varpi_2u_3, u_3)^T \qquad (22b)$$

Here

$$\varpi_1 = \frac{H_0H_5[p\beta_1+(1-p)C_2]}{C_2H_0H_5+(d_0+\theta)\bar{B}}$$

$$\varpi_2 = \frac{\beta_1H_0H_5 + (d_0+\theta)[(1-p)S^*B_4 + \beta_1{H_3}^2]}{(C_2H_0H_5+(d_0+\theta)\bar{B})}$$

While u_3 be any nonzero real number. Again due to condition (10a) then $\varpi_1 > 0$ and $\varpi_2 > 0$.

Now according to the Sotomayor's theorem system (3) undergoes a saddle node bifurcation near the endemic equilibrium point if and only if $H_{\beta_2}\left(E_1, \tilde{\beta}_2\right) \neq 0$ and $U^TD^2H\left(E_1, \tilde{\beta}_2\right)(\mathcal{Z},\mathcal{Z}) \neq 0$. Since

$$\frac{\partial H}{\partial\beta_2} = H_{\beta_2}(Y,\beta_2) = \left(\frac{\partial h_1}{\partial\beta_2}, \frac{\partial h_2}{\partial\beta_2}, \frac{\partial h_3}{\partial\beta_2}\right)^T = (0,0,I)^T$$

Then $H_{\beta_2}(E_1, \tilde{\beta}_2) = (0,0,I^*)^T$, and hence

$$U^T H_{\beta_2}(E_1, {\beta_2}^*) = -I^* u_3 \neq 0$$

Further, by substituting the values of $(E_1, \tilde{\beta}_2)$ and the eigenvector $\mathcal{Z}$ in Eq. (16) and doing some algebraic manipulations gives that

$$u_{11} = {z_3}^2\big[\mathrm{e}_1{\xi_1}^2 - (\mathrm{e}_2 + \mathrm{e}_4)\xi_1\xi_2 - (\mathrm{e}_3 + \mathrm{e}_7)\xi_1 + 2\mathrm{e}_5{\xi_2}^2 - (\mathrm{e}_6 + \mathrm{e}_8)\xi_2 + \mathrm{e}_9\big]$$

$$u_{21} = p{z_3}^2\big[-\mathrm{e}_1{\xi_1}^2 + (\mathrm{e}_2 + \mathrm{e}_4)\xi_1\xi_2 + (\mathrm{e}_3 + \mathrm{e}_7)\xi_1 - 2\mathrm{e}_5{\xi_2}^2 + (\mathrm{e}_6 + \mathrm{e}_8)\xi_2 - \mathrm{e}_9\big]$$

$$u_{31} = (1-p){z_3}^2\big[-e_1{\xi_1}^2 + (e_2 + e_4)\xi_1\xi_2 + (e_3 + e_7)\xi_1 - 2e_5{\xi_2}^2 + (e_6 + e_8)\xi_2 - (e_9 + \tfrac{ab}{(1+bI^*)^2})\big]$$

Here

$$\mathrm{e}_1 = \frac{2\gamma_1 H_0 H_5}{{H_3}^3}, \qquad \mathrm{e}_2 = \frac{H_3 B_6 - 2\gamma_1 S^* B_4}{{H_3}^3},$$

$$e_3 = \frac{H_3 B_7 - 2\gamma_1 S^* B_5}{{H_3}^3}, e_4 = \frac{H_3 B_8 - 2\gamma_2 H_0 H_5}{{H_3}^3}$$

$$\mathrm{e}_5 = \gamma_2 S^*\left(\frac{B_4}{{H_3}^3}\right), e_6 = S^*\left(\frac{(\gamma_2\alpha_2 - \alpha_1\gamma_3)H_3 - 2\gamma_2 B_5}{{H_3}^3}\right)$$

$$\mathrm{e}_7 = \frac{H_3 B_9 - 2\gamma_3(H_0 H_5)}{{H_3}^3},$$

$$e_8 = S^*\left(\frac{(\gamma_3\alpha_1 - \alpha_2\gamma_2)H_3 - 2\gamma_3 B_4}{{H_3}^3}\right), e_9 = \gamma_3 S^*\left(\frac{B_5}{{H_3}^3}\right)$$

With $H_0 = (1 + \gamma_2 {I_C}^* + \gamma_3 I^*); H_1 = (1 + \gamma_1 S^* + \gamma_3 I^*)$

$H_2 = (1 + \gamma_1 S^* + \gamma_2 {I_C}^*)$,

$H_3 = (1 + \gamma_1 S^* + \gamma_2 {I_C}^* + \gamma_3 I^*)$

$H_4 = d_2 + \beta_2 + \dfrac{a}{(1 + bI^*)}$

$H_5 = (\alpha_1 {I_C}^* + \alpha_2 I^*)$

$H_6 = (1 + 2\gamma_1 S^* + \gamma_3 I^*)$

$H_7 = (1 + 2\gamma_1 S^* + \gamma_2 {I_C}^*)$

$H_8 = (1 + 2\gamma_2 {I_C}^* + \gamma_3 I^*)$

$H_9 = (1 + \gamma_2 {I_C}^* + 2\gamma_3 I^*)$

while $B_6 = \alpha_1 H_6 - \alpha_2\gamma_2 I^*$; $B_7 = \alpha_2 H_7 - \alpha_1\gamma_3 {I_C}^*$

$B_8 = \alpha_1 H_8 + \alpha_2\gamma_2 I^*$; $B_9 = \alpha_2 H_9 + \alpha_1\gamma_3 {I_C}^*$

Clearly here conditions (10a)-(10b) guarantee that $B_6 > 0$ and $B_7 > 0$. Consequently straightforward computation gives that

$$U^T D^2 H(E_1, {\beta_2}^*)(\mathcal{Z}, \mathcal{Z}) = {z_3}^2 u_3\big[(\varpi_1 - p\varpi_2 - (1-p))\big(e_1{\xi_1}^2 - (e_2 + e_4)\xi_1\xi_2 - (e_3 + e_7)\xi_1 + e_5{\xi_2}^2 - (e_6 + e_8)\xi_2 + e_9\big) - (1-p)\tfrac{ab}{(1+bI^*)^2}\big]$$

Now, since

$$e_2 + e_4 = 2\frac{\alpha_1 {H_3}^2 - (\gamma_1 S^* B_4 + \gamma_2 H_0 H_5)}{{H_3}^3}$$

$$\mathrm{e}_3 + \mathrm{e}_7 = 2\frac{\alpha_2 {H_3}^2 - (\gamma_1 S^* B_5 + \gamma_3 H_0 H_5)}{{H_3}^3}$$

$$\mathrm{e}_6 + \mathrm{e}_8 = -2S^*\left(\frac{\gamma_2 B_5 + \gamma_3 B_4}{{H_3}^3}\right)$$

and

$$\varpi_1 - p\varpi_2 - (1-p) = \frac{-(d_0 + \theta){H_3}^2\,[\beta_1 p + (1-p)C_2]}{C_2 H_0 H_5 + (d_0 + \theta)\bar{B}}$$

Obviously the sufficient condition (21b) ensures the positivity of $(e_2 + e_4)$ and $(\mathrm{e}_3 + \mathrm{e}_7)$. Therefore $U^T D^2 H(E_1, {\beta_2}^*)(\mathcal{Z}, \mathcal{Z}) \neq 0$ and hence saddle node bifurcation takes place, which complete the proof.

■

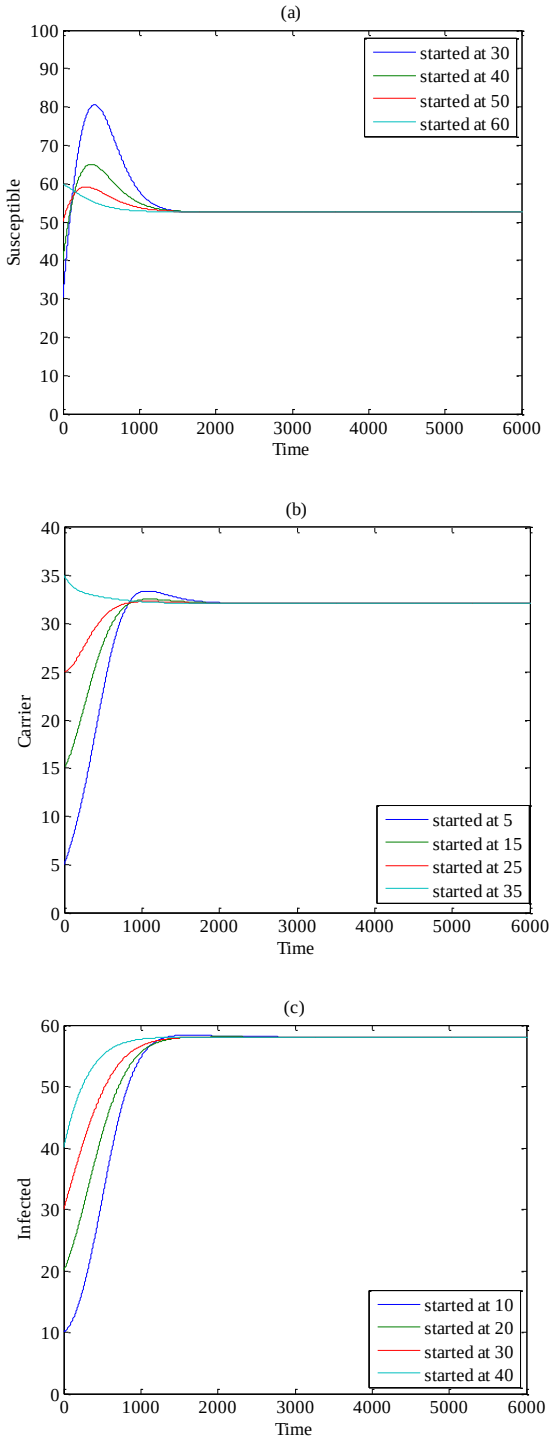
5. NUMERICAL SIMULATIONS AND DISCUSSION

In this section, the global dynamical behavior of system (1) is investigated numerically in order to detect about the occurrence of bifurcation and confirm our obtained analytical results.

For the following biologically reasonable set of parameter values system (1) is solved numerically starting from different sets of initial conditions using six order Runge-Kutta method together with Predector-Corrector method and then the trajectories of the system are drawn in the form of time series as shown below

It is observed that for the following set Eq. (23) of parameter values system (1) is solved numerically starting from different sets of initial conditions and then the trajectories are drawn in Fig. (1), which is clearly shows the global convergence of the solution of system (1) to the endemic equilibrium point. Moreover the trajectory of system (1) starting from a specific initial set is drawn in Fig. (2) as a function of time.

$$A = 40, p = 0.5, d_0 = 0.1, d_1 = d_2 = 0.3, \\ d_3 = 0.2, \beta_1 = 0.2, \beta_2 = 0.1, \alpha_1 = \alpha_2 = 0.2, \\ \gamma_1 = \gamma_2 = \gamma_3 = 0.2, \theta = 0.05, a = b = 0.4 \qquad (23)$$



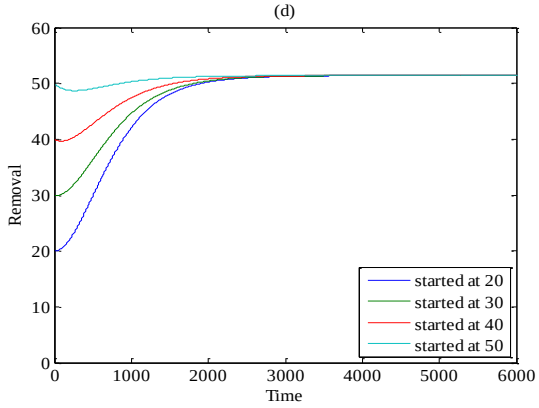


Figure 1: Globally asymptotically stable an endemic equilibrium point $E_1 = (52.62, 32.1, 58.07, 51.4)$ of system (1) for the parameters set (23), started from different sets of initial point. (a) Trajectory of susceptible population. (b) Trajectory of carrier population. (c) Trajectory of infected population. (d) Trajectory of removal population.

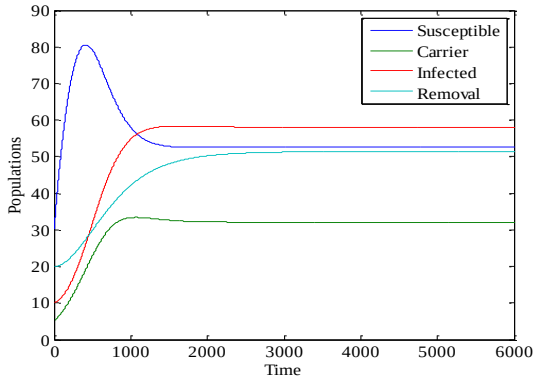


Figure 2: The trajectory of system (1) for the data (23) approaches asymptotically to the endemic equilibrium point, started from sets of initial point (30, 5, 10, 20).

Obviously, the above two figures confirm our obtained analytical results regarding to the destabilizing of the disease free equilibrium point when the basic reproduction number becomes $R_0 > 1$, which can be easily verified, and the existence of an unique endemic equilibrium point.

Now in order to determine the effects of varying the parameters values on the dynamics of system (1) and specify the control set of parameter values in system (1), the system is solved numerically for the parameter set (23) with varying one parameter each time. Consequently, system (1) is solved numerically for data (23) with varying the influx rate of susceptible individuals at $A = 5$ and $A = 0.5$ respectively, and then the trajectory of the system is drawn in Fig. (3).

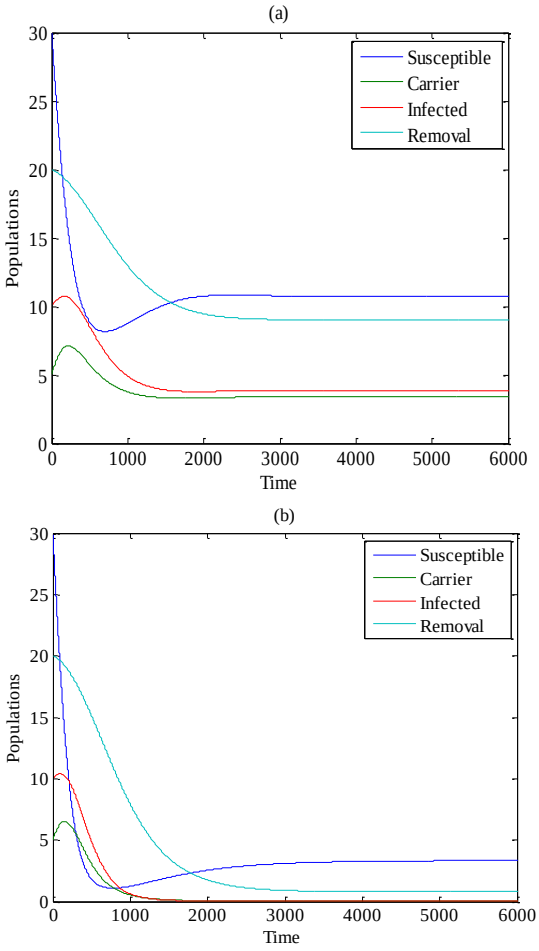


Figure 3: The trajectory of system (1) for the data (23) with different values of A. (a) The system approaches asymptotically to a stable endemic equilibrium point $\mathbf{E_1}$ for $\mathbf{A = 5}$. (b) The system approaches asymptotically to stable disease free equilibrium point $\mathbf{E_0}$ for $\mathbf{A = 0.5}$.

It's observed that for the values $A > 0.85$ we have $R_0 > 1$ and hence the solution converges asymptotically to E_1 as shown in the typical figure given by Fig. (3a) in which the value of the basic reproduction number is $R_0 = 1.63$. However for the values of $A < 0.85$ it's observed that $R_0 < 1$ and then the solution converges asymptotically to E_0, as in the typical figure given by Fig. (3b) which has the basic reproduction number given by $R_0 = 0.75$.

Now, varying the contact rate between the susceptible individuals and carrier individuals represented by α_1 at $\alpha_1 = 0.05$ and $\alpha_1 = 0.01$ with the rest of parameters given by (23) and then the resulting trajectories of system (1) are drawn in Fig.(4) and Fig.(5) respectively.

According to the last two figures, the trajectory of system (1) approaches asymptotically to the endemic equilibrium point for different values of contact rates. In fact, it's observed that for the data used in Fig. (4), the value of the basic reproduction number is $R_0 = 1.104 > 1$, while its $R_0 = 0.907 < 1$ for the data used in Fig. (5).

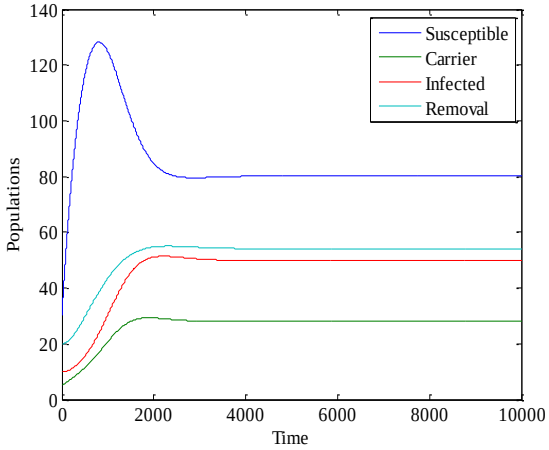


Figure 4: The trajectory of system (1) for the data (23) with $\alpha_1 = 0.05$.

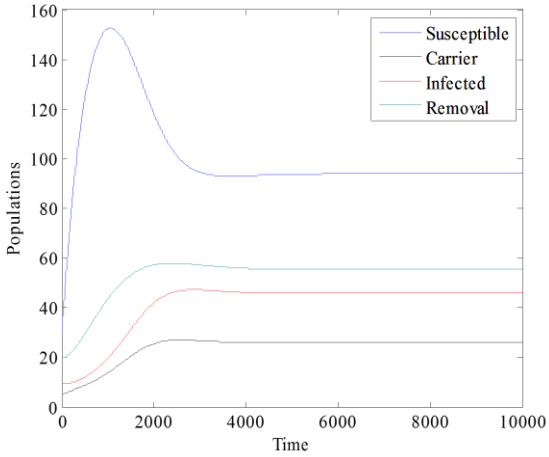


Figure 5: The trajectory of system (1) for the data (23) with $\alpha_1 = 0.01$

Further, it's observed that, although for the values $\alpha_1 < 0.02$ the value of the basic reproduction number is in the range $R_0 < 1$, the solution of system (1) still approaches asymptotically to the endemic equilibrium point E_1.

This is due to the existence of endemic equilibrium point simultaneously with the disease free equilibrium point, which may leads to occurrence of backward bifurcation[35]. Indeed this is due to the use of saturated treatment function in the form of system (1) that leads at most to special case called bi-stable and hence the backward bifurcation take place.

Now in order to explain the role of using the saturated treatment function in the occurrence of backward bifurcation, system (1) is solved numerically for the same data used in Fig. (5), treating $b = 0$ and then the resulting trajectory is drawn in Fig. (6) below.

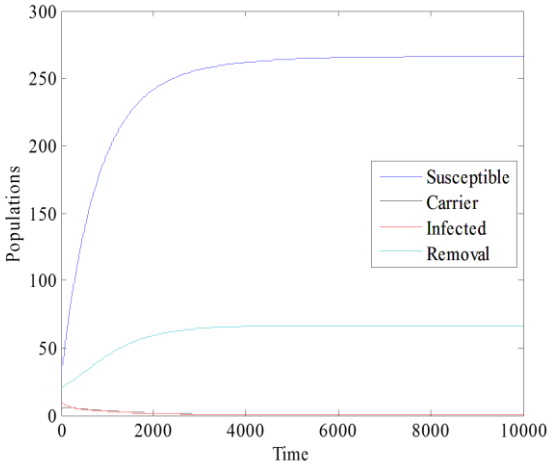


Figure 6: The trajectory of system (1) for the data (23) with $\alpha_1 = 0.01$ and $b = 0$.

Although the parameter b is not included in the form of basic reproduction number and their value still fixed at $R_0 = 0.907 < 1$, the trajectory of the system (1) approaches asymptotically to the disease free E_0 equilibrium point instead of endemic equilibrium point E_1 as in Fig. (5).

This is mean that replacing the saturated treatment function by the classical linear treatment function, which occurs by setting $b = 0$, prevent the occurrence of bi-stable (existence of disease free equilibrium point and the endemic equilibrium point simultaneously) hence backward bifurcation doesn't occur and the trajectory of system (1) approaches asymptotically to disease free equilibrium point when $R_0 < 1$ as proved analytically.

Further investigation for the effect of varying other parameters on the dynamical behavior of system (1) is carried out and the results are summarized in the following table. According to the given table, all the obtained numerical results are coincide with those obtained analytically except the death rate parameter of the carrier compartment. That is because the occurrence of backward bifurcation as that shown in Fig. (5) for the varying of the contact rate between the susceptible individuals and carrier individuals.

Table (2): Dynamical behavior of system (1) as a function of a specific parameter with the rest of parameters for data (23)

parameter	Range	R_0	Behavior
α_2	$\alpha_2 > 0.004$	$R_0 > 1$	E_1 is asymptotically stable.
	$\alpha_2 < 0.004$	$R_0 < 1$	E_0 is asymptotically stable.
β_2	$\beta_2 > 36.63$	$R_0 < 1$	E_0 is asymptotically stable
	$\beta_2 < 36.63$	$R_0 > 1$	E_1 is asymptotically stable.
θ	$\theta > 6.9$	$R_0 < 1$	E_0 is asymptotically stable.
	$\theta < 6.9$	$R_0 > 1$	E_1 is asymptotically stable.
d_0	$d_0 > 6.95$	$R_0 < 1$	E_0 is asymptotically stable.
	$d_0 < 6.95$	$R_0 > 1$	E_1 is asymptotically stable.
d_1	$d_1 > 1.38$	$R_0 < 1$	
	$d_1 < 1.38$	$R_0 > 1$	E_1 is asymptotically stable.
d_2	$d_2 > 36.83$	$R_0 < 1$	$E_0$is asymptotically stable.
	$d_2 < 36.83$	$R_0 > 1$	$E_1$is asymptotically stable.
a	$a > 36.93$	$R_0 < 1$	$E_0$is asymptotically stable.
	$a < 36.93$	$R_0 > 1$	$E_1$is asymptotically stable.
γ_1	$\gamma_1 > 0.37$	$R_0 < 1$	$E_0$is asymptotically stable.
	$\gamma_1 < 0.37$	$R_0 > 1$	$E_1$is asymptotically stable.
p	$p > 0$	$R_0 > 1$	$E_1$is asymptotically stable
β_1	$\beta_1 > 0$	$R_0 > 1$	$E_1$is asymptotically stable
b	$b > 0$	$R_0 > 1$	$E_1$is asymptotically stable
γ_2	$\gamma_2 > 0$	$R_0 > 1$	$E_1$is asymptotically stable
γ_3	$\gamma_3 > 0$	$R_0 > 1$	$E_1$is asymptotically stable
d_3	$d_3 > 0$	$R_0 > 1$	$E_1$is asymptotically stable

Finally, from the above analysis, it's observed that the system (1) approaches asymptotically either to the disease free equilibrium point or to the endemic equilibrium point, depending on the value of basic reproduction number except for the parameters α_1 and d_1, in which the backward bifurcation occurs. Further

it's observed too that the system (1) for the data given by (23) has no periodic dynamics rather than that it has one type of attractors (point attractor) represented by disease free equilibrium point or endemic equilibrium point as shown above.

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