CERTAIN CLASS OF ANALYTIC AND UNIVALENT FUNCTIONS WITH SOME BASIC GEOMETRIC PROPERTIES

Kassim Abdulhameed Jassim[1] & Ali Abd Ali Mohammed[2]
[1,2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
kasimmathphd@gmail.com[1] & iraqiraq.ii50@gmail.com[2]

ABSTRACT: *The subclass* $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ *of* $\boldsymbol{N}$ *, the class of analytic and univalent functions defined on unit disk of the form:*
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *,have been considered .Sharp results concerning Coefficients, Distortion and Growth theorem, Radii of starlikeness and Convexity, Hadamard product property, Convex set , Arithmetic mean , Neighborhood and convolution operator of function belonging to the subclass* $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.

KEYWORDS. Univalent function, Distortion and Growth theorem, Radii of starlikeness and convexity, Hadamard product, Convex set and Arithmetic mean and Neighborhood property.

1. INTRODUCTION

Let $\boldsymbol{N}$ denotes a class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are univalent and analytic in open unit disk U={z ∈ ℂ: |z|< 1}. Assume $\boldsymbol{S}$ denotes a subclass of $\boldsymbol{N}$ consisting of functions f of the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0. \qquad (1.2)$$

If a function f is given by (1.2) and g defined by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \ b_n \geq 0, \qquad (1.3)$$

hence convolution (or hadamard product) of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \ , z \in U. \qquad (1.4)$$

We must recall that a function f is called univalent if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$, and f is said to be starlike and convex if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$ and $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho$,where $0 \leq \rho < 1$, [4,5].

Our aim of this pape is to study the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ consisting functions f of the form (1.2) which is satisfying

$$\left|\frac{\frac{zf''(z)}{f'(z)}}{(B-A)\gamma\left[\frac{zf''(z)}{f'(z)}+(1-\alpha)\right]+B\left[\frac{zf''(z)}{f'(z)}\right]}\right| < \beta, z \in U, \qquad (1.5)$$

where $-1 \leq A < B \leq 1$,$0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$.

Many scholars have discussed functions which are univalent and analytic in the unit disk and getting many geometric properties like, 'Aouf' and 'Mostafa'[2], 'Darus' [3] , 'Kharnar and Meena' [6,7] and'Ruscheweyh' [10] .

2. COEFFICIENT INEQUALITY

Now,we obtain a necessary and sufficient conditions for function f to be in the class $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ as follows.

Theorem(2.1). Let the function f be defined by (1.2) .Then $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} n[(n-1)(1+\beta B) + \beta(B-A)\gamma(n-\alpha)]\, a_n \leq \beta(B-A)\gamma\,(1-\alpha)$$

where $-1 \leq A < B \leq 1$, $0 \leq \alpha < 1$,$0 < \beta \leq 1$,$0 < \gamma \leq 1$. (2.1)

The result is sharp for the function

$$f(z) = z - \frac{\beta(B-A)\gamma(1-\alpha)}{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]} z^n, \ n \geq 2$$

where $-1 \leq A < B \leq 1$, $0 \leq \alpha < 1$,$0 < \beta \leq 1$,$0 < \gamma \leq 1$. (2.2)

Proof.Assume that $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$and $|z| = 1$. So , we want to prove

$$|zf''(z)| - \beta|(B-A)\gamma[zf''(z) + (1-\alpha)f'(z)] - Bzf''(z)| \geq 0,$$ where

$-1 \leq A < B \leq 1$,$0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$.

Therefore,

$$|\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}| - \beta| - (B-A)\gamma \sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1} + (B-A)\gamma(1-\alpha)z - (B-A)\gamma(1-\alpha)\sum_{n=2}^{\infty} n\, a_n z^{n-1} - B\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}|$$

$$= |\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}| - \beta|(B-A)\gamma(1-\alpha)z - \sum_{n=2}^{\infty} n\,[(B-A)\gamma(n-1) + (B-A)\gamma(1-\alpha) + B(n-1)]\, a_n z^{n-1}| \leq$$

$$\sum_{n=2}^{\infty} n\,[(n-1)(1+\beta B) + \beta(B-A)\gamma(n-\alpha)]\, a_n - \beta(B-A)\gamma(1-\alpha) \leq 0,$$

by hypothesis. Therefore by principle maximum modulus, we get the result.

Conversely, suppose that the inequality (1.5) holds. Since Re(z) ≤ $|z|$ and letting z→ 1 − , then

$$\sum_{n=2}^{\infty} n\,(n-1)a_n < \beta((B-A)\gamma(1-\alpha) - \sum_{n=2}^{\infty} n\,[(B-A)\gamma(n-1) + (B-A)\gamma(1-\alpha) + B(n-1)]$$

So, we get the condition (2.1).

Corollary(2.1). Let the function f be defined by (1.2) and $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$. Then

$$a_n \leq \frac{\beta(B-A)\gamma(1-\alpha)}{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]},$$

where

$-1 \leq A < B \leq 1$,$0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$..

The following property has studied by 'Silverman' [11].

3. DISTORTION AND GROWTH THEOREM

Now,we present the distortion and growth theorems for functions f in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.

Theorem (3.1). Let $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$. Then

$$|z| - \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z|^2 \leq |f(z)|$$ and

$$|f(z)| \leq |z| + \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z|^2 \qquad (3.1)$$

The result is sharp and attained

$$f(z) = z + \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}z^2 \qquad (3.2)$$

Proof. Assume $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. By taking the absolute value for f, we get

$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z^n|$.

$\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$.By theorem (2.1) , we get

$$|f(z)| \le |z| + \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z|^2$$

Also ,

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^2 \ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\ge |z| - \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z|^2$$

Hence, we get the result.

Theorem (3.2). Let $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$. Then

$$1 - \frac{\beta(B-A)\gamma(1-\alpha)}{[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z| \le |f'(z)| \text{ and}$$

$$|f'(z)| \le 1 + \frac{\beta(B-A)\gamma(1-\alpha)}{[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}|z| \quad (3.3)$$

4. RADII OF STARLIKENESS AND CONVEXITY

In the following theorems , we obtain the radii of starlikeness and convexlity for the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.

Theorem(4.1). Let $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.Then f is univalent starlike function of order λ $(0 \le \lambda < 1)$ in the disk $|z| < r_1$where

$$r_1 = inf\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)](1-\lambda)}{\beta(B-A)\gamma(1-\alpha)(n-\lambda)}\right)^{\frac{1}{n-1}}, \quad n \ge 2$$

(4.1)

Proof. It is sufficient to show that

$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\lambda$,for $|z| < r_1$. Therefore,

$$\left|\frac{z'f(z)}{f(z)} - 1\right| = \left|\frac{zf'(z)-f(z)}{f(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} na_nz^n - \sum_{n=2}^{\infty} a_nz^n}{z+\sum_{n=2}^{\infty} a_nz^n}\right|$$

$$= \frac{|\sum_{n=2}^{\infty}(n-1)a_nz^n|}{|z-\sum_{n=2}^{\infty} a_nz^n|} \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n|z|^{n-1}}$$

The last expression must bounded by $1-\lambda$ if

$$\sum_{n=2}^{\infty}\frac{(n-\lambda)}{1-\lambda}a_n|z|^{n-1} \le 1 \quad (4.2)$$

The last inequality (4.2) will be true if

$$\frac{n-\lambda}{1-\lambda}|z|^{n-1} \le \frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\beta(B-A)\gamma(1-\alpha)}.$$

Hence ,

$$|z| \le \left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)](1-\lambda)}{\beta(B-A)\gamma(1-\alpha)(n-\lambda)}\right)^{\frac{1}{n-1}},$$

$n \ge 2.$

Putting $|z| < r_1$, we get the result.

Theorem (4.2). Let $f \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$. Then f is univalent convex function of order λ $(0 \le \lambda < 1)$, where $r_2 = \inf\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)](1-\lambda)}{n\beta(B-A)\gamma(1-\alpha)(n-\lambda)}\right)^{\frac{1}{n-1}}$

(4.3)

Proof. It is enough to show that

$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\lambda$,for $|z| < r_1$. Therefore,

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)a_nz^{n-1}}{1-\sum_{n=2}^{\infty} na_nz^{n-1}}\right|$$

$$= \frac{|\sum_{n=2}^{\infty} n(n-1)a_nz^{n-1}|}{|1-\sum_{n=2}^{\infty} na_nz^{n-1}|}$$

$$\le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}$$

The last expression must bounded by 1-λ if

$$\sum_{n=2}^{\infty}\frac{n(n-\lambda)}{1-\lambda}a_n|z|^{n-1} \le 1. \quad (4.4)$$

Hence by theorem (2.1) , will be true if

$$\frac{n(n-\lambda)}{1-\lambda}|z|^{n-1} \le \frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\beta(B-A)\gamma(1-\alpha)}$$

which implies that

$$|z| \le \left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)](1-\lambda)}{n\beta(B-A)\gamma(1-\alpha)(n-\lambda)}\right)^{\frac{1}{n-1}}, n \ge 2.$$

Putting $|z| < r_2$, we get the result.

5 . CONVOLUTION PROPERTY

In the following theorem , we obtain the convolution results for function f belonging to the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.

Theorem (5.1). Let $f(z) = z - \sum_{n=2}^{\infty} a_nz^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_nz^n$ are in the subclass $(A,B,\alpha,\beta,\gamma)$. Then the hadamard product $f * g$ is in the class $\boldsymbol{S}(A,B,\alpha,\beta,\sigma)$, where

$$\sigma \ge [\gamma\beta(B-A)(1-\alpha)][(n-1)(1+\beta B)]\gamma/([(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)] * [(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]) - [\gamma\beta(B-A)(1-\alpha)\beta(B-A)(n-\alpha)\gamma] \quad (5.1)$$

Proof. We must find a smallest σ such that

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\sigma(n-\alpha)]}{\sigma\beta(B-A)(1-\alpha)}a_nb_n \le 1 \quad (5.2)$$

Let $f,g \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$, therefore

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)(1-\alpha)}a_n \le 1 \quad (5.3)$$

and

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)(1-\alpha)}b_n \le 1. \quad (5.4)$$

By using Cauchy – Schwarz inequality , we get

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)(1-\alpha)}\sqrt{a_nb_n} \le 1. \quad (5.5)$$

To prove our theorem , we have to show that

$$\frac{n[(n-1)(1+\beta B)+\beta(B-A)\sigma(n-\alpha)]}{\sigma\beta(B-A)(1-\alpha)}a_nb_n \le \frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)(1-\alpha)}\sqrt{a_nb_n}. \quad (5.6)$$

That is

$$\sqrt{a_nb_n} \le \frac{[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\sigma}{[(n-1)(1+\beta B)+\beta(B-A)\sigma(n-\alpha)]\gamma} \quad (5.7)$$

From (5.5) , we have

$$\sqrt{a_nb_n} \le \frac{\gamma\beta(B-A)(1-\alpha)}{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}. \quad (5.8)$$

It is sufficient to show

$$\frac{\gamma\beta(B-A)(1-\alpha)}{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]} \le \frac{[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\sigma}{[(n-1)(1+\beta B)+\beta(B-A)\sigma(n-\alpha)]\gamma} \quad (5.9)$$

Therefore

$$\sigma \ge [\gamma\beta(B-A)(1-\alpha)][(n-1)(1+\beta B)]\gamma/([(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)] * [(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]) - [\gamma\beta(B-A)(1-\alpha)\beta(B-A)(n-\alpha)\gamma]$$

Theorem(5.2). Let the function $f(z) = z - \sum_{n=2}^{\infty} a_nz^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_nz^n$ be in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$. Then the function

$$h(z) = z - \sum_{n=2}^{\infty}(a_n^2 + b_n^2)z^n \quad (5.10)$$

is in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\tau)$, where

$$\tau \ge 2(\gamma\beta(B-A)(1-\alpha))^2(n[(n-1)(1+\beta B))/(n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]^2$$

$(\beta(B-A)\,(1-\alpha))$- $\beta(B-A)(n-\alpha))$
$(2(\gamma\beta(B-A)\,(1-\alpha))^2\ )$ (5.11)

Proof. By theorem (2.1), and $f, g \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$, we have

$$\sum_{n=2}^{\infty}\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}\right)^2 a_n^2 \le \left(\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}a_n\right)^2 \le 1 \qquad (5.12)$$

and

$$\sum_{n=2}^{\infty}\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}\right)^2 b_n^2 \le \left(\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}b_n\right)^2 \le 1 \qquad (5.13)$$

It follows from (5.12) and (5.13) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}\right)^2 (a_n^2+b_n^2) \le 1 \qquad (5.14)$$

But $h(z) \in \boldsymbol{S}(A,B,\alpha,\beta,\tau)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n[(n-1)(1+\beta B)+\beta(B-A)\tau(n-\alpha)]}{\tau\beta(B-A)\,(1-\alpha)}(a_n^2+b_n^2) \le 1 \qquad (5.15)$$

The last inequality satisfied if

$$\frac{n[(n-1)(1+\beta B)+\beta(B-A)\tau(n-\alpha)]}{\tau\beta(B-A)\,(1-\alpha)} \le \frac{1}{2}\left(\frac{n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma\beta(B-A)\,(1-\alpha)}\right)^2 \qquad (5.16)$$

This implies the result.

6. Convex set

Theorem (6.1). The subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ is convex set.

Proof . Let functions f and g are in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$, then for every 0≤t≤1 , we must show that

$$(1-t)f(z)+tg(z)\ \epsilon\ \boldsymbol{S}(A,B,\alpha,\beta,\gamma).$$

We have

$(1-t)f(z)+tg(z) = z - \sum_{n=2}^{\infty}[(1-t)a_n + tb_n]z^n$.

Therefore by theorem (2.1) we get

$\sum_{n=2}^{\infty} n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\,[(1-t)a_n+tb_n]$
$=(1-t)\sum_{n=2}^{\infty} n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\,a_n + t\sum_{n=2}^{\infty} n\,[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\,b_n$
$\le (1-t)\gamma\beta(B-A)\,(1-\alpha) + t\,\gamma\beta(B-A)\,(1-\alpha)$
$= \gamma\beta(B-A)\,(1-\alpha)$

7. ARITHMETIC MEAN

Theorem (7.1). Let f_1(z) , f_2(z) … f_k(z) defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n.i}z^n\ , \qquad (7.1)$$

where ($a_{n.i} \ge 0$, i=1,…,k) be in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$, then arithmetic mean of $f_i(z)$ (i=1,…,k) is defined by

$$h(z) = \frac{1}{k}\sum_{i=1}^{k} f_i(z) \qquad (7.2)$$

is also in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$.

Proof. By using equations (7.1) and (7.2) , we can write

$h(z) = \frac{1}{k}\sum_{i=1}^{k}\ (z - \sum_{n=2}^{\infty} a_{n.i}z^n) = z - \sum_{n=2}^{\infty}(\frac{1}{k}\sum_{i=1}^{k} a_{n.i})z^n$

Since $f_i \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ for every (i=1,…,k) , then by using theorem(2.1) we get

$\sum_{n=2}^{\infty} n[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\,(\frac{1}{k}\sum_{i=1}^{k} a_{n.i})$
$=\frac{1}{k}\sum_{i=1}^{k}(\sum_{n=2}^{\infty} n\,[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]\,a_{n.i})$
$\le \frac{1}{k}\sum_{i=1}^{k}\gamma\beta(B-A)\,(1-\alpha) = \gamma\beta(B-A)\,(1-\alpha)$.

The following Neighborhood property has studied by Owa [9].

8. Neighborhood property

Now, we define the (n,δ)-neighborhood of a function $f\in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ by

$N_{n,\delta}(f) = \{g \in\ \boldsymbol{S}(A,B,\alpha,\beta,\gamma): g(z) = z + \sum_{n=2}^{\infty} b_n\, z^n$ and $\sum_{n=2}^{\infty} n|a_n - b_n| \le \delta, 0 \le \delta < 1\}$ (8.1)

For the identity function e(z)=z , we get

$N_{n,\delta}(e) = \{g \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma): g(z)=z+\sum_{n=2}^{\infty} b_n\, z^n$ and $\sum_{n=2}^{\infty} n|b_n| \le \delta\}$.

Definition(8.1). A function $f\in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ is said to be in the subclass $S^{\eta}(A,B,\alpha,\beta,\gamma)$ if there exists a function $g \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$,such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\eta\ \ , (z\in U, 0 \le \eta < 1). \qquad (8.2)$$

Theorem(8.1). If $g \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ and

$$\eta = 1 - \frac{2\delta[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]-\beta(B-A)\gamma(1-\alpha)}. \qquad (8.3)$$

Then $N_{n,\delta}(g) \subset S^{\eta}(A,B,\alpha,\beta,\gamma)$.

Proof: Let $f\in N_{n,\delta}(g)$. We want to find from (8.1) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \delta,$$

Which implies the following coefficient inequality

$$\sum_{n=2}^{\infty}|a_n - b_n| \le \delta, \quad (n\in N) \qquad (8.4)$$

Since $\in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$, we have from theorem(2.1)

$$\sum_{n=2}^{\infty} b_n \le \frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]} \qquad (8.5)$$

Therefore,

$$\left|\frac{f(z)}{g(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty} b_n} \le \frac{\delta}{1-\frac{\beta(B-A)\gamma(1-\alpha)}{2[(1+B)+\beta(B-A)\gamma(2-\alpha)]}}$$

$$\le \frac{2\delta[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]}{2[(1+\beta B)+\beta(B-A)\gamma(2-\alpha)]-\beta(B-A)\gamma(1-\alpha)}$$

$$=1-\eta$$

9. Convolution Operator

Definition(9.1)[8]. The Gaussian hypergeometric function which is defined by

${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n\, z^n}{(c)_n\, n!}$, $|z| < 1$, where $c > b > 0, c > a+b$ and

$(x)_n = \begin{cases} x(x-1)(x+2)\ldots\ldots(x+n-1) & for\ n=1,2,3,\ldots\ldots \\ 1 & for\ n=0\ . \end{cases}$

Definition(9.2)[1]. For every $f \in C$, we define the convolution operator $W_{a,b,c}(f)(z)$ as below :

$$W_{a,b,c}(f)(z) = {}_2F_1(a,b,c;z) * f(z) = z - \sum_{n=2}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} a_n z^n \ . \quad (9.1)$$

Theorem (9.1). Let the function f is defined by (1.2) and in the subclass $\boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ .Then convolution operator $W_{a,b,c}(f)(z) \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$ for $|z| \le r(\gamma,\tau)$,where

$$r(\gamma,\tau) = \inf\left[\frac{\tau[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\gamma n[(n-1)(1+\beta B)+\beta(B-A)\tau(n-\alpha)]\frac{(a)_n(b)_n}{(c)_n\, n!}}\right]^{\frac{1}{n-1}} \quad (9.2)$$

The result is sharp for the function

$$f_n(z) = z - \frac{\beta(B-A)\gamma(1-\alpha)}{n[(n-1)(1+\beta B)+\beta[(B-A)\gamma(n-\alpha)]} z^n \ , \quad n = 2,3,\ldots\ .$$

Proof. Since $f(z) \in \boldsymbol{S}(A,B,\alpha,\beta,\gamma)$,so we have $\sum_{n=2}^{\infty} \frac{n[(n-1)(1+B\beta)+\beta[(B-A)\gamma(n-\alpha)]}{\beta(B-A)\gamma(1-\alpha)}\ a_n \le 1$ .It is sufficient to show that

$$\sum_{n=2}^{\infty} \frac{n[(n-1)(1+\beta B)+\beta(B-A)\tau(n-\alpha)]\frac{(a)_n(b)_n}{(c)_n\, n!}}{\beta(B-A)\tau(1-\alpha)}\ a_k \le 1. \quad (9.3)$$

Note that (9.3) is satisfied if

$$\frac{[(n-1)(1+\beta B)+\beta(B-A)\tau(n-\alpha)]\frac{(a)_n(b)_n}{(c)_n\, n!}}{\beta(B-A)\tau(1-\alpha)}\ |z|^{n-1} \le \frac{[(n-1)(1+\beta B)+\beta(B-A)\gamma(n-\alpha)]}{\beta(B-A)\gamma(1-\alpha)}$$

. Therefore, we get the result.

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