COUNTING THE NUMBER OF DISCONNECTED VERTEX LABELLED GRAPHS WITH ORDER MAXIMAL FOUR

Amanto[1], Wamiliana[1], Mustofa Usman[1], and Reni Permata Sari[1]
[1] Department of Mathematics, Universitas Lampung, Indonesia
Email: wamiliana.1963@fmipa.unila.ac.id

ABSTRACT : *An undirected graph G(V,E) with n vertices and m edges is called to be connected if every pair of the vertices in G exists a path connecting them, otherwise, G is disconnected. A graph can be labeled if only the vertices are labelled and it is called vertex labelling, if only edges are labelled then it is called as edge labelling, and if both vertices and edges are labelled, it is called as total labelling. The edge that has the same starting and end point is called a loop, and two or more edges that connect the same vertices are called parallel edges. A graph is called simple if there is neither loop nor parallel edge on that graph. Given n vertices and m edges there are a lot of possible graphs can be constructed either connected or not, simple or not. In this research we discuss about counting the number of disconnected vertex labelled graph for graph with order maximal four and* $m \geq 1$.

Keywords : counting graph, vertex labelled graph, disconnected, graph order

1. INTRODUCTION

An undirected graph G(V,E) is a graph with V as the set of vertices and E as the set of edges, where |V| = n and |E| = m, is called connected if there is a path connecting every pair of vertices on that graph, otherwise it is disconnected. A graph is called simple if it does not contain any loops or parallel edges. Loop is an edge having the same initial and end vertex, while parallel edges are two or more edges that connect the same vertices.

A graph can be labelled on its vertices or on its edges. If only the vertices are labelled, the graph has vertex labelling, and if only edges are labelled, then the graph has edges labeling. If both vertices and edges are labeled, the graph has total labeling. If given |V| vertices and |E| edges, many graphs can be constructed. The graphs are constructed either connected or disconnected and simple or not simple. This research tries to consider finding the formula for disconnected graph of order maximal four. In this case, any loops or parallel edges are accepted.

Some works have already been done related to graph counting; for example Cayley [1] who did the application of graph enumeration for finding hydrocarbon structure (tree concepts in graph).

In this paper the formula for counting the number of disconnected vertex labelled graphs with order maximal four will be discussed.

2. BRIEF HISTORY ON GRAPH COUNTING

Graph theoretic concept has been used as one of important tools for solving daily life problems. For problem related with graph enumeration, Cayley [1] investigated the number of hydrocarbon structure. This work was known as the counting the number of hydrocarbon isomers. These isomers are specific trees which are rooted trees with n vertices. In this problem tree represents the hydrocarbon structure [1].

Graph theory is not only used to represent the real-life problems into simple structure of graph, but also used as tools to solve the problem. In [2] comprehensive techniques and applications of graph theory in networks were given and in [3] a diverse of applications of graph theory in the area of engineering (electrical, civil, and industrial), operations research and science were exposed.

Motivated by the work of Cayley [1], some chemical structures which are isomorphic to certain graph structures are investigated [4], and the method for counting and labelling graph was given [5]. The detail of generating function for finding formulas in combinatoric was given by Stanley [6-7] and a various combinatorial counting technique was given in [8].

Given the number of edges, the formula for counting the number of graphs for simple graph is given [9] and the graphs constructed are all graphs either connected or disconnected. The formula for counting the number of disconnected vertex labeled graphs of order five without parallel edges was investigated [10]. In this paper formula for counting disconnected vertex labelled graphs with order maximal four will be discussed.

3. PATTERNS INVESTIGATION

Given undirected graph with n vertices, n is ≤ 4 and m is number of edges. Vertex labelled graph with order one or two the case is trivial. This research only considers graphs with order three or four. Before doing the counting/enumeration method, the graphs are observed by constructing the patterns and calculating the number of different disconnected vertex labeled graphs in every pattern. In this research it is assumed that g_i as the number of edges (not loops) where the number of parallel edges that connects the same pair of vertices is counted as one.

Fig. 1. Example of graph with g_1 for n=4

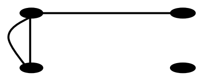


Fig. 2. Example of graph with g_2 for n = 4

For n = 3.

Because there are many graphs can be constructed, some examples of the patterns of possible disconnected graphs are displayed in the following table:

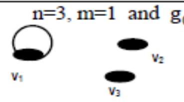
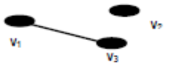
One example of the pattern of the graph	The number of graphs can be formed
n=3 and m=1	6
n=3, m=1 and g_0 	3
n=3, m=1 and g_1 	3

Fig. 3. Number of disconnected vertex labeled graphs for n=3, m = 1

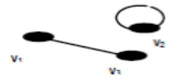
One example of the pattern of the graph	The number of graphs can be formed
n=3 dan m=2	18
n=3, m=2 and g_0	
	3
	3
n=3, m=2 , g_1	
	6
	3
	3

Fig. 4. Number of disconnected vertex labeled Graphs for n=3, m=2

Due to the space limitation, the tables for m > 2 are not displayed in this paper.

The following figures shows some examples of the patterns of graphs with different g_i for n = 4

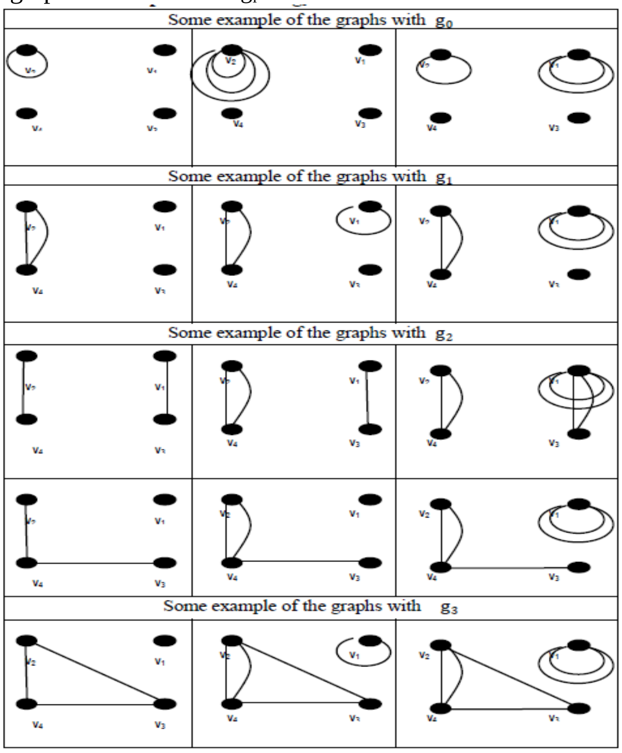


Fig. 5. Some examples of patterns constructed and the assumptions made for g_i For n = 4

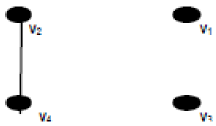
One example of the pattern of the graph	The number of graphs can be formed
n=4 and m=1	10
n=4, m=1 and g_1 	6
n=3, m=1 and g_0 	4

Fig. 6. Number of disconnected vertex labeled graphs for n=4, m =1

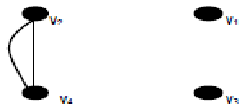
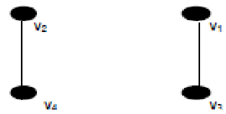
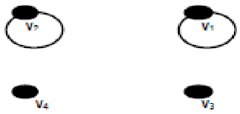
One example of the pattern of the graph	The number of graphs can be formed
n=4 and m=2	55
n=4, m=2 , g_1	
	12
	12
n=4, m=2 , g_1	
	6
n=4, m=2 , g_2	
	3
	12
n=4, m=2 and g_0	
	4
	6

Fig. 7. Number of disconnected vertex labeled graphs for n=4, m =2

Due to the space limitation, the tables for n=4, m>3 are not presented here.

4. RESULTS AND DISCUSSION

Given n is the number of vertices and m is the number of edges. Let g_i as the number of edges (not loops, where the number of parallel edges that connects the same pair of vertices is counted as one); i = 0,1,2,3.

After the construction and observation of all possible patterns for n = 3, the following table is acquired:

Table 1. Number of graphs with n =3 dan m ≥ 1

m	1	2	3	4	5	6	7	8	9	10	...
The number of graphs	6	18	40	75	126	196	288	405	550	726	...

Note $G_{n,m}^{D}$ is as disconnected vertex labeled graph with n vertices and m edges and $N(G_{n,m}^{D})$ is as the number of disconnected vertex labeled graph with n vertices and m edges.

Theorem 1

Given n vertices, $n = 3$ and $m \geq 1$ then the number of disconnected vertex labeled graphs is

$$N(G_{3,m}) = (m+1)\binom{m+2}{2}.$$

Proof:

Consider the following sequence:

6 18 40 75 126 196 288 405 550 726 ...

12 22 35 51 70 92 117 145 176 ...

10 13 16 19 22 25 28 31 ...

3 3 3 3 3 3 3 ...

Note that at the fix difference (3) of the sequence occurs at the last line. This sequence forms an arithmetic sequence of order three. The general form of the equation for coefficients of arithmetics sequence of order three is:

$$a_m = Am^3 + Bm^2 + Cm + D$$

For m = 1

$a_1 = A(1)^3 + B(1)^2 + C(1) + D$

6 = A + B + C + (1)

For m = 2

$a_2 = A(2)^3 + B(2)^2 + C(2) + D$

18 = 8A + 4B + 2C + D (2)

For m = 3

$a_3 = A(3)^3 + B(3)^2 + C(3) + D$

40 = 27A + 9B + 3C + D (3)

For m = 4

$a_4 = A(4)^3 + B(4)^2 + C(4) + D$

75 = 64A + 16B + 4C + D (4)

By using elimination method for Eq.(1) and Eq.(2), Eq.(2) and Eq.(3), Eq.(3) and Eq.(4), the following equations are derived:

12 = 7A + 3B + C (5) 22

= 19A + 5B + C (6) 35 =

37A + 7B + C (7)

By using elimination method for Eq.(5) and Eq.(6), Eq.(6) and Eq.(7), the following equations are derived:

10 = 12A + 2B (8)

13 = 18A + 2B (9)

By using elimination method for Eq.(8) and Eq.(9) it yields $A = \frac{1}{2}$

Using substitution, it yields B = 2 , $C = \frac{5}{2}$, and D = 1.

By substituting this values to the general form, t yields:

$$\begin{aligned} a_m &= \frac{1}{2}m^3 + 2m^2 + \frac{5}{2}m + 1 \\ &= \frac{1}{2}(m^3 + 4m^2 + m + 2) \\ &= \frac{1}{2}\{(m^3 + 3m^2 + 2m) + (m^2 + 3m + 2)\} \\ &= \frac{1}{2}\{(m(m^2 + 3m + 2)) + ((m+2)(m+1))\} \\ &= \frac{1}{2}\{(m(m+2)(m+1)) + ((m+2)(m+1))\} \\ &= \frac{1}{1.2}\{(m(m+2)(m+1)) + ((m+2)(m+1))\} \\ &= \frac{m(m+2)(m+1)}{1.2} + \frac{(m+2)(m+1)}{1.2} \\ &= m\binom{m+2}{2} + \binom{m+2}{2} \\ &= (m+1)\binom{m+2}{2} \end{aligned}$$

Therefore, by the above calculation it yields

$$N(G_{3,m}) = (m+1)\binom{m+2}{2}$$

■

For n=4, consider the following table which shows the total number of graphs constructed.

Table 2. Number of disconnected vertex labeled graphs with paralel edges or loops for n=4, m ≥ 1, and $0 \leq g_i \leq 3$, i = 0,1,2,3

m	g_0	g_1	g_2	g_3	Total number of graphs
1	4	6	-	-	10
2	10	30	15	-	55
3	20	90	90	4	204
4	35	210	315	28	588
5	56	420	840	112	1428
6	84	756	1890	336	3066
7	120	1260	3780	840	6000
8	165	1980	6930	1848	10923
9	220	2970	11880	3696	18766
10	286	4290	19305	6864	30745

The following Table is the modification of Table 2 above:

Table 3. Other form for Table 2

m	g_0	g_1	g_2	g_3	Total number of graphs
1	4	6 x 1	-	-	10
2	10	6 x 5	15 x 1	-	55
3	20	6 x 15	15 x 6	4 x 1	204
4	35	6 x 35	15 x 21	4 x 7	588
5	56	6 x 70	15 x 56	4 x 28	1428
6	84	6 x 126	15 x 126	4 x 84	3066
7	120	6 x 210	15 x 252	4 x 210	6000
8	165	6 x 330	15 x 462	4 x 462	10923
9	220	6 x 495	15 x 792	4 x 942	18766
10	286	6 x 715	15 x 1287	4 x 1716	30745

Note (G_{n,m,g_i}^{D}) is as disconnected vertex labeled graph with n vertices m edges and g_i edges (not loops, and parallel edges are counted as one); and $N(G_{n,m,g_i}^{D})$ is as the number of disconnected vertex labelled graph with n vertices, m edges and g_i are non loops.

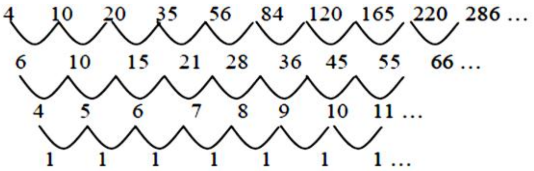
Theorem 2

Given n vertices, n = 4 and $m \geq 1$, and g_0 then the number of disconnected vertex labeled graphs is

$$N(G_{4,m,g_0}) = \binom{m+3}{3}$$

Proof:

Consider the following sequence.



Because fixed difference appears at the last line, then again this sequence forms an arithmetic sequence of order three. Using similar method to prove Theorem 1, it yields

$a_m = \frac{1}{6}m^3 + m^2 + \frac{11}{6}m + 1$

$= \frac{1}{6}(m^3 + 6m^2 + 11m + 6)$

$= \frac{1}{6}\{(m+3)(m+2)(m+1)\}$

$= \frac{1}{3.2.1}\{(m+3)(m+2)(m+1)\}$

$= \frac{(m+3)(m+2)(m+1)}{3.2.1}$

$= \binom{m+3}{3}$

Therefore, the number of disconnected vertex labelled graphs for n = 4 and $m \geq 1$, and g_0 is

$$N(G_{4,m,g_0}) = \binom{m+3}{3}$$ ■

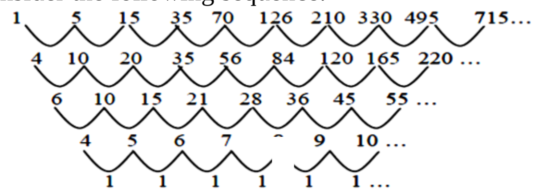
Theorem 3

Given n vertices, n = 4 and $m \geq 1$, and g_1 , the number of disconnected vertex labeled graphs is

$$NG^D_{n,m,g_i}) = \frac{3}{2}m\binom{m+3}{3}$$

Proof :

Consider the following sequence:



Since the fixed difference in the sequence occurs in the fourth order, the sequence is an arithmetic sequence of order four. Therefore, the coefficients of the sequence can be calculated using the following equation:

$$a_m = Am^4 + Bm^3 + Cm^2 + Dm + E \qquad (10)$$

By substituting m = 1,2,3,4, and 5 to Equation (10) the the following is acquired:

$m = 1 \rightarrow\ 5 = A + B + C + D + E$ (11)

$m = 2 \rightarrow 15 = 16A + 8B + 4C + 2D + E$ (12)

$m = 3 \rightarrow 35 = 81A + 27B + 9C + 3D + E$ (13)

$m = 4 \rightarrow 70 = 256A + 64B + 16C + 4D + E$ (14)

$m = 5 \rightarrow 126 = 625A + 125B + 25C + 5D + E$ (15)

By applying elimination and substitution methods to the above equations, A = $\frac{6}{24}$, B =$\frac{36}{24}$, C= $\frac{66}{24}$, D = $\frac{36}{24}$, and E = 0 is acquired.

$a_m = Am^4 + Bm^3 + Cm^2 + Dm + E$

$a_m = \frac{6}{24}m^4 + \frac{36}{24}m^3 + \frac{66}{24}m^2 + \frac{36}{24}m + 0$

$= \frac{6}{24}m^4 + \frac{36}{24}m^3 + \frac{66}{24}m^2 + \frac{36}{24}m$

$= \frac{6m}{24}(m^3 + 6m^2 + 11m + 6)$

$= \frac{6m}{24}(m+3)(m+2)(m+1)$

$= \frac{6m}{4}\frac{1}{6}(m+3)(m+2)(m+1)$

$= \frac{3m}{2}\frac{(m+3)(m+2)(m+1)}{3.2.1}$

$= \frac{3m}{2}\binom{m+3}{3}$

Therefore, the number of disconnected vertex labeled graphs for n =4 , $m \geq 1$, and g_1 is

$N(G_{4,m,g_1}) = \frac{3m}{2}\binom{m+3}{3}$ ■

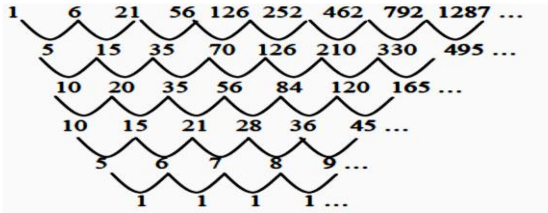
Theorem 4

Given n vertices, n = 4 and $m \geq 1$ and g_2 then the number of disconnected vertex labeled graphs is

$$N(G_{4,m,g_2}) = 15\binom{m+3}{5}$$

Proof:

Consider the following sequence:



From the sequence, it can be shown that the fixed difference occurs at the fifth order. Therefore, by using the general form of arithmetic sequence for fifth order $a_m = Am^5 + Bm^4 + Cm^3 + Dm^2 + Em + F,$ elimination and substitution methods the following is got:

A = $\frac{1}{8}$; B = $\frac{5}{8}$, C = $\frac{5}{8}$, D = $-\frac{5}{8}$, E =$-\frac{1}{8}$, and F = 0.

Thus,

$a_m = \frac{1}{8}m^5 + \frac{5}{8}m^4 + \frac{5}{8}m^3 - \frac{5}{8}m^2 - \frac{6}{8}m + 0$

$= \frac{1}{8}m^5 + \frac{5}{8}m^4 + \frac{5}{8}m^3 - \frac{5}{8}m^2 - \frac{6}{8}m$

$= \frac{1}{8}(m^5 + 5m^4 + 5m^3 - 5m^2 - 6m)$

$= \frac{\frac{120}{8}}{120}(m^5 + 5m^4 + 5m^3 - 5m^2 - 6m)$

$= \frac{\frac{120}{8}(m^5+5m^4+5m^3-5m^2-6m)}{120}$

$= \frac{15(m+3)(m+2)(m+1)m(m-1)}{120}$

$= \frac{15(m+3)(m+2)(m+1)m(m-1)}{5.4.3.2.1}$

$= \frac{15(m+3)(m+2)(m+1)m(m-1)(m-2)!}{5.4.3.2.1\ (m-2)!}$

$= 15\binom{m+3}{5}.$

Therefore, the number of disconnected vertex labeled graphs for n =4 , $m \geq 1$, and g_2 is

$$N(G_{4,m,g_2}) = 15\binom{m+3}{3}$$

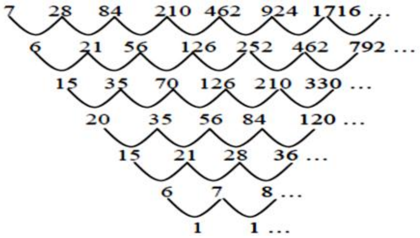
Theorem 5

Given n vertices, n = 4 and $m \geq 1$, and g_2, then the number of disconnected vertex labeled graphs is

$$N(G_{4,m,g_3}) = 4\binom{m+3}{6}.$$

Proof :

Consider the following sequence:



Since the fixed difference in the sequence occurs in the sixth order, the sequence is an arithmetic sequence of order four. Therefore, the coefficients of the sequence can be calculated using the following equation:
$a_m = Am^6 + Bm^5 + Cm^4 + Dm^3 + Em^2 + Fm + G.$ By doing some eliminations and substitution methods it acquires : A = $\frac{4}{720}$; B = $\frac{2}{120}$, C = $\frac{-20}{720}$, D = $-\frac{10}{120}$, E =$\frac{16}{720}$, F = $\frac{8}{120}$, and G =0. Thus

$$a_m = \frac{4}{720}m^6 + \frac{2}{120}m^5 - \frac{20}{720}m^4 - \frac{10}{120}m^3 + \frac{16}{720}m^2 + \frac{8}{120}m + 0$$
$$= \frac{4}{720}m^6 + \frac{2}{120}m^5 - \frac{20}{720}m^4 - \frac{10}{120}m^3 + \frac{16}{720}m^2 + \frac{8}{120}m$$
$$= \frac{4}{720}m^6 + \frac{2}{120}m^5 - \frac{20}{720}m^4 - \frac{10}{120}m^3 + \frac{16}{720}m^2 + \frac{8}{120}m$$
$$= \frac{1}{720}(4m^6 + 12m^5 - 20m^4 - 60m^3 + 16m^2 + 48m)$$
$$= \frac{4}{720}(m^6 + 3m^5 - 5m^4 - 15m^3 + 4m^2 + 12m)$$
$$= \frac{4}{720}(m+3)(m+2)(m+1)m(m-1)(m-2)$$
$$= \frac{4\,(m+3)(m+2)(m+1)m(m-1)(m-2)}{6.5.4.3.2.1}$$
$$= \frac{4\,(m+3)(m+2)(m+1)m(m-1)(m-2)(m-3)!}{6.5.4.3.2.1(m-3)!}$$
$$= 4\binom{m+3}{6}$$

Therefore, the number of disconnected vertex labeled graphs for n =4 , $m \geq 1$, and g_3 is

$$N(G_{4,m,g_3}) = 4\binom{m+3}{6}$$ ■

5. CONCLUSION

From the above discussion it can be concluded that the number of disconnected vertex labelled graphs with order maximal four is:

1. For n = 3 and $m \geq 1$, the number of disconnected vertex labelled graphs is:
$N(G_{3,m}) = (m+1)\binom{m+2}{2}$.
2. For $n = 4$, $m \geq 1$, and $0 \leq g_i \leq 3$, i = 0,1,2,3 the number of disconnected vertex labelled graphs is:
$N(G_{4,m,g_i} = N(G_{4,m,g_0}) + N(G_{4,m,g_1}) + N(G_{4,m,g_2}) + N(G_{4,m,g_3})$
$N(G_{4,m,g_i}) = \binom{m+3}{3} + \frac{3}{2}m\binom{m+3}{3} + 15\binom{m+3}{5} + 4\binom{m+3}{6}$

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