ANALYTIC-NUMERIC SIMULATION OF SHOCK WAVE EQUATION USING REDUCED DIFFERENTIAL TRANSFORM METHOD

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***ABSTRACT**: In this work, we apply the reduced differential transform method (RDTM) to solve the shock wave equation that describes the flow of most gases. The new approach provides the solution in a form of rapidly convergent series with easily computable components and not at grid points. The reported numerical results reveal a reliability of the proposed algorithm.*

Key words: Reduced differential transform method; Variational iteration method; Adomian decomposition method; Shock wave equation; Conservation law; Soliton solutions.

1. INTRODUCTION

The notion of conservation laws plays an important role in the study of differential equations which are of great importance in many areas of physics and engineering. They are essential since they allow us to draw conclusions of physical system under study in an efficient way. The mathematical idea of conservation laws comes from the formulation of familiar physical laws of conservation of energy, conservation of momentum and so on.

Consider the one-dimensional scalar conservation law with source term h (called balance laws) and flux f:

$$u_t + f(u)_x = h(x,t),\ (x,t) \in \mathbb{R} \times [0,T]. \qquad (1)$$

Subject to initial condition

$$u(x,0) = u_0(x),\ x \in \mathbb{R}, \qquad (2)$$

Recently, many methods were developed for solving nonlinear partial differential equations. The homotopy analysis method (HAM) [1-4], the homotopy perturbation method (HPM) [5-10], the variational iteration method (VIM) [11-14], the Adomian decomposition method (ADM) [15-20] the differential transform method (DTM)[20-25] and the RDTM [25-31]are some examples of the methods.

In this paper, we discuss the analytic-numeric solutions of the one-dimensional scalar equation of conservation law (1), which is known as shock wave equation and describes the flow of most gases, and given by [16]

$$u_t + \left(\frac{1}{c_0}u - \frac{\gamma+1}{4c_0^2}u^2\right)_x = 0,\ (x,t) \in \mathbb{R} \times [0,T], \qquad (3)$$

where c_0, γ are constants and γ is the specific heat. For the study case under consideration, we take $c_0 = 2$ and $\gamma = \frac{3}{2}$ correspond to the flow of air which results in the following shock wave equation:

$$u_t + \left(\frac{1}{2}u - \frac{5}{16}u^2\right)_x = 0,\ (x,t) \in \mathbb{R} \times [0,T], \qquad (4)$$

and the initial condition is assumed to be

$$u(x,0) = e^{-x^2/2},\ x \in \mathbb{R}, \qquad (5)$$

The series solution for Eq.(4) subject to initial condition (5) can be obtained if $c_0 >> \frac{1}{2}(\gamma+1)u$, as shown in [32], and is given by

$$u(x,t) = \sum_{k=0}^{\infty}\left((-Bt)^n \frac{(n+1)^{\frac{n}{2}}}{(n+1)!} H_n\left(\sqrt{n+1}\right) e^{-\frac{1}{2}\left(x-\frac{t}{2}\right)^2(n+1)}\right), \qquad (6)$$

where $B = (\gamma+1)/2c_0^2$ and H_n is the Hermit polynomial of order n.

Several numerical techniques were employed to solve Eq. (4). The ADM [16], and the HAM [33] in comparison to the VIM [34]. Approximately, same errors were obtained. In this work, we compare a relatively new technique, the RDTM, with some existing method as ADM and VIM using more terms of series solutions for each method.

This paper is organized as follows: In section two, we begin with some basic definitions, theorems and explain the use of the proposed method for solving one-dimensional scalar conservation law (1). An analytic solution and numerical simulation for shock wave equation (3), to show the effectiveness of proposed method, are given in Sections 3 and 4 respectively. Conclusions are drawn in section 5.

2. REDUCED DIFFERENTIAL TRANSFORM METHOD

Recently, the RDTM is presented to overcome the demerit of complex calculations of the DTM for solving nonlinear partial differential equations. The basic definitions and operations of RDTM [26-27] are defined as follows:

Definition 2.1. If the function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let,

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \qquad (7)$$

where the $t-$dimensional spectrum function $U_k(x)$ is the transformed function.

In this work, the lowercase $u_k(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition 2.2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \qquad (8)$$

Combining Eq.(7) and (8) yields

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \qquad (9)$$

One can easily obtained that the reduced differential transform is derived from the power series expansion. Next, some basic theorems and generalized formulas of reduced differential transform are listed.

Theorem 2.1. The reduced differential transform is linear.

Theorem 2.2. If $u(x,t)=x^m t^n$ then,

$$U_k(x)=x^m\delta(k-n),\ \delta(k)=\begin{cases}1 & ,k=0\\ 0 & ,k\neq 0\end{cases}. \tag{10}$$

Theorem 2.3. If $u(x,t)=x^m t^n v(x,t)$ then,

$$U_k(x)=x^m V_{k-n}(x). \tag{11}$$

Theorem 2.4. If $u(x,t)=\frac{\partial^r}{\partial t^r}v(x,t)$ then,

$$U_k(x)=(k+1)\ldots(k+r)V_{k+r}(x). \tag{12}$$

Theorem 2.5. If $u(x,t)=\frac{\partial^r}{\partial x^r}v(x,t)$ then,

$$U_k(x)=\frac{\partial^r}{\partial x^r}V_k(x). \tag{13}$$

Az-Zo'bi and Al Dawoud [25-26] proved the following generalized reduced differential transforms:

Theorem 2.6. If $u(x,t)=v^n(x,t)$, $n\in\mathbb{N}$, then,

$$U_k(x)=\sum_{r_1=0}^{k}\sum_{r_2=0}^{k-r_1}\cdots\sum_{r_{n-1}=0}^{k-\sum_{t=1}^{n-2}r_t}V_{r_1}(x)V_{r_2}(x)\cdots V_{k-\sum_{t=1}^{n-1}r_t}(x). \tag{14}$$

Theorem 2.7. If $u(x,t)=v^{n-m}(x,t)\left(\frac{\partial}{\partial x}v(x,t)\right)^m$, $n,m\in\mathbb{N}$, then,

$$U_k(x)=\sum_{r_1=0}^{k}\sum_{r_2=0}^{k-r_1}\cdots\sum_{r_{n-1}=0}^{k-\sum_{t=1}^{n-2}r_t}V_{r_1}(x)V_{r_2}(x)\cdots V_{r_{n-m}}(x)\frac{\partial}{\partial x}V_{r_{n-m+1}}\cdots(x)V_{k-\sum_{t=1}^{n-1}r_t}(x). \tag{15}$$

Theorem 2.8. If $u(x,t)=\frac{\partial^{n+m}}{\partial x^n\partial t^m}v(x,t)$ then,

$$U_k(x)=(k+1)\ldots(k+m)\frac{\partial^n}{\partial x^n}V_{k+m}(x). \tag{16}$$

To illustrate the methodology of the proposed method, we consider the first order equation of conservation law (1) in the standard operator form

$$L(u)+N(u)=h(x,t), \tag{17}$$

where $L=\frac{\partial}{\partial t}$ is a linear partial differential operator, N is a nonlinear analytic operator, and $h(x,t)$ is the inhomogeneous term.

Operating the reduced differential transform to Eq.(10), we get the following recurrence relation:

$$(k+1)U_{k+1}(x)=H_k(x)-N(U_k(x)), \tag{18}$$

where $N(U_k(x))$ and $H_k(x)$ are the transforms of the functions $N(u)$ and $h(x,t)$ respectively.

From initial condition (2), we write

$$U_0(x)=f(x). \tag{19}$$

Substituting Eq.(19) into Eq.(18) and using straight forward recursive calculations, we get the following $U_k(x)$ values. The approximate solution of order n

$$u_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k, \tag{20}$$

can be found using the inverse transform of the set of values $\{U_k(x)\}_{k=0}^{n}$. Therefore, the exact solution is given by

$$u(x,t)=\lim_{n\to\infty}u_n(x,t). \tag{21}$$

Provided the series solution converge.

3. Solution of the shock wave equation

To construct the analytic solution of Eq. (4), we apply the RDTM as presented in the previous section. The transformed of Eq.(4) is

$$\begin{aligned}(k+1)U_k(x)=&-\frac{1}{2}\frac{\partial}{\partial x}U_k(x)\\ &+\frac{5}{16}\sum_{r=0}^{k}U_r(x)\frac{\partial}{\partial x}U_{k-r}(x)\end{aligned},\ k\geq 0. \tag{22}$$

Using starting value

$$U_0(x)=e^{\frac{-x^2}{2}}. \tag{23}$$

Substituting (23) into (22) and using the symbolic computation software *Mathematica*, the first few terms of the series in Eq.(22), are as following:

$$U_0(x)=e^{-x^2/2},$$

$$U_1(x)=\frac{1}{2}\mathrm{e}^{-x^2/2}\left(1-\frac{5}{8}\mathrm{e}^{-x^2/2}\right)x,$$

$$U_2(x)=-\frac{1}{8}\mathrm{e}^{-x^2/2}\left(1-\frac{5}{4}\mathrm{e}^{-x^2/2}+\frac{25}{64}\mathrm{e}^{-x^2}-x^2\left(1-\frac{5}{2}\mathrm{e}^{-x^2/2}+\frac{75}{64}\mathrm{e}^{-x^2}\right)\right),$$

$$\begin{aligned}U_3(x)=-\frac{1}{16}\mathrm{e}^{-x^2/2}\Bigg[&\left(1-\frac{15}{4}\mathrm{e}^{-x^2/2}+\frac{225}{64}\mathrm{e}^{-x^2}-\frac{125}{128}\mathrm{e}^{-3x^2/2}\right)\\ &-x^2\left(\frac{1}{4}-\frac{5}{2}\mathrm{e}^{-x^2/2}+\frac{225}{64}\mathrm{e}^{-x^2}-\frac{125}{96}\mathrm{e}^{-3x^2/2}\right)\Bigg]x,\end{aligned}$$

$$\vdots$$

For calculations at hand, we have calculated the first 20 terms of Eq.(22). The differential inverse transforms of $U_k(x)$ gives the approximate solution

$$u_{appr}(x,t)=\sum_{k=0}^{20}U_k(x)t^k. \tag{24}$$

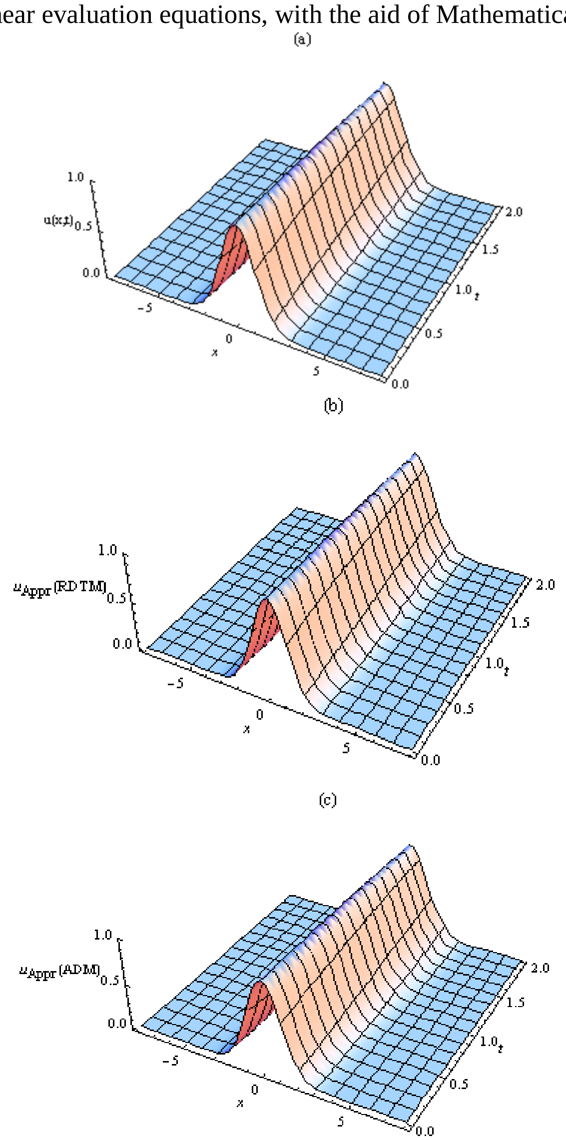
To demonstrate the accuracy of RDTM, we calculate the first 20 terms of the exact solution $u(x,t)$ at Eq. (6), 7 iterations of the VIM and 20 iterations of the ADM. Figure 1 show the surface plot of the exact solution (6), the approximate solutions using RDTM, ADM and VIM respecively. The absolute errors which is defined at any point as

$$E_{Abs}=\left|u(x,t)-u_{appr}(x,t)\right| \tag{25}$$

using mentioned methods, for $-8\leq x\leq 8$ and $t\in[0,2]$ are shown in Figure 2. Numerically, Table 2 show the comparisons of absolute errors for Eq.(4) using 20-terms of the ADM [16], 7 iterations of the VIM [34] and 20 iterations of the RDTM. These results reveal that, with less and an easier computation, the RDTM is effective, accurate and convenient, has the same results as ADM and better than the other method.

5. CONCLUSIONS

In this paper, we consider the non-linear shock wave equation that describes the flow of air for finding an analytic solution via Reduced Differential Transform Method (RDTM). In this example, a high accuracy analytic approximate solution was obtained. It may be concluded that the RDTM methodology is very powerful and efficient technique in finding analytical solutions for wide classes of problems and can be also easy to be extended to other non-linear evaluation equations, with the aid of Mathematica.



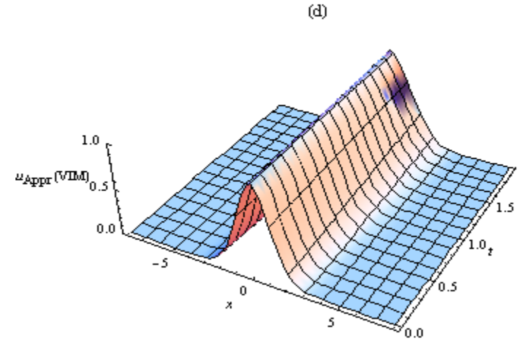


Fig.1. Behavior of: (a) the exact solution, approximate solutions using (b) RDTM, (c) ADM and (d) VIM for $-8 \le x \le 8$, $0 \le t \le 2$.

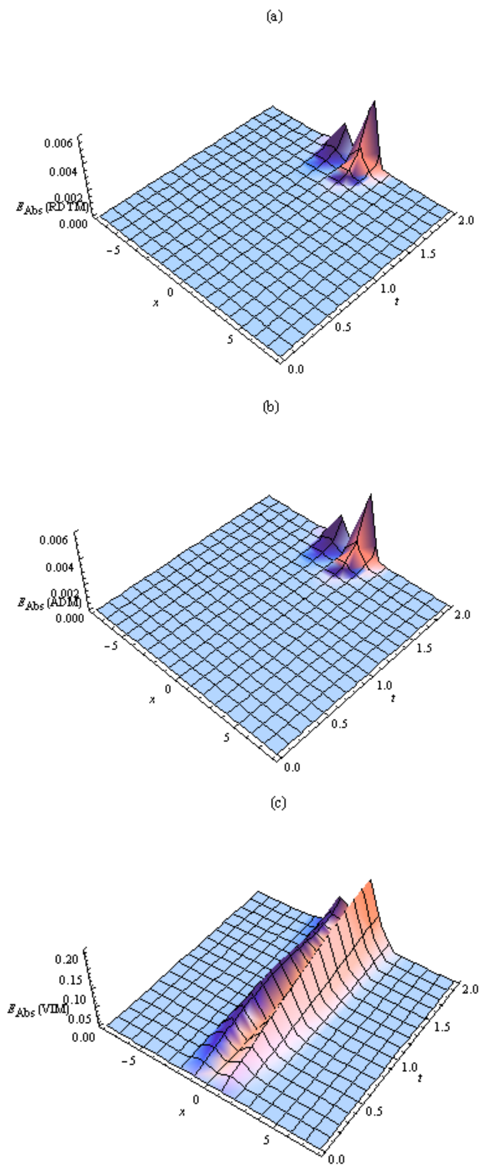


Fig.2. Surface plot of absolute error of solutions obtained by: (a) RDTM, (b) ADM and (c) VIM for for $-8 \le x \le 8$, $0 \le t \le 2$ ..

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Table 2. The absolute error (25) for solving Shock wave equation (4) by 20-terms of ADM, 7 iterations of VIM and 20 iterations of RDTM

x	t	ADM	VIM	RDTM
-3	0	0.	0.	0.
	0.4	4.33681×10^{-18}	1.44151×10^{-5}	5.20417×10^{-18}
	0.8	2.03670×10^{-14}	5.46557×10^{-6}	1.11382×10^{-14}
	1.2	6.72544×10^{-11}	4.39829×10^{-5}	4.67019×10^{-11}
	1.6	2.14116×10^{-8}	3.44805×10^{-4}	1.65289×10^{-8}
	2	1.10100×10^{-2}	1.25945×10^{-2}	1.10103×10^{-2}
-2	0	0.	0.	0.
	0.4	4.16334×10^{-17}	2.41974×10^{-3}	2.77556×10^{-17}
	0.8	5.37251×10^{-12}	2.19746×10^{-3}	1.79814×10^{-13}
	1.2	7.62415×10^{-9}	1.40723×10^{-3}	3.58617×10^{-9}
	1.6	2.95242×10^{-7}	2.29719×10^{-3}	2.87532×10^{-6}
	2	2.35000×10^{-4}	1.15395×10^{-2}	4.09089×10^{-4}
-1	0	0.	0.	0.
	0.4	1.11022×10^{-16}	4.64532×10^{-2}	1.11022×10^{-16}
	0.8	1.32605×10^{-11}	8.86868×10^{-2}	5.10875×10^{-12}
	1.2	2.13084×10^{-8}	1.13138×10^{-1}	1.89467×10^{-8}
	1.6	5.64300×10^{-6}	1.00519×10^{-1}	3.76712×10^{-6}
	2	7.28268×10^{-5}	5.96668×10^{-2}	8.29648×10^{-5}
0	0	0.	0.	0.
	0.4	1.11022×10^{-16}	1.60125×10^{-3}	2.22045×10^{-16}
	0.8	3.11602×10^{-11}	6.86785×10^{-3}	3.11601×10^{-11}
	1.2	1.72054×10^{-8}	1.71889×10^{-2}	1.72055×10^{-8}
	1.6	3.19746×10^{-6}	3.49810×10^{-2}	3.19749×10^{-6}
	2	1.13863×10^{-4}	6.43383×10^{-2}	1.13861×10^{-4}
1	0	0.	0.	0.
	0.4	0.	4.42555×10^{-2}	0.
	0.8	1.95342×10^{-11}	8.39973×10^{-2}	7.91367×10^{-13}
	1.2	1.99126×10^{-8}	1.18758×10^{-1}	2.11708×10^{-8}
	1.6	1.08523×10^{-4}	1.48681×10^{-1}	9.89165×10^{-5}
	2	6.91144×10^{-3}	1.82942×10^{-1}	7.57033×10^{-3}
2	0	0.	0.	0.
	0.4	8.32667×10^{-17}	7.44470×10^{-3}	5.55112×10^{-17}
	0.8	1.50907×10^{-12}	2.11534×10^{-2}	4.19026×10^{-12}
	1.2	1.05646×10^{-8}	4.04125×10^{-2}	2.28982×10^{-8}
	1.6	7.00734×10^{-6}	6.30800×10^{-2}	9.85306×10^{-6}
	2	8.97890×10^{-4}	8.66403×10^{-2}	1.09041×10^{-3}
3	0	0.	0.	0.
	0.4	6.93889×10^{-18}	1.33758×10^{-4}	1.04083×10^{-17}
	0.8	1.20529×10^{-14}	6.86928×10^{-4}	2.04767×10^{-14}
	1.2	9.27564×10^{-11}	2.32743×10^{-3}	1.11524×10^{-10}
	1.6	4.44960×10^{-8}	6.20207×10^{-3}	4.89566×10^{-8}
	2	4.97185×10^{-6}	1.39489×10^{-2}	5.28289×10^{-6}

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