MODIFIED TWO-STEP FIXED POINT ITERATIVE METHOD FOR SOLVING NONLINEAR FUNCTIONAL EQUATIONS

Ashaq Ali [1], **Muhmmad Saeed Ahmad** [2], **Muhmmad Tanveer** [3], **Qaisar Mehmood** [3], **Waqas Nazeer** [4*]

[1]Department of Mathematics, Govt. College Kamoke, Gujranwala, Pakistan
E-mail: arhamusman2001@gmail.com

[2]Department of Mathematics and Statistics, University of Lahore, Lahore, Pakistan
E-mail: saeedkhan07@live.com

[3] Department of Mathematics, Lahore Leads University, Lahore 54810, Pakistan
E-mail: tanveer.7955180@yahoo.com, qaisar47@hotmail.com

[4*] Division of Science and Technology, University of Education, Lahore, Pakistan
E-mail: waqaster@yahoo.com

***ABSTRACT**: In this paper, we present a modified two-step fixed point iterative method for solving nonlinear functional equations and analyzed. The modified two-step fixed point iterative method has convergence of order two. The modified two-step fixed point iterative method converges faster than the fixed point method. The comparison table demonstrates the faster convergence of modified two-step fixed point method.*

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1 INTRODUCTION

The problem, to recall, is solving equations in one variable. We are given a function f and would like to find atleast one solution of the equation $f(x)=0$. Note that, we do not put any restrictions on the function f; we need to be able to evaluate the function; otherwise, we cannot even check that a given $x=\xi$ is true, that is $f(r)=0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior". The more we assume, the more potential we have, on the one hand, to develop fast iteration scheme for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This a fundamental paradigm in numerical analysis.

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method [1].

In the fixed-point iteration method for solving nonlinear equation $f(x)=0$, the equation is usually rewritten as

$$x=g(x), \quad (1)$$

where

(i) there exists $[a,b]$ such that $g(x)\in[a,b]$ for all $x\in[a,b]$,

(ii) there exists $[a,b]$ such that $|g'(x)|\le L<1$ for all $x\in[a,b]$.

Considering the following iteration scheme

$$x_{n+1}=g(x_n), n=0,1,2\ldots, \quad (2)$$

and starting with a suitable initial approximation x_0, we built up a sequence of approximations, say $\{x_n\}$, for the solution of nonlinear equation, say ξ. The scheme will be converge to ξ, provided that

(i) the initial approximation x_0 is chosen in the interval $[a,b]$,

(ii) $|g'(x)|<1$ for all $x\in[a,b]$,

(iii) $a\le g(x)\le b$ for all $x\in[a,b]$.

It is well known that the fixed point method has first order convergence.

Shin et al. described a new second order iterative method for solving nonlinear equations [18] extracted from fixed point method by following the approach of [8] as follows:

If $g'(x)\neq 1$,we can modify (1) by adding $\theta\neq -1$ to both sides as:

$$\theta x+x=\theta x+g(x),$$
$$(1+\theta)x=\theta x+g(x),$$

which implies that

$$x=\frac{\theta x+g(x)}{1+\theta}=g_\theta(x). \quad (3)$$

In order for $g_\theta(x)$ to be efficient, we can choose θ such that $g'_\theta(x)=0$,we yields

$$\theta=-g'(x),$$

so that (3) takes the form

$$x=\frac{-xg'(x)+g(x)}{1-g'(x)}.$$

For a given x_0, we calculate the approximation solution x_{n+1}, by the iteration scheme

$$x_{n+1}=\frac{-x_n g'(x_n)+g(x_n)}{1-g'(x_n)}, g'(x_n)\neq 1.$$

This is so-called a new second order iterative method for solving nonlinear equations, which converges quadratically.

During the last century, the numerical techniques for solving nonlinear equations have been successfully applied (see, e. g., [2-18] and the references therein).

Theorem 1.[8]

Suppose $g(x)\in C^p[a,b]$. If $g^{(k)}(x)=0$, for $k=1,2,\ldots,p-1$ and $g^{(p)}(x)\neq 0$, then the sequence $\{x_n\}$ is of order p.

In this paper, we presented a modified two-step fixed point iterative method for solving nonlinear functional equations having (MFPIM) convergence of order 2 extracted from fixed point iterative method for solving nonlinear equations motivated by the technique of Fernando et al. [11]. The proposed modified two-step fixed point iterative method applied to solve some problems in order to assess its validity and accuracy.

2 *Iterative Method*

Let $f{:}X\subset R\rightarrow R$ for an open interval X is a scalar function and consider that the nonlinear equation $f(x)=0$ (or $x=g(x)$) ,where $g(x){:}X\subset R\rightarrow R$, then we have a new second order iterative method [18]

$$x_{n+1}=\frac{-x_n g'(x_n)+g(x_n)}{1-g'(x_n)},\ g'(x_n)\neq 1. \qquad (4)$$

By following the approach of Fernando et al.[11], we develop a modified two-step fixed point iterative method by replacing $g'(x_n)$ by arithmetic mean $\frac{g'(x_n)+g'(v_n)}{2}$ as follows:

$$x_{n+1}=x_n+\frac{2(g(x_n)-x_n)}{2-g'(x_n)-g'(v_n)}, \qquad (5)$$

$$v_n=g(x_n).$$

Where $g'(x)+g'(v)\neq 2.$

3 *Convergence Analysis*

Theorem 3.1.

Let $f{:}X\subset R\rightarrow R$ for an open interval X and consider that the nonlinear equation $f(x)=0$ (or $x=g(x)$) has a simple root $\alpha\in X$, where $g(x){:}X\subset R\rightarrow R$ be sufficiently smooth in the neighborhood of α; then the convergence order of modified two-step fixed point iterative method given in (4) is at least two.

Proof.

To analysis the convergence of the modified two-step fixed point iterative method (5), let

$$H(x)=x+\frac{2(g(x)-x)}{2-g'(x)-g'(v)};\ g'(x)+g'(v)\neq 2$$

Table: Comparison of FPM and MFPIM

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
$f(x)=x+\ln(x-2), g(x)=2+e^{-x}$				
$x_0=2.2$				
FPM	32	32	5.410786e-30	2💻12002823898764122948468797527 2
MFPIM	4	12	6.258792e-50	2💻120028238987641229484687975272
$f(x)=x^3+4x^2-10, g(x)\sqrt{\frac{10}{4+x}}$				
$x_0=1.5$				
FPM	34	34	6.189210e-30	1💻365230013414096845760806828982
MFPIM	4	12	8.333249e-48	1💻365230013414096845760806828982
$f(x)=x^2-e^x-3x+2, g(x)=\ln(x^2-3x+2)$				
$x_0=0.8$				
FPM			Diverged	
MFPIM	6	18	1.3688869e-55	0💻257530285439860760455367304937
$f(x)=x^3+x^2-3x-3, g(x)=(3+3x-x^2)^{\frac{1}{3}}$				
$x_0=1$				
FPM	24	24	9.368499e- 30	1💻732050807568877293527446341506
MFPIM	4	12	4.065230e- 36	1💻732050807568877293527446341506
$f(x)=x^3+4x^2+8x+8, g(x)=-(1+\frac{1}{2}x^2+\frac{1}{8}x^3)$				
$x_0=-1.9$				
FPM	97	97	5.570340e- 30	-2.000000000000000000000000000000
MFPIM	5	15	1.378876e- 50	-2.000000000000000000000000000000

Let α be a simple zero of f and $f(\alpha)=0$ (or $g(\alpha)=\alpha$), then we can easily deduce by using the software Maple that

$$H(\alpha)=\alpha,$$

$$H'(\alpha)=0,$$

$$H''(\alpha)=\frac{g'(\alpha)g''(\alpha)}{-1+g'(\alpha)}.$$

Now, from above equation it can easily be seen that $H''(\alpha)\neq 0$, the according to theorem 1, modified two-step fixed point iterative method (5) has second order convergence.

4 Applications

In this section we included some nonlinear functions to illustrate the efficiency of our developed modified two-step fixed point iterative method (MFPIM). We compare the MFPIM with Fixed point method (FPM) as shown in Table given at last page.

Table above shows the numerical comparisons of modified two-step fixed point iterative method (MFPIM) with fixed point method (FMP). The columns represent the number of iterations N and the number of functions or derivatives evaluations N_f required to meet the stopping criteria, and the magnitude $|f(x)|$ of $f(x)$ at the final estimate x_n.

5 CONCLUSIONS

A new MFPIM for solving nonlinear functions has been established. We can conclude from table that

1. The modified MFPIM has convergence of order two.
2. By using some examples the performance of MFPIM is also discussed. The MFPIM is performing very well in comparison to FPM as discussed in Table below.

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