HADAMARD'S TYPE INEQUALITIES FOR FUNCTIONS WHOSE ABSOLUTE VALUES OF FIRST DERIVATIVE ARE s-CONVEX

Abdul Waheed[1], Muhammad Iqbal Bhatti[1] Fakhar Haider[1]

Corresponding Author:waheeduet1@gmail.com

[1]Department of Mathematics, University of Engineering and Technology Lahore, Pakistan

ABSTRACT: In this paper, we find new inequalities for functions whose absolute values of first derivative are s-convex and deduce some relations related to special mean.

Keywords: Integral Inequalities; Hermite-Hadamard Inequality; s-convex function; Beta and incomplete Beta Functions; Holder's Inequality; Means for real numbers.

1. INTRODUCTION

The term "convex'' stems from a result obtained by Hermite in 1881 and published in 1883 as a short note in Mathesis, a journal of elementary mathematics. In a letter sent on November 22, 1881, to the journal *Mathesis* (and published there two years later), Ch. Hermite noted that every convex function $f : [a, b] \to \mathbf{R}$ satisfies the inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

The left-hand side inequality was rediscovered ten years later by J. Hadamard. Nowadays, the double inequality (1.1) is called *the Hermite-Hadamard inequality*. The interested reader can find its complete story in [2,3,7,8]. Number of articles are published on generalization, refinements and extensions of inequality (1.1).

We include some basic definitions which will be necessary in the forthcoming theorems.

Definition 1.1[4] *Let X be a convex set in a real vector space. A function $f : X \to \mathbb{R}$ is said to be convex if*

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$$

$\forall x_1, x_2 \in X$ *and* $t \in [0,1]$.

Definition 1.2[4] *A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s-convex$(in second sense) if,*

$$f(tx_1 + (1-t)x_2) \le t^s f(x_1) + (1-t)^s f(x_2)$$

holds $\forall x_1, x_2 \in [0,\infty)$, $t \in [0,1]$ *and some fixed* $s \in (0,1]$.

The class of s-convex (in second sense) functions are usually denoted by k_s^2. It can be easily seen that for $s = 1$, $s-convexity$ reduces to ordinary convexity.

Dragomir [1] has extended inequality (1.1) as:

Theorem1.1. *Let $f: I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, and $a, b \in I^0$ with $b > a$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right) \tag{1.2}$$

Another result has been published in [1] as:

Theorem 1.2. *Let $f: I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, $a, b \in I^0$ with $b > a$ and $p > 1$. If mapping $|f'(x)|^{\frac{p}{p-1}}$ is convex on $[a, b]$ for $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{2(p+1)^{\frac{1}{q}}}\left(|f'(x)|^{\frac{p}{p-1}} + |f'(x)|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \tag{1.3}$$

U. S. Kirmaci in [5] refine inequality (1.1) as:

Theorem 1.3. *Let $f: I \subset [0,\infty)$ be a differentiable mapping on I^0 and $a, b \in I$ with $b > a$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{2+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} \tag{1.4}$$

And also in [5]:

Theorem 1.4. *Let $f: I \subset [0,\infty)$ be a differentiable mapping on I^0 and $a, b \in I$ with $b > a$ .If $|f'(x)|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\frac{1}{s+1}\right]^{\frac{1}{q}}\left\{\left[|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{\frac{1}{q}}\right\} \tag{1.5}$$

In this paper, we find new inequalities for those functions whose absolute values of first derivative are s-convex and apply these inequalities to discover inequalities for special mean.

2. MAIN RESULT

We use the following lemma, to introduce our main results.

Lemma 2.1 ([6]): *Let I be an interval and $f: I \to \mathbb{R}$ be a differentiable on I^0, $a, b \in I$ with $a < b$ and $\lambda, \mu \in \mathbb{R}$. If $f' \in L^1[a, b]$ then the following inequality holds:*

$$(1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = (b-a)\left\{\int_0^{1/2}(\lambda-t)f'(ta+(1-t)b)\,dt + \int_{1/2}^1(\mu-t)f'(ta+(1-t)b)dt\right\} \tag{2.1}$$

Theorem 2.1: *Let $f: I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0, $a, b \in I$ with $b > a$, $1 \ge \mu \ge 1/2 \ge \lambda \ge 0$*

and $f' \in L^1[a,b]$. *If* $|f'(x)|$ *is s-convex on* $[a,b]$, *then the following inequality holds.*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{(s+1)}\left[\left\{\frac{2\lambda^{s+2}}{(s+2)}-\frac{\lambda}{2^{s+1}}+\frac{2\mu^{s+2}}{(s+2)}-\mu\left(\frac{1}{2^{s+1}}+1\right)+\frac{s+1}{s+2}\left(\frac{1}{2^{s+1}}+1\right)\right\}|f'(a)|+\left\{\frac{2(1-\lambda)^{s+2}}{s+2}+\lambda\left(1+\frac{1}{2^{s+1}}\right)-\frac{1}{2^{s+2}}\left(\frac{1}{s+2}+1\right)-\frac{1}{s+2}+\frac{2(1-\mu)^{s+2}}{s+1}-\frac{1}{2^{s+2}}\left(1+\frac{1}{s+2}\right)+\frac{\mu}{2^{s+1}}\right\}|f'(b)|\right] \quad (2.2)$$

Proof: Let

$$I=\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

By taking modulus on both sides of (2.1), we get the inequality

$$I \le (b-a)\left\{\int_0^{1/2}|\lambda-t||f'(ta+(1-t)b)|\,dt+\int_{1/2}^1|\mu-t||f'(ta+(1-t)b)|dt\right\}$$

As $|f'(x)|$ is s-convex, so

$$I \le (b-a)\left\{\int_0^{1/2}|\lambda-t|\left(t^s|f'(a)|+(1-t)^s|f'(b)|\right)dt+\int_{1/2}^1|\mu-t|\left(t^s|f'(a)|+(1-t)^s|f'(b)|\right)dt\right\}$$
$$=(b-a)[I_1+I_2] (2.3)$$

As

$$I_1=|f'(a)|\left\{\frac{2\lambda^{s+2}}{(s+1)(s+2)}-\frac{\lambda}{2^{s+1}(s+1)}+\frac{1}{2^{s+2}(s+2)}\right\}+\frac{|f'(b)|}{s+1}\left\{\left\{\frac{2(1-\lambda)^{s+2}}{s+2}+\lambda\left(1+\frac{1}{2^{s+1}}\right)-\frac{1}{2^{s+2}}\left(\frac{1}{s+2}+1\right)-\frac{1}{s+2}\right\}\right\}$$

$$I_2=|f'(a)|\left\{\frac{2\mu^{s+2}}{(s+1)(s+2)}-\frac{\mu}{s+1}\left(\frac{1}{2^{s+1}}+1\right)+\frac{1}{s+2}\left(\frac{1}{2^{s+2}}+1\right)\right\}+\frac{|f'(b)|}{s+1}\left\{\frac{2(1-\mu)^{s+2}}{s+2}-\frac{1}{2^{s+2}}\left(1+\frac{1}{s+2}\right)+\frac{\mu}{2^{s+1}}\right\}$$

Using above values of I_1 and I_2 in (3.2), we get the $R.H.S$of (2.3).

$$=\frac{(b-a)}{(s+1)}\left[\left\{\frac{2\lambda^{s+2}}{(s+2)}-\frac{\lambda}{2^{s+1}}+\frac{2\mu^{s+2}}{(s+2)}-\mu\left(\frac{1}{2^{s+1}}+1\right)+\frac{s+1}{s+2}\left(\frac{1}{2^{s+1}}+1\right)\right\}|f'(a)|+\left\{\frac{2(1-\lambda)^{s+2}}{s+2}+\lambda\left(1+\frac{1}{2^{s+1}}\right)-\frac{1}{2^{s+2}}\left(\frac{1}{s+2}+1\right)-\frac{1}{s+2}+\frac{2(1-\mu)^{s+2}}{s+1}-\frac{1}{2^{s+2}}\left(1+\frac{1}{s+2}\right)+\frac{\mu}{2^{s+1}}\right\}|f'(b)|\right] \ (Proved)$$

Corollary 3.1.1: *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b>a$, $1 \ge \mu \ge 1/2 \ge \lambda \ge 0$*and* $f' \in L^1[a,b]$. *If* $|f'(x)|$ *is convex on* $[a,b]$, *then the following inequality holds.*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{24}\{(10-3\lambda+8\lambda^3-15\mu+8\mu^3)|f'(a)|+(8-9\lambda+24\lambda^2-8\lambda^3-21\mu+24\mu^2-8\mu^3)|f'(b)|\}$$

Proof :Proof of this corollary is simple, by putting $s=1$ in theorem 3.1.

Theorem 3.2: *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b>a, 1>\mu \ge 1/2 \ge \lambda > 0$ *and* $f' \in L^1[a,b]$. *If* $|f'(x)|^q$ *for* $q>1$ *is s-convex on* $[a,b]$, *then*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le (b-a)\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}}\left[\left(\lambda^{\frac{2q-p-1}{q-1}}+\left(\frac{1}{2}-\lambda\right)^{\frac{2q-p-1}{q-1}}\right)^{1-\frac{1}{q}}\left\{|f'(a)|^q\lambda^{s+p+1}\left[B(s+1,p+1)+(-1)^{p+1}B\left(1-\frac{1}{2\lambda};p+1,s+1\right)\right]+|f'(b)|^q(1-\lambda)^{s+p+1}\left[(-1)^{p+1}B\left(\frac{\lambda}{\lambda-1};p+1,s+1\right)+B\left(\frac{1-2\lambda}{2(1-\lambda)};p+1,s+1\right)\right]\right\}^{\frac{1}{q}}+\left(\left(\mu-\frac{1}{2}\right)^{\frac{2q-p-1}{q-1}}+(1-\mu)^{\frac{2q-p-1}{q-1}}\right)^{1-\frac{1}{q}}\left\{|f'(a)|^q\mu^{s+p+1}\left[B\left(1-1/2\mu;p+1,s+1\right)+(-1)^{p+1}B\left(\frac{\mu-1}{\mu};p+1,s+1\right)\right]+|f'(b)|^q(1-\mu)^{s+p+1}\left[(-1)^{p+1}B\left(\frac{1-2\mu}{2(1-\mu)};p+1,s+1\right)+B(p+1,s+1)\right]\right\}^{\frac{1}{q}}\right]$$

Where

$$B(p,s)=\int_0^1 t^{p-1}(1-t)^{s-1}dt$$

and $B(v;p,s)=\int_0^v t^{p-1}(1-t)^{s-1}dt$

Proof:By lemma 2.1 and using the properties of modulus, we have

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le (b-a)\left\{\int_0^{1/2}|\lambda-t||f'(ta+(1-t)b)|\,dt+\int_{1/2}^1|\mu-t||f'(ta+(1-t)b)|dt\right\}$$

Using Holder's inequality,

$$\le (b-a)\left\{\left(\int_0^{1/2}|\lambda-t|^{\frac{q-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\lambda-t|^p|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}+\left(\int_{1/2}^1|\mu-t|^{\frac{q-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1|\mu-t|^p|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\}$$
$$=(b-a)[I_1+I_2] \quad (2.4)$$

By using the beta and incomplete beta function, we have

$$I_1 = \left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}} \left(\lambda^{\frac{2q-p-1}{q-1}} + \left(\frac{1}{2}-\lambda\right)^{\frac{2q-p-1}{q-1}}\right)^{1-\frac{1}{q}} \Big\{\left|f'(a)\right|^q \lambda^{s+p+1}\Big[B(s+1,p+1) + (-1)^{p+1}B\Big(1-\frac{1}{2\lambda};p+1,s+1\Big)\Big] + \left|f'(b)\right|^q (1-\lambda)^{s+p+1}\Big[(-1)^{p+1}B\Big(\frac{\lambda}{\lambda-1};p+1,s+1\Big) + B\Big(\frac{1-2\lambda}{2(1-\lambda)};p+1,s+1\Big)\Big]\Big\}^{\frac{1}{q}} \quad (2.5)$$

and

$$I_2 = \left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}} \left(\left(\mu-\frac{1}{2}\right)^{\frac{2q-p-1}{q-1}} + (1-\mu)^{\frac{2q-p-1}{q-1}}\right)^{1-\frac{1}{q}} \times \Big\{\left|f'(a)\right|^q \mu^{s+p+1}\Big[B\Big(1-{}^{1}/_{2\mu};p+1,s+1\Big) + (-1)^{p+1}B\Big(\frac{\mu-1}{\mu};p+1,s+1\Big)\Big] + \left|f'(b)\right|^q (1-\mu)^{s+p+1}\Big[(-1)^{p+1}B\Big(\frac{1-2\mu}{2(1-\mu)};p+1,s+1\Big) + B(p+1,s+1)\Big]\Big\}^{\frac{1}{q}} \quad (2.6)$$

Using (2.5) *and* (2.6) *in* (2.4). We get the desire result.

Corollary 3.2.1: *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b > a, 1 \geq \mu \geq {}^{1}/_{2} \geq \lambda \geq 0$ *and* $f' \in L^1[a,b]$. *If* $\left|f'(x)\right|^q$ *for* $q \geq 1$ *is* s*-convex on* $[a,b]$, *then*

$$\left|(1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{\frac{1}{q}}2^{3\left(1-\frac{1}{q}\right)}}\Big[(8\lambda^2-4\lambda+1)^{1-\frac{1}{q}}\Big\{\left|f'(a)\right|^q\Big[\frac{\lambda^{s+2}}{s+2} + \frac{1-2\lambda}{2^{s+2}} + \frac{1}{s+2}\Big(\lambda^{s+2} - \frac{1}{2^{s+2}}\Big)\Big] + \left|f'(b)\right|^q\Big[\lambda - \frac{1}{s+2}\{1-(1-\lambda)^{s+2}\} - \frac{1}{2^{s+1}}\Big(\frac{1}{2}-\lambda\Big) - \frac{1}{s+2}\Big\{\frac{1}{2^{s+2}} - (1-\lambda)^{s+2}\Big\}\Big]\Big\}^{\frac{1}{q}} + (8\mu^2-12\mu+5)^{1-\frac{1}{q}}\Big\{\left|f'(a)\right|^q\Big[\frac{1-2\mu}{2^{s+2}} - \frac{1}{s+2}\Big(\frac{1}{2^{s+2}} + 1 - 2\mu^{s+2}\Big) + 1 - \mu\Big] + \left|f'(b)\right|^q\Big[\frac{1}{2^{s+1}}\Big(\mu-\frac{1}{2}\Big) - \frac{1}{s+2}\Big\{\frac{1}{2^{s+2}} - 2(1-\mu)^{s+2}\Big\}\Big]\Big\}^{\frac{1}{q}}\Big]$$

Proof: Putting $p = 1$ in theorem 3.2.

Corollary 3.2.2: *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b > a, m > 0, m \geq 2l \geq 0$ *and* $f' \in L^1[a,b]$. *If* $\left|f'(x)\right|^q$ *for* $q \geq 1$ *is* s*-convex on* $[a,b]$, *then*

$$\left|\frac{l}{m}\{f(a)+f(b)\} + \frac{m-2l}{m}f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)m^{s+2}2^{s-1}}\right)^{\frac{1}{q}}\left(\frac{8l^2-4ml+m^2}{m^2}\right)^{1-\frac{1}{q}} \times \Big[\Big\{\left|f'(a)\right|^q[(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}] + \left|f'(b)\right|^q[2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}]\Big\}^{\frac{1}{q}} + \Big\{\left|f'(a)\right|^q[2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}] + \left|f'(b)\right|^q[(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}]\Big\}^{\frac{1}{q}}\Big]$$

Proof: By putting $\lambda = 1-\mu = \frac{l}{m}$ in corollary 3.2.1.

Corollary 3.2.3: *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b > a$ *and* $f' \in L^1[a,b]$. *If* $\left|f'(x)\right|^q$ *for* $q \geq 1$ *is* s*-convex on* $[a,b]$, *then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}}\Big[\{(s+1)\left|f'(a)\right|^q + (2^{s+2}-3-s)\left|f'(b)\right|^q\}^{\frac{1}{q}} + \{(2^{s+2}-s-3)\left|f'(a)\right|^q + (s+1)\left|f'(b)\right|^q\}^{\frac{1}{q}}\Big] \quad (2.7)$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}}\Big[\{\left|f'(a)\right|^q + (2^{s+1}s+1)\left|f'(b)\right|^q\}^{\frac{1}{q}} + \{(2^{s+1}s+1)\left|f'(a)\right|^q + \left|f'(b)\right|^q\}^{\frac{1}{q}}\Big] \quad (2.8)$$

$$\left|\frac{1}{6}\left\{f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72}\left[\frac{1}{5(s+1)(s+2)3^s2^{s-1}}\right]^{\frac{1}{q}}\Big[\{[3^{s+1}(2s+1)+2]\left|f'(a)\right|^q + [2.5^{s+2} + 3^{s+1}\{2^{s+1}(s-4)-2s-7\}]\left|f'(b)\right|^q\}^{\frac{1}{q}} + \{[2.5^{s+2} + 3^{s+1}\{2^{s+1}(s-4)-2s-7\}]\left|f'(a)\right|^q + [3^{s+1}(2s+1)+2]\left|f'(b)\right|^q\}^{\frac{1}{q}}\Big] \quad (2.9)$$

Proof: By putting $(l,m) = (0,1), (1,2)$ and $(1,6)$ in corollary 3.2.2.

Corollary 3.2.4: *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0, $a,b \in I$ *with* $b > a$ *and* $f' \in L^1[a,b]$. *If* $\left|f'(x)\right|$ *is* s*-convex on* $[a,b]$, *then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \quad (2.10)$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(s+1)(s+2)2^{s+1}}(2^s s+1)\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \quad (2.11)$$

$$\left|\frac{1}{6}\left\{f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(s+1)(s+2)2^{s+2}3^{s+2}}[3^{s+1}\{2^{s+1}(s-4)-6\} + 2(1+5^{s+2})]\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \quad (2.12)$$

Proof: By putting $q = 1$ in corollary 3.2.3.

Remark: By putings $s = 1$ in corollary 3.2.4, we reform the inequality (1.2) and inequalities of [6] as follows.

Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^0, $a,b\in I$ with $b>a$and $f'\in L^1[a,b]$. If $|f'(x)|$is convex on $[a,b]$, *then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[|f'(a)|+|f'(b)|\right] \quad (2.13)$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[|f'(a)|+|f'(b)|\right] \quad (2.14)$$

$$\left|\frac{1}{6}\left\{f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right\}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72}\left[|f'(a)|+|f'(b)|\right] \quad (2.15)$$

Theorem 2.3: *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0, $a,b\in I$ with $b>a$, $1>\mu\geq 1/2\geq\lambda>0$ and $f'\in L^1[a,b]$. If $|f'(x)|^q$ for $q>1$ is s-convex on* $[a,b]$, *then*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\left(\frac{1}{2^{sp+1}(sp+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[|f'(a)|\left\{\left(\lambda^{q+1}+\left(\frac{1}{2}-\lambda\right)^{q+1}\right)^{\frac{1}{q}}+(2^{sp+1}-1)^{\frac{1}{p}}\left(\left(\mu-\frac{1}{2}\right)^{q+1}+(1-\mu)^{q+1}\right)^{\frac{1}{q}}\right\}+|f'(b)|\left\{(2^{sp+1}-1)^{\frac{1}{p}}\left(\lambda^{q+1}+\left(\frac{1}{2}-\lambda\right)^{q+1}\right)^{\frac{1}{q}}+\left(\left(\mu-\frac{1}{2}\right)^{q+1}+(1-\mu)^{q+1}\right)^{\frac{1}{q}}\right\}\right]$$

Proof: Using lemma 2.1, convexity of $f'(x)$ on $[a,b]$ and Holder's inequality

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left\{\int_0^{1/2}|\lambda-t||f'(ta+(1-t)b)|\,dt+\int_{1/2}^1|\mu-t||f'(ta+(1-t)b)|dt\right\}$$
$$\leq (b-a)\left\{\int_0^{1/2}|\lambda-t|\{t^s|f'(a)|+(1-t)^s|f'(b)|\}\,dt+\int_{1/2}^1|\mu-t|\{t^s|f'(a)|+(1-t)^s|f'(b)|\}\,dt\right\}$$
$$=(b-a)\{I_1+I_2\} \quad (2.16)$$

Now

$$I_1=|f'(a)|\int_0^{1/2}t^s|\lambda-t|\,dt+|f'(b)|\int_0^{1/2}(1-t)^s|\lambda-t|\,dt$$

Using Holder's inequality

$$I_1 \leq |f'(a)|\left[\left(\int_0^{\frac{1}{2}}t^{sp}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\lambda-t|^q dt\right)^{\frac{1}{q}}\right]+|f'(b)|\left[\left(\frac{1}{2^{sp+1}(sp+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}(2^{sp+1}-1)^{\frac{1}{p}}\left\{\lambda^{q+1}+\left(\frac{1}{2}-\lambda\right)^{q+1}\right\}^{\frac{1}{q}}\right]$$

$$I_1 \leq \left(\frac{1}{2^{sp+1}(sp+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[|f'(a)|\left\{\lambda^{q+1}+\left(\frac{1}{2}-\lambda\right)^{q+1}\right\}^{\frac{1}{q}}+|f'(b)|(2^{sp+1}-1)^{\frac{1}{p}}\left\{\lambda^{q+1}+\left(\frac{1}{2}-\lambda\right)^{q+1}\right\}^{\frac{1}{q}}\right]$$

Similarly we have

$$I_2 \leq \left(\frac{1}{2^{sp+1}(sp+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[|f'(a)|(2^{sp+1}-1)^{\frac{1}{p}}\left\{\left(\mu-\frac{1}{2}\right)^{q+1}+(1-\mu)^{q+1}\right\}^{\frac{1}{q}}+|f'(b)|\left\{\left(\mu-\frac{1}{2}\right)^{q+1}+(1-\mu)^{q+1}\right\}^{\frac{1}{q}}\right]$$

Using above values of I_1 and I_2 in (2.16), the required result follows.

Corollary 2.3.1:*Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0, $a,b\in I$ with $b>a, m>0, m\geq 2l\geq 0$ and$f'\in L^1[a,b]$. If $|f'(x)|^q$ for $q>0$ is s-convex on* $[a,b]$, *then*

$$\left|\frac{l[f(a)+f(b)]+(m-2l)f\left(\frac{a+b}{2}\right)}{m}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)\left\{1+(2^{sp+1}-1)^{\frac{1}{p}}\right\}}{\{2^{sp+1}(sp+1)\}^{\frac{1}{p}}}\left\{\frac{2^{q+1}l^{q+1}+(m-2l)^{q+1}}{(q+1)2^{q+1}m^{q+1}}\right\}^{\frac{1}{q}}\left[|f'(a)|+|f'(b)|\right]$$

Proof: The result is obtained, if we put$\lambda=1-\mu=\frac{l}{m}$ in theorem 2.3.

3. APPLICATION TO SPECIAL MEAN.

For $a>0$and $b>0$, means are defined as:

Arithmetic Mean:$A(a,b)=\frac{a+b}{2}$

Geometric Mean:$G(a,b)=\sqrt{ab}$

Harmonic Mean:$H(a,b)=\frac{2ab}{a+b}$

Logarithmic Mean:$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if\, a\neq b\\ a & if\, a=b\end{cases}$

and

Generalized logarithmic mean

$$L_n(a,b)=\begin{cases}\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & , \ n\neq -1,0\\ \frac{b-a}{\ln b-\ln a} & , \ n=-1\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & , \ n=0\end{cases}$$

Now, we deduce relations between means.

Theorem 3.1: *Let* $b>a>0, m>0, m\geq 2l\geq 0$ *and* $0<s\leq 1$, $q\geq 1$, *then*

$$\left|\frac{2lA(a^{s+1},b^{s+1})+(m-2l)A^{s+1}(a,b)-mL_{s+1}^{s+1}(a,b)}{m(s+1)}\right|$$

$$\leq \frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)m^{s+2}2^{s-1}}\right)^{\frac{1}{q}}\left(\frac{8l^2-4ml+m^2}{m^2}\right)^{1-\frac{1}{q}}\left[\{a^{sq}[(s+1)m^{s+2}-2l(s+2)m^{s+1}+2^{s+3}l^{s+2}]+b^{sq}[2^{s+3}(m-l)^{s+2}+2l(s+2)(1+2^{s+1})m^{s+1}-(s+3+2^{s+2})m^{s+2}]\}^{\frac{1}{q}}+\{a^{sq}[2^{s+3}(m-l)^{s+2}+2l(s+2)(1+2^{s+1})m^{s+1}-(s+3+2^{s+2})m^{s+2}]+b^{sq}[(s+1)m^{s+2}-2l(s+2)m^{s+1}+2^{s+3}l^{s+2}]\}^{\frac{1}{q}}\right]$$

Proof: Take $f(x)=\frac{x^{s+1}}{s+1}$. As $f'(x)=x^s$ is s-convex function. Appling $f(x)$ on corollary 2.2.2. We get the desire result.

Corollary 3.1.1: *Let*$b>a>0, m>0, m\geq 2l\geq 0$, *and* $1\geq s>0$ *then*

$$\left|\frac{2lA(a^{s+1},b^{s+1})+(m-2l)A^{s+1}(a,b)-mL_{s+1}^{s+1}(a,b)}{m(s+1)}\right|\leq\frac{b-a}{(s+1)(s+2)2^s m^{s+2}}[2^{s+2}\{l^{s+2}+(m-l)^{s+2}\}+2^{s+1}(s+2)lm^{s+1}-(s+2^{s+1})m^{s+2}]A(a^s,b^s)$$

Proof: Put $q=1$ in theorem 3.1.

Corollary 3.1.2: *Let* $b>a>0$, *and* $1\geq s>0$, $q\geq 1$, *then*

$$\left|\frac{A^{s+1}(a,b)-L_{s+1}^{s+1}(a,b)}{(s+1)}\right|\leq\frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}}[\{(s+1)a^{sq}+(2^{s+2}-3-s)b^{sq}\}^{\frac{1}{q}}+\{(2^{s+2}-s-3)a^{sq}+(s+1)b^{sq}\}^{\frac{1}{q}}] \quad (3.1)$$

$$\left|\frac{A(a^{s+1},b^{s+1})-L_{s+1}^{s+1}(a,b)}{(s+1)}\right|\leq\frac{(b-a)}{8}\left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}}[\{a^{sq}+(2^{s+1}s+1)b^{sq}\}^{\frac{1}{q}}+\{(2^{s+2}s+1)a^{sq}+b^{sq}\}^{\frac{1}{q}}] \quad (3.2)$$

$$\left|\frac{2A(a^{s+1},b^{s+1})+A^{s+1}(a,b)-3L_{s+1}^{s+1}(a,b)}{3(s+1)}\right|\leq\frac{5(b-a)}{72}\left[\frac{1}{5(s+1)(s+2)3^s2^{s-1}}\right]^{\frac{1}{q}}[\{[3^{s+1}(s-1)+2^{s+3}]a^{sq}+[2^{2s+5}+3^{s+1}\{2^{s+2}(s-1)-s-5\}]b^{sq}\}^{\frac{1}{q}}+\{[2^{2s+5}+3^{s+1}\{2^{s+2}(s-1)-s-5\}]a^{sq}+[3^{s+1}(s-1)+2^{s+3}]b^{sq}\}^{\frac{1}{q}}] \quad (3.3)$$

$$\left|\frac{A(a^{s+1},b^{s+1})+A^{s+1}(a,b)-2L_{s+1}^{s+1}(a,b)}{2(s+1)}\right|\leq\frac{(b-a)}{32}\left[\frac{1}{(s+1)(s+2)2^{2(s-1)}}\right]^{\frac{1}{q}}[\{[2^s s+1]a^{sq}+[3^{s+2}+2^s\{2^{s+1}(s-2)-s-4\}]b^{sq}\}^{\frac{1}{q}}+\{[3^{s+2}+2^s\{2^{s+1}(s-2)-s-4\}]a^{sq}+[2^s s+1]b^{sq}\}^{\frac{1}{q}}] \quad (3.4)$$

$$\left|\frac{2A(a^{s+1},b^{s+1})+3A^{s+1}(a,b)-5L_{s+1}^{s+1}(a,b)}{5(s+1)}\right|\leq\frac{b-a}{200}\left[\frac{1}{(s+1)(s+2)5^s2^{s-1}}\right]^{\frac{1}{q}}[\{[5^{s+1}(3s+1)+2^{s+3}]a^{sq}+[2^{3s+7}+5^{s+1}\{2^{s+2}(s-3)-3s-11\}]b^{sq}\}^{\frac{1}{q}}+\{[2^{3s+7}+5^{s+1}\{2^{s+2}(s-3)-3s-11\}]a^{sq}+[5^{s+1}(3s+1)+2^{s+3}]b^{sq}\}^{\frac{1}{q}}] \quad (3.5)$$

$$\left|\frac{4A(a^{s+1},b^{s+1})+A^{s+1}(a,b)-5L_{s+1}^{s+1}(a,b)}{5(s+1)}\right|\leq\frac{17(b-a)}{200}\left[\frac{1}{17(s+1)(s+2)5^s2^{s-1}}\right]^{\frac{1}{q}}[\{[5^{s+1}(s-3)+2^{2s+5}]a^{sq}+[2^{s+3}3^{s+2}+5^{s+1}\{2^{s+2}(2s-1)-s-7\}]b^{sq}\}^{\frac{1}{q}}+\{[2^{s+3}3^{s+2}+5^{s+1}\{2^{s+2}(2s-1)-s-7\}]a^{sq}+[5^{s+1}(s-3)+2^{2s+5}]b^{sq}\}^{\frac{1}{q}}] \quad (3.6)$$

$$\left|\frac{A(a^{s+1},b^{s+1})+2A^{s+1}(a,b)-3L_{s+1}^{s+1}(a,b)}{3(s+1)}\right|\leq\frac{5(b-a)}{72}\left[\frac{1}{5(s+1)(s+2)3^s2^{s-1}}\right]^{\frac{1}{q}}[\{[3^{s+1}(2s+1)+2]a^{sq}+[2.5^{s+2}+3^{s+1}\{2^{s+1}(s-4)-2s-7\}]b^{sq}\}^{\frac{1}{q}}+\{[2.5^{s+2}+3^{s+1}\{2^{s+1}(s-4)-2s-7\}]a^{sq}+[3^{s+1}(2s+1)+2]b^{sq}\}^{\frac{1}{q}}] \quad (3.7)$$

Proof:By putting$(l,m)=(0,1),(1,2),(1,3),(1,4),(1,5),(2,5)and(1,6)$ in theorem 3.1.

Theorem 3.2:*Let*$b>a>0, m>0, m\geq 2l\geq 0$ *and* $q\geq 1$, *then*

$$\left|\frac{1}{m}[2lA(\ln a,\ln b)+(m-2l)\ln A(a,b)]-\ln I(a,b)\right|\leq\frac{(b-a)}{24m^3G^2(a,b)}\{3m(8l^2-4ml+m^2)\}^{1-\frac{1}{q}}\left[\{(2m^3-8l^3+24l^2m-9lm^2)a^q+(m^3+8l^3-3lm^2)b^q\}^{\frac{1}{q}}+\{(m^3+8l^3-3lm^2)a^q+(2m^3-8l^3+24l^2m-9lm^2)b^q\}^{\frac{1}{q}}\right]$$

Proof: Take $s=1$ and $f(x)=lnx$ in corollary 2.2.3. We got the desire result.

Corollary 3.2.1: *Let* $b>a>0, m>0$, *and* $m\geq 2l\geq 0$, *then*

$$\left|\frac{1}{m}[2lA(\ln a,\ln b)+(m-2l)\ln A(a,b)]-\ln I(a,b)\right|\leq\frac{b-a}{4m^2H(a,b)}(8l^2-4lm+m^2)$$

Proof: Byputting $q=1$ in theorem 3.2.

Theorem 3.3: *Let* $b > a > 0, m > 0, m \geq 2l \geq 0, 1 \geq s > 0$ *and* $q > 1$, *then*

$$\left|\frac{2lA(a^{s+1}, b^{s+1}) + (m-2l)A^{s+1}(a,b) - mL_{s+1}^{s+1}(a,b)}{m(s+1)}\right| \leq \frac{2(b-a)\left\{1+(2^{sp+1}-1)^{\frac{1}{p}}\right\}}{\{2^{sp+1}(sp+1)\}}\left\{\frac{2^{q+1}l^{q+1}+(m-2l)^{q+1}}{(q+1)2^{q+1}m^{q+1}}\right\}^{\frac{1}{q}} A(a^s, b^s)$$

Proof: Take $f(x) = \frac{x^{s+1}}{s+1}$. As $f'(x) = x^s$ is s-convex function. Appling $f(x)$ on corollary 2.3.1. We get the desire result.

Corollary 3.3.1: *Let* $b > a > 0, m > 0, m \geq 2l \geq 0$, $1 \geq s > 0$ *and* $q \geq 1$, *then*

$$\left|\frac{2lA(a^{s+1}, b^{s+1}) + (m-2l)A^{s+1}(a,b) - mL_{s+1}^{s+1}(a,b)}{m(s+1)}\right|$$

$$\leq \frac{2(b-a)}{s+1}\left\{\frac{2^{q+1}l^{q+1}+(m-2l)^{q+1}}{(q+1)2^{q+1}m^{q+1}}\right\}^{\frac{1}{q}} A(a^s, b^s)$$

Proof: By putting $p = 1$ in theorem 3.3.

REFERENCES

[1] Dragomir S. S and R. P. Agarwal, "*Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula*," Applied Mathematics Letters, Vol. 11, No. 5, 1998, pp. 91-95. doi:10.1016/S0893-9659(98)00086-X.

[2] Dragomir S. S, J. Pecaric and L.-E. Persson, "*Some Inequalities of Hadamard Type*," Soochow Journal of Mathematics, Vol. 21, No. 3, 1995, pp. 335-341.

[3] Dragomir S. S and C. E. M. Pearce, "*Selected Topics on Hermite-Hadamard Type Inequalities and Applications*," RGMIA Monographs, Victoria University, Melbourne, 2000.

[4] Hudzik. H and L. Maligranda, "*Some remarks on s-convex functions*," AequationesMathematicae, vol.48, No.1, 1994, pp.100-111.

[5] Kirmaci U. S, "*Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Midpoint Formula*," Applied Mathematics and Computation, Vol. 147, No. 1, 2004, pp. 137-146. doi:10.1016/S0096-3003(02)00657-4

[6] Qi Feng, Tian-yuzhang, and Bo-yan "*Hermite-Hadamard type integral inequalities for functions whose first derivatives are of convexity*" arXiv:1305.5933v1,25 May 2013

[7] Xi. B.-Y and F. Qi, "*Some Hermite-Hadamard Type Inequalities for Differentiable Convex Functions and Applications*," Hacettepe Journal of Mathematics and Statistics, Vol. 42, 2013, in Press.

[8] Xi. B.-Y and F. Qi, "*Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means*," Journal of Function Spaces and Applications, Vol. 2012, 2012, 14 pp. doi:10.1155/2012/980438