ABSTRACT. A graph \( G(V, E) \) has an \( H \)-covering if every edge in \( E \) belongs to a subgraph of \( G \) isomorphic to \( H \). Suppose \( G \) admits an \( H \)-covering. An \( H \)-magic labeling is a total labeling \( \lambda \) from \( V(G) \cup E(G) \) onto the integers \( \{1, 2, \ldots, |V(G) \cup E(G)|\} \) with the property that, for every subgraph \( A \) of \( G \) isomorphic to \( H \) there is a positive integer \( c \) such that \( \sum_{v \in V(A)} \lambda(v) + \sum_{e \in E(A)} \lambda(e) = c \). A graph that admits such a labeling is called \( H \)-magic. In addition, if \( \{\lambda(v)\}_{v \in V} = \{1, \ldots, |V|\} \), then the graph is called \( H \)-supermagic. Moreover a graph is said to be \( H \)-(\( a, d \)) antimagic if the magic constant for an arithmetic progression with initial value \( a \) and common difference \( d \). In this paper we formulate cycle anti-supermagic labeling for the disjoint union of isomorphic copies of ladder and triangular ladder graphs.

Key Words: \( H \)-anti-supermagic labeling, cycles graphs, disjoint union of graphs.

1. INTRODUCTION

Let \( G = (V, E) \) be a finite, simple, planar, connected and undirected graph, where \( V \) and \( E \) are its vertex-set and edge-set respectively. Let \( H \) and \( G = (V, E) \) be finite simple graphs with the property that every edge of \( G \) belongs to at least one subgraph isomorphic to \( H \). A bijection \( \lambda : V \cup E \to \{1, \ldots, |V| + |E|\} \) is an \( H \)-magic labeling of \( G \) if there exist a positive integer \( c \) (called the magic constant), such that for any subgraph \( H'(V', E') \) of \( G \) isomorphic to \( H \), the sum \( \sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e) \) is equal to magic constant \( c \). This sum is also known as weight of \( H \). A graph is \( H \)-magic if it admits an \( H \)-magic labeling. In addition, if \( H \)-magic labeling \( \lambda \) has the property that \( \{\lambda(v)\}_{v \in V} = \{1, \ldots, |V|\} \), then the graph is \( H \)-supermagic. In our terminology, super edge-magic total labeling is a \( K_2 \)-supermagic labeling. The notion of \( H \)-magic graph was introduced by A.Gutierrez and A. Llado [11], as an extension of the magic valuation given by Rosa [15], which corresponds to the case \( H \cong K_2 \).

There are many results for \( H \)-supermagic graphs. Llado and Moragas [16] studied cycle-magic labeling of wheels, prisms, books, wind mill graphs and subdivided wheels by using the technique of partitioning sets of integers. Ngurah et al. [20] constructed cycle-supermagic labelings for some families of graphs namely fans, ladders, and books etc. For any connected graph \( G \), Maryati et al. [17], proved that disjoint union of \( k \)-isomorphic copies of \( G \) is a \( G \)-supermagic graph if and only if \( |V(G)| + |E(G)| \) is even or \( k \) is odd. For a disconnected graph \( G \), having at least two vertices in its every component, the result was proved again by Maryati et al. [19]. In [22] Rizvi et al. discussed \( H \)-supermagic labeling of the disconnected graphs. They formulated cycle-supermagic labelings for the disjoint union of isomorphic copies of different families of graphs. They also proved that disjoint union of non-isomorphic copies of fans and ladders are cycle-supermagic. Hartsfeld and Ringel [12] introduced antimagic graphs in 1990. A graph with \( q \) edges is called antimagic if its edges can be labeled with \( 1, 2, \ldots, q \) such that the sums of the labels of the edges incident to each vertex are distinct. Among the graphs they prove are antimagic are: \( P_n(n \geq 3) \), cycles and wheels. The concept of an \( (a, d) \)-antimagic labelings was introduced by Bodendiek and Walther [6] in 1993. A connected graph \( G = (V, E) \) is said to be \( (a, d) \)-antimagic if there exist positive integers \( a \) and \( d \), and a bijection \( f : E \to \{1, 2, 3, \ldots, |E|\} \) such that the induced mapping \( g_f : V \to \mathbb{N} \), defined by \( g_f(v) = \sum_{uv \in E(G)} f(uv) \), is injective and \( g_f(V) = a, a + d, a + (|V| - 1)d \). In [5] Baca and Hollander proved that necessary conditions for \( C_n \times P_2 \) to be \( (a, d) \)-antimagic. Bodendiek and Walther [7] proved that the following graphs are not \( (a, d) \)-antimagic: even cycles; paths of even order; stars; trees of odd order. Javaid et al. [13] and Javaid et al. [14] constructed super \( (a, d) \)-edge-antimagic total labeling for \( w \)-trees and extended \( w \)-trees as well as super \( (a, d) \)-edge-antimagic total labeling for disjoint union of isomorphic and non-isomorphic copies of extended \( w \)-trees. Baca [4] gave antimagic labeling for the antiprism.
In this paper, we formulate cycle anti-supermagic labeling for the disjoint union of isomorphic copies of ladder graphs for different value of $d$. Also we prove cycle supermagic labeling for the disjoint union of isomorphic copies of triangular ladder graphs as well as cycle anti-supermagic labeling for the disjoint union of isomorphic copies of triangular ladder graphs for different value of $d$.

2. MAIN RESULTS

In the next two theorems, we prove that disjoint union of isomorphic copies of ladder graphs admit cycle anti-supermagic labeling with different values of $d$.

**Theorem.** For every $m \geq 1$, $n \geq 2$, the graph $G \cong mL_n$ is $C_4$-anti-supermagic with distance $d = 2$ and $a = 18mn - 5m + 5$.

**Proof.**

Let $v = |V(G)| = 2mn$, and $e = |E(G)| = m(3n - 2)$. We denote the vertex and edge sets of $G$ as follows:

$$V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$$E(G) = \{u_i^jv_j^i : 1 \leq i \leq n, 1 \leq j \leq m\}$$

Define a total labeling

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, m(5n - 2)\}$$

as follows:

For $1 \leq i \leq n, 1 \leq j \leq m$,

$$\lambda(u_i^j) = 2m(i-1) + j$$

$$\lambda(v_j^i) = m(2i-1) + j$$

$$\lambda(u_i^jv_j^i) = v + e - m(i-1) - (j-1),$$

where $v = 2mn$ and $e = m(3n - 2)$.

We can easily calculate the values of $a$ of sub cycle such that

$$a = \sum C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_i^{j+1}) + \lambda(v_j^i) + \lambda(v_j^{i+1})$$

$$+ \lambda(u_i^{j+1}v_j^{i+1}) + \lambda(u_i^{j+1}v_j^{i+1}) + \lambda(u_i^{j+1}v_j^{i+1})$$

$$= 18mn - 5m + 5.$$

Hence $mL_n$ is $C_4$-anti-supermagic with distance $d = 2$.

**Theorem.** For every $m \geq 1, n \geq 2$, the graph $G \cong mL_n$ is $C_4$-anti-supermagic with distance $d = 6$ and $a = 16mn - 3m + 7$.

**Proof.**

Let $v = |V(G)| = 2mn$, and $e = |E(G)| = m(3n - 2)$. We denote the vertex and edge sets of $G$ as follows:

$$V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$$\cup \{v_j^i : 1 \leq i \leq n, 1 \leq j \leq m\};$$

$$E(G) = \{u_i^jv_j^i : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$$\cup \{u_i^{j+1}u_i^{j+1} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$$

$$\cup \{v_j^{i+1}v_j^{i+1} : 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$
\( \lambda(u_i^j) = 2m(i-1) + j \)
\( \lambda(v_j) = m(2i-1) + j \)
\( \lambda(u_i^j_v_j^j) = m(6n-3) - 2m(i-1) - (j - 1) \)
\( \lambda(u_i^j_v_j^j) = m(6n-4) - 2m(i-2) - (j - 1) \)
\( \lambda(u_i^j_u_i^j) = m(4n-2) - 2m(i-1) - (j - 1) \)
\( \lambda(v_j^j_u_i^j) = m(4n-3) - 2m(i-1) - (j - 1) \).
We can easily calculate the value of \( a \) of subcycle such that 
\[ a = C_{4n}^{(i-j)} = \lambda(u_i^j) + \lambda(u_i^j) + \lambda(v_j^j) + \lambda(v_j^j) + \lambda(v_j^j) = 20mn - 7m + 4 \]
Hence \( mTL_n \) is \( C_4 \)-antimagic.

Theorem. The graph \( G \cong mTL_n \) admits \( C_4 \)-anti-supermagic labeling for any \( m, n \).

Proof. Let \( v = |V(G)| \) and \( e = |E(G)| \). Then \( v = 2mn \), \( e = m(4n-3) \). We denote the vertex and edge sets of \( G \) as follows:
\[ V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_j^j : 1 \leq j \leq m\}, \]
\[ E(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j_u_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_j^j_u_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_j^j_u_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}. \]

Define a total labeling 
\[ \lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., 6mn-3m\} \] as follows:
For \( 1 \leq i \leq n, 1 \leq j \leq m \).
\[ \lambda(u_i^j) = 2m(i-1) + j \]
\[ \lambda(v_j^j) = m(2i-1) + j \]
\[ \lambda(u_i^j_v_j^j) = 2m(2n-1) + 1 + 2m(i-1) + (j - 1) \]
\[ \lambda(u_i^j_v_j^j) = 2m(2n-1) + 3 + 2m(i-1) + (j - 1) \]
\[ \lambda(u_i^j_u_i^j) = m(4n-2) - 2m(i-1) - (j - 1) \]
\[ \lambda(v_j^j_u_i^j) = m(4n-3) - 2m(i-1) - (j - 1) \]
It is easy to verify that for every subcycle 
\[ C_{4n}^{(i-j)} = \lambda(u_i^j) + \lambda(u_i^j) + \lambda(v_j^j) + \lambda(v_j^j) + \lambda(v_j^j) = 16mn - m + 6 \]
Hence \( mTL_n \) is \( C_4 \)-(anti)-supermagic.

3. Conclusion and open problems
In this paper we studied the problem that if a ladder graph \( G \) has a cycle anti-supermagic labeling then either disjoint union of isomorphic copies of \( G \) will have a cycle anti-supermagic labeling or not? We have studied the case for the disjoint union of isomorphic copies for \( m \geq 2 \) of ladders and triangular ladders graphs. We also proved that disjoint union of isomorphic copies of triangular ladders are cycle-supermagic.

We have not found a \( C_4 \)-anti-supermagic labeling for disjoint union of non-isomorphic copies of ladder and triangular ladder yet. Therefore, we propose the following open problem.

For any even integer \( n \geq 2 \) and every \( m, k \geq 1 \), determine whether there is \( C_4 \)-anti-supermagic labeling for 
\[ G \cong mL_n \cup kTL_{n+1} \]
For any even integer \( n \geq 2 \) and every \( m, k \geq 1 \), determine whether there is \( C_4 \)-anti-supermagic labeling for 
\[ G \cong mL_n \cup kTL_{n+1} \]

REFERENCES


