ON CYCLE ANTI-SUPERMAGIC LABELLING OF ISOMORPHIC COPIES OF LADDER AND TRIANGULAR LADDER

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ABSTRACT. A graph $G(V,E)$ *has an* H*-covering if every edge in* E *belongs to a subgraph of* G *isomorphic to* H*. Suppose* G *admits an* H*-covering. An* H*-magic labeling is a total labeling* λ *from* $V(G)\cup E(G)$ *onto the integers* $\{1,2,\cdots,|V(G)\cup E(G)|\}$ *with the property that, for every subgraph* A *of* G *isomorphic to* H *there is a positive integer* c *such that* $\sum A=\sum_{v\in V(A)}\lambda(v)+\sum_{e\in E(A)}\lambda(e)=c$*. A graph that admits such a labeling is called* H-magic. *In addition, if* $\{\lambda(v)\}_{v\varepsilon V}=\{1,\cdots,|V|\}$ *, then the graph is called* H *-supermagic. Moreover a graph is said to be* H-(a,d) *antimagic if the magic constant for an arithmetic progression with initial value* a *and common difference* d*. In this paper we formulate cycle anti-supermagic labeling for the disjoint union of isomorphic copies of ladder and triangular ladder graphs.*

Key Words: H -anti-supermagic labeling, cycles graphs, disjoint union of graphs.

1. INTRODUCTION

Let $G=(V,E)$ be a finite, simple, planar, connected and undirected graph, where V and E are its vertex-set and edge-set respectively. Let H and $G=(V,E)$ be finite simple graphs with the property that every edge of G belongs to at least one subgraph isomorphic to H. A bijection $\lambda : V\cup E\rightarrow\{1,...,|V|+|E|\}$ is an H-magic labeling of G if there exist a positive integer c (called the magic constant), such that for any subgraph $H'(V',E')$ of G isomorphic to H, the sum $\sum_{v\in V'}\lambda(v)+\sum_{e\in E'}\lambda(e)$ is equal to magic constant c. This sum is also known as weight of H. A graph is H-magic if it admits an H-magic labeling. In addition, if H-magic labeling λ has the property that $\{\lambda(v)\}_{v\varepsilon V}=\{1,\cdots,|V|\}$, then the graph is H-supermagic. In our terminology, super edge-magic total labeling is a K_2 -supermagic labeling. The notion of H-magic graph was introduced by A.Gutierrez and A. Llado [11], as an extension of the magic valuation given by Rosa [15], which corresponds to the case $H\cong K_2$. There are many results for H -supermagic graphs. Llado and Moragas [16] studied cycle-magic labeling of wheels, prisms, books, wind mill graphs and subdivided wheels by using the technique of partitioning sets of integers. Ngurah et al. [20] constructed cycle-supermagic labelings for some families of graphs namely fans, ladders, and books etc. For any connected graph G, Maryati et al. [17], proved that disjoint union of k-isomorphic copies of G is a G-supermagic graph if and only if $|V(G)|+|E(G)|$ is even or k is odd. For a disconnected graph G, having at least two vertices in its every component, the result was proved again by Maryati et al. [19]. In [22] Rizvi et al. discussed H-supermagic labeling of the disconnected graphs. They formulated cycle-supermagic labelings for the disjoint union of isomorphic copies of different families of graphs. They also proved that disjoint union of non-isomorphic copies of fans and ladders are cycle-supermagic. Hartsfeld and Ringel [12] introduced antimagic graphs in 1990. A graph with q edges is called antimagic if its edges can be labeled with $1,2,3,...,q$ such that the sums of the labels of the edges incident to each vertex are distinct. Among the graphs they prove are antimagic are: $P_n(n\geq 3)$, cycles and wheels. The concept of an (a,d)-antimagic labelings was introduced by Bodendiek and Walther [6] in 1993. A connected graph $G=(V,E)$ is said to be (a,d)-antimagic if there exist positive integers a,d and a bijection $f : E\rightarrow 1,2,3,...,|E|$ such that the induced mapping $g_f : V\rightarrow N$, defined by $g_f(v)=\sum_{uv\varepsilon E(G)}f(uv)$, is injective and $g_f(V)=a,a+d,...,a+(|V|-1)d$. In [5] Baca and Hollander proved that necessary conditions for $C_n\times P_2$ to be $(a,\ d)$-antimagic. Bodendiek and Walther [7] proved that the following graphs are not $(a,\ d)$-antimagic: even cycles; paths of even order; stars; trees of odd order. Javaid et al. [13] and Javaid et al. [14] constructed super $(a,\ d)$-edge-antimagic total labeling for w-trees and extended w-trees as well as super $(a,\ d)$-edge-antimagic total labeling for disjoint union of isomorphic and non-isomorphic copies of extended w-trees. Baca [4] gave antimagic labeling for the antiprism.

In this paper, we formulate cycle anti-supermagic labeling for the disjoint union of isomorphic copies of ladder graphs for different value of d. Also we prove cycle supermagic labeling for the disjoint union of isomorphic copies of triangular ladder graphs as well as cycle anti-supermagic labeling for the disjoint union of isomorphic copies of triangular ladder graphs for different value of d.

2. MAIN RESULTS

In the next two theorems, we prove that disjoint union of isomorphic copies of ladder graphs admit cycle anti-supermagic labeling with different values of d.

Theorem. *For every $m \geq 1$, $n \geq 2$, the graph $G \cong mL_n$ is C_4-anti-supermagic with distance $d = 2$ and* $a = 18mn - 5m + 5$.

Proof.

Let $v = |V(G)| = 2mn,$ and $e = |E(G)| = m(3n-2)$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{u_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\};$$

$$E(G) = \{u_i^j v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$

Define a total labeling

$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(5n-2)\}$ as follows:

For $1 \leq i \leq n, 1 \leq i \leq m.$

$$\lambda(u_i^j) = 2m(i-1) + j$$

$$\lambda(v_i^j) = m(2i-1) + j$$

$$\lambda(u_i^j v_i^j) = v + e - m(i-1) - (j-1),$$

where $v = 2mn$ and $e = m(3n-2)$

$$\lambda(u_i^j u_{i+1}^j) = v + e - mn - 2m(i-1) - (j-1)$$

$$\lambda(v_i^j v_{i+1}^j) = v + e - mn - 2mi + j.$$

We can easily calculate the value of a of sub cycle such that

$$a = \sum C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_i^j) + \lambda(v_{i+1}^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(v_i^j v_{i+1}^j) + \lambda(u_{i+1}^j v_{i+1}^j) = 18mn - 5m + 5.$$

Hence mL_n is C_4-anti-supermagic with distance $d = 2$.

Theorem. *For every $m \geq 1$, $n \geq 2$, the graph $G \cong mL_n$ is C_4-anti-supermagic with distance $d = 6$ and* $a = 16mn - 3m + 7.$

Proof.

Let $v = |V(G)| = 2mn,$ and $e = |E(G)| = m(3n-2)$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{u_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\};$$

$$E(G) = \{u_i^j v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$

Define a total labeling

$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(5n-2)\}$ as follows:

For $1 \leq i \leq n, 1 \leq i \leq m.$

$$\lambda(u_i^j) = 2m(i-1) + j$$

$$\lambda(v_i^j) = m(2i-1) + j$$

$$\lambda(u_i^j v_i^j) = v + e - m(i-1) - (j-1),$$

where $v = 2mn$ and $e = m(3n-2)$

$$\lambda(u_i^j u_{i+1}^j) = v + e - 2mn + m + i + (n-1)(j-1)$$

$$\lambda(v_i^j v_{i+1}^j) = v + e - 2mn - (m-2) - (i-1)(n-1)(j-1).$$

We can easily calculate the value of a of sub cycle such that

$$a = \sum C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_i^j) + \lambda(v_{i+1}^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(v_i^j v_{i+1}^j) + \lambda(u_{i+1}^j v_{i+1}^j) = 16mn - 3m + 7.$$

Hence mL_n is C_4-anti-supermagic with distance $d = 6$.

In the next two theorems, we prove that disjoint union of isomorphic copies of triangular ladder graphs, admit cycle supermagic labeling and cycle anti-supermagic labeling with different values of d.

Theorem. *The graph $G \cong mTL_n$ admits C_4 -supermagic labeling for any $m, n.$*

Proof.

Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = 2mn$, $e = m(4n-3)$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{u_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\},$$

$$E(G) = \{u_i^j v_i^j \ : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_{i+1}^j v_i^j \ : 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$

Define a total labeling

$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., 6mn - 3m\}$ as follows:

For $1 \leq i \leq n, 1 \leq j \leq m.$ $1 \leq i \leq n, 1 \leq j \leq m$

$\lambda(u_i^j) = 2m(i-1) + j$

$\lambda(v_i^j) = m(2i-1) + j$

$\lambda(u_i^j v_i^j) = m(6n-3) - 2m(i-1) - (j-1)$

$\lambda(u_{i+1}^j v_i^j) = m(6n-4) - 2m(i-2) - (j-1)$

$\lambda(u_i^j u_{i+1}^j) = m(4n-2) - 2m(i-1) - (j-1)$

$\lambda(v_i^j v_{i+1}^j) = m(4n-3) - 2m(i-1) - (j-1)$.

We can easily calculate the value of a of sub cycle such that

$$a = C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_i^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(u_{i+1}^j v_i^j) = 20mn - 7m + 4.$$

Hence mTL_n is C_4-supermagic.

Theorem. *The graph* $G \cong mTL_n$ *admits* C_4*-anti-supermagic labeling for any* $m, n.$

Proof.

Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = 2mn$, $e = m(4n-3)$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{u_i^j \ : 1 \le i \le n, 1 \le j \le m\} \cup \{v_i^j \ : 1 \le i \le n, 1 \le j \le m\},$$

$$E(G) = \{u_i^j v_i^j \ : 1 \le i \le n, 1 \le j \le m\} \cup \{u_i^j u_{i+1}^j \ : 1 \le i \le n-1, 1 \le j \le m\} \cup \{v_i^j v_{i+1}^j \ : 1 \le i \le n-1, 1 \le j \le m\} \cup \{u_{i+1}^j v_i^j \ : 1 \le i \le n-1, 1 \le j \le m\}.$$

Define a total labeling

$\lambda \ : V(G) \cup E(G) \to \{1, 2, ..., 6mn - 3m\}$ as follows:

For $1 \le i \le n, 1 \le j \le m.$

$\lambda(u_i^j) = 2m(i-1) + j$

$\lambda(v_i^j) = m(2i-1) + j$

$\lambda(u_i^j v_i^j) = 2m(2n-1) + 1 + 2m(i-1) + (j-1)$

$\lambda(u_{i+1}^j v_i^j) = 2m(2n-1) + 3 + 2m(i-1) + (j-1)$

$\lambda(u_i^j u_{i+1}^j) = m(4n-2) - 2m(i-1) - (j-1)$

$\lambda(v_i^j v_{i+1}^j) = m(4n-3) - 2m(i-1) - (j-1)$

It is easy to verify that for every sub cycle

$$C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_i^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(u_{i+1}^j v_i^j) = 16mn - m + 6.$$

Hence mTL_n is C_4-(anti)-supermagic.

3. Conclusion and open problems

In this paper we studied the problem that if a ladder graph G has a cycle anti-supermagic labeling then either disjoint union of isomorphic copies of G will have a cycle anti-supermagic labeling or not? We have studied the case for the disjoint union of isomorphic copies for $m \ge 2$ of ladders and triangular ladders graphs. We also proved that disjoint union of isomorphic copies of triangular ladders are cycle-supermagic.

We have not found a C_4 -anti-supermagic labeling for disjoint union of non-isomorphic copies of ladder and triangular ladder yet. Therefore, we propose the following open problem.

For any even integer $n \ge 2$ and every $m, k \ge 1$, determine whether there is C_4-anti-supermagic labeling for $G \cong mL_n \cup kL_{n+1}$.

For any even integer $n \ge 2$ and every $m, k \ge 1$, determine whether there is C_4-anti-supermagic labeling for $G \cong mTL_n \cup kTL_{n+1}$.

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