ANTIMAGIC TOTAL LABELING OF SUBDIVISION OF CATERPILLAR

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ABSTRACT. *Let $G=(V(G),E(G))$ be a graph with $v=|V(G)|$ vertices and $e=|E(G)|$ edges. An (a,d)- edge antimagic total (EAT) labeling is a bijective function λ from $V(G)\cup E(G)$ to the set of consecutive positive integers $\{1,2,\ldots,v+e\}$ such that the weights of the edge $\{w(xy): xy\in E(G)\}$ form an arithmetic sequence starting with first term a and having common difference d, where $w(xy)=\lambda(x)+\lambda(y)+\lambda(xy)$. And, if $\lambda(V)=\{1,2,...,v\}$ then G is super (a,d)-edge antimagic total graph. In this paper we formulate super edge antimagic total labeling of subdivion of caterpillar for different values of the parameter d.*

Key Words : super (a,d)-edge antimagic total labeling, caterpillar, subdivision of caterpillar.
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1 INTRODUCTION

All graphs in this paper are finite, simple and undirected. The graph G has the vertex-set $V(G)$ and edge-set $E(G)$. A general reference for graph-theoretic ideas can be seen in [20]. A *labeling* (or *valuation*) of a graph is a mapping that carries graph elements to numbers (usually to positive or non-negative integers). In this paper the domain will be the set of all vertices and edges and such a labeling is called *a total labeling*. Some labelings use the vertex-set only, or the edge-set only, and we shall call them *vertex-labelings* and *edge-labelings* respectively. A graph G is called (a,d)-edge antimagic total ((a,d)-EAT) if there exist integers $a>0$, $d\geq 0$ and a bijectve function $\lambda: V(G)\cup E(G)\rightarrow\{1,2,...,v+e\}$ such that $W=\{w(xy): xy\in E(G)\}$ forms an arithmetic sequence starting from a with common difference d, where $w(xy)=\lambda(x)+\lambda(y)+\lambda(xy)$. W is called the set of edge-weights of the graph G. And, if $\lambda(V(G))=\{1,2,...,v\}$ then G is super (a,d)-edge antimagic total. A number of classification studies on edge antimagic total graphs has been intensively investigated. For further detail see a book [4] on antimagic labeling and recent survey [12] of graph labelings. The subject of edge-magic total labeling of graphs has its origin in the work of Kotzig and Rosa [15, 16], on what they called magic valuations of graphs. The notion of super edge-magic total labeling was introduced by Enomoto et al. in [7] and they proposed following conjecture:

Conjecture 1 *Every tree admits a super edge-magic total labeling.*

In the effort of attacking this conjecture, many authors have considered super edge-magic total labeling for some particular classes of trees for example [1, 3, 2, 5, 6, 11, 9, 13, 14, 18]. However, this conjecture still remains open. Lees and Shah [17] have verified this conjecture for trees on at most 17 vertices with a computer help. Kotzig and Rosa in [15] proved that every caterpillar is super edge-magic total. In [19] Sugeng et al. proved some results related to super (a,d)-edge antimagic total labeling of stars and caterpillars for different values of the parameter d. In [1] Baca et al. proved that disjoint union of caterpillars also admits super (a,d)-edge antimagic total labeling. In [3] Baca et al. proved that if a tree with order greater or equal to 2 is super (a,d)-edge antimagic total then d must be less or equal to 3. In this paper we formulate super edge antimagic total labeling on subdivided caterpillar for $d=\{0,1,2\}$.

Definition 1 *In a caterpillar, if we subdivide the end edges then the resulting graph is known as a subdivided caterpillar. It is denoted by* $G\cong\zeta(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n:n,l)$, *where* $\alpha_1=(m_{11},m_{12},m_{13},...,m_{1l})$, $\alpha_2=(m_{21},m_{22},m_{23},...,m_{2l})$,...., $\alpha_n=(m_{n1},m_{n2},m_{n3},...,m_{nl})$. *The set of vertices and edges are defined as*

$$V(G)=\{c_i:1\leq i\leq n\}\cup$$
$$\{a_{ir}^{p_{ir}}:1\leq i\leq n, 1\leq p_{ir}\leq m_{ir}, 1\leq r\leq l\}.$$

And

$$E(G)=\{c_ic_{i+1}:1\leq i\leq n-1\}\cup$$
$$\{a_{ir}^{p_{ir}}a_{ir}^{p_{ir}+1}:1\leq i\leq n, 1\leq p_{ir}\leq m_{ir}-1, 1\leq r\leq l\}$$
$$\cup\{a_{ir}^{1}c_i:1\leq i\leq n, 1\leq r\leq l\}.$$

If $\alpha_1=\alpha_2=...=\alpha_n$ then $G\cong\ \zeta(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n:n,l)$ is denoted by $G\cong\ \zeta(m_1,m_2,m_3,...,m_l:n,l)$ and if $m_1=,m_2=...=m_l=m$ then $\zeta(m_1,m_2,m_3,...,m_l:n,l)$ becomes $G\cong\ \zeta(m,n,l)$.

2 MAIN RESULTS

Before giving our main results, let us consider the following lemma found in [8] that gives a necessary and sufficient condition for a graph to be super edge-magic total.

Lemma 1. *A graph G with v vertices and e edges is super edge-magic total if and only if there exists a bijective function $f:V(G)\rightarrow\{1,2,\cdots,v\}$ such that the set $S=\{f(x)+f(y)\,|\,xy\in E(G)\}$ consists of e consecutive integers. In such a case, f extends to a super edge-magic total labeling of G with the magic constant $c=v+e+s$, where $s=min(S)$ and*

$$S\ =\{f(x)+f(y)\,|\,xy\in E(G)\}$$
$$\quad=\{c-(v+1),c-(v+2),\cdots,c-(v+e)\}.$$

Theorem 1 *For* $m\geq 3\,odd$, $n\geq 2$, $l=5$, $\alpha_1=(m+1,m,m-1,m,2m)$ *and* $\alpha_2=\alpha_3=...=\alpha_n=(2m,2m-1,m,m,2m)$,

$G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n : n, 5)$ *admits super* $(a,0)-$*edge antimagic total labeling with* $a = 2v + s - 1$ *and super* $(a,2)-$*edge antimagic total labeling with* $a = v + s + 1$,*where* $s = (3m+1) + (4m-1)\lfloor \frac{n}{2} \rfloor + (4m+1)(\lceil \frac{n}{2} \rceil - 1) + 2$ *and* $v = |V(G)|$.

Proof. If $v = |V(G)|$ and $e = |E(G)|$ then $v = 8mn - 2m + 1$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{a_{ir}^{p_{ir}} : 1 \le i \le n, a1 \le p_{ir} \le m_{ir}, 1 \le r \le 5\} \cup \{c_i : 1 \le i \le n\}$$

$$E(G) = \{c_i c_{i+1} : 1 \le i \le n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir}+1} : 1 \le i \le n, 1 \le p_{ir} \le m_{ir} - 1, a1 \le r \le 5\} \cup \{a_{ir}^1 c_i : 1 \le i \le n, 1 \le r \le 5\}.$$

Now we define the labeling $\lambda : V \to \{1, 2, ..., v\}$ as follows:

We shall cosider the expression as

$\alpha = 8m$ and

$$\eta = (3m+1) + (4m-1)\lfloor \frac{n}{2} \rfloor + (4m+1)(\lceil \frac{n}{2} \rceil - 1).$$

$$\lambda(c_i) = \begin{cases} \eta + m + 1 & \text{for } i = 1, \\ \eta + \frac{\alpha}{2}(i-3) + 9m + 1 & \text{for } i \ge 3 \text{ odd}, \\ \frac{\alpha}{2}(i-2) + 5m + 1 & \text{for } i = \text{even}. \end{cases}$$

When $i = 1$ and $1 \le r \le 5$:

for $p_{1r} = 1, 3, 5, ..., m_{1r}$;

$$\lambda(u) = \begin{cases} \frac{p_{11}+1}{2} & \text{for } u = a_{11}^{p_{11}}, \\ (m+2) - \frac{p_{12}+1}{2} & \text{for } u = a_{12}^{p_{12}}, \\ (m+1) + \frac{p_{13}+1}{2} & \text{for } u = a_{13}^{p_{13}}, \\ 2(m+1) - \frac{p_{14}+1}{2} & \text{for } u = a_{14}^{p_{14}}, \\ (3m+2) - \frac{p_{15}+1}{2} & \text{for } u = a_{15}^{p_{15}}, \end{cases}$$

and for $p_{1r} = 2, 4, 6, ..., m_{1r} - 1$;

$$\lambda(u) = \begin{cases} \eta + \frac{p_{11}}{2} & \text{for } u = a_{11}^{p_{11}}, \\ \eta + (m+1) - \frac{p_{12}}{2} & \text{for } u = a_{12}^{p_{12}}, \\ \eta + (m+1) + \frac{p_{13}}{2} & \text{for } u = a_{13}^{p_{13}}, \\ \eta + (2m+1) - \frac{p_{14}}{2} & \text{for } u = a_{14}^{p_{14}}, \\ \eta + (3m+1) - \frac{p_{15}}{2} & \text{for } u = a_{15}^{p_{15}}. \end{cases}$$

When i is even and $1 \le r \le 5$:

for $p_{ir} = 1, 3, 5, ..., m_{ir}$;

$$\lambda(u)=\begin{cases}\eta+\alpha(\frac{i-2}{2})+3m+\frac{p_{i1}+1}{2} & \text{for } u=a_{11}^{p_{i1}},\\ \eta+\alpha(\frac{i-2}{2})+(5m+1)-\frac{p_{i2}+1}{2} & \text{for } u=a_{12}^{p_{i2}},\\ \eta+\alpha(\frac{i-2}{2})+5m+\frac{p_{i3}+1}{2} & \textit{for } u=a_{13}^{p_{i3}},\\ \eta+\alpha(\frac{i-2}{2})+(6m+2)-\frac{p_{i4}+1}{2} & \textit{for } u=a_{14}^{p_{i4}},\\ \eta+\alpha(\frac{i-2}{2})+(7m+2)-\frac{p_{i5}+1}{2} & \textit{for } u=a_{15}^{p_{i5}},\end{cases}$$

and for $p_{ir}=2,4,6,...,m_{ir}-1$;

$$\lambda(u)=\begin{cases}\alpha(\frac{i-2}{2})+(3m+1)+\frac{p_{i1}}{2} & \text{for } u=a_{11}^{p_{i1}},\\ \alpha(\frac{i-2}{2})+(5m+1)-\frac{p_{i2}}{2} & \text{for } u=a_{12}^{p_{i2}},\\ \alpha(\frac{i-2}{2})+(5m+1)+\frac{p_{i3}}{2} & \text{for } u=a_{13}^{p_{i3}},\\ \alpha(\frac{i-2}{2})+(6m+1)-\frac{p_{i4}}{2} & \text{for } u=a_{14}^{p_{i4}},\\ \alpha(\frac{i-2}{2})+(7m+1)-\frac{p_{i5}}{2} & \text{for } u=a_{15}^{p_{i5}}.\end{cases}$$

When $i\geq 3$ odd and $1\leq r\leq 5$:
for $p_{ir}=1,3,5,...,m_{ir}$;

$$\lambda(u)=\begin{cases}\alpha(\frac{i-3}{2})+7m+\frac{p_{i1}+1}{2} & \text{for } u=a_{11}^{p_{i1}},\\ \alpha(\frac{i-3}{2})+(9m+1)-\frac{p_{i2}+1}{2} & \text{for } u=a_{12}^{p_{i2}},\\ \alpha(\frac{i-3}{2})+9m+\frac{p_{i3}+1}{2} & \text{for } u=a_{13}^{p_{i3}},\\ \alpha(\frac{i-3}{2})+(10m+2)-\frac{p_{i4}+1}{2} & \textit{for } u=a_{14}^{p_{i4}},\\ \alpha(\frac{i-3}{2})+(11m+2)-\frac{p_{i5}+1}{2} & \text{for } u=a_{15}^{p_{i5}},\end{cases}$$

and for $p_{ir}=2,4,6,...,m_{ir}-1$;

$$\lambda(u)=\begin{cases}\eta+\alpha(\frac{i-3}{2})+(7m+1)+\frac{p_{i1}}{2} & \text{for } u=a_{11}^{p_{i1}},\\ \eta+\alpha(\frac{i-3}{2})+(9m+1)-\frac{p_{i2}}{2} & \text{for } u=a_{12}^{p_{i2}},\\ \eta+\alpha(\frac{i-3}{2})+(9m+1)+\frac{p_{i3}}{2} & \text{for } u=a_{13}^{p_{i3}},\\ \eta+\alpha(\frac{i-3}{2})+(10m+1)-\frac{p_{i4}}{2} & \text{for } u=a_{14}^{p_{i4}},\\ \eta+\alpha(\frac{i-3}{2})+(11m+1)-\frac{p_{i5}}{2} & \text{for } u=a_{15}^{p_{i5}}.\end{cases}$$

The set of all edge-sums generated by the above labeling scheme forms a consecutive integer sequence $s=(\eta+1)+1,(\eta+1)+2,\cdots,(\eta+1)+e$. Therefore, by Lemma 1, λ can be extended to a super (a,0)-edge antimagic total labeling and we obtain the magic constant $a=2v+s-1=\eta+16mn-4m+3$. Similarly, λ can be extended to a super (a,2)-edge antimagic total labeling and we obtain the magic constant $a=v+1+s=\eta+8mn-2m+4$.

Theorem 2. *For* $m \geq 3$ *odd,* $n \geq 2,$ $l = 5,$ $\alpha_1 = (m+1, m, m-1, m, 2m)$ *and* $\alpha_2 = \alpha_3 = ... = \alpha_n = (2m, 2m-1, m, m, 2m),$ $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n : n, 5)$ *admits super* $(a,1)-$*edge antimagic total labeling with* $a = s + \frac{3}{2}v$ *if* v *is even, where*

$$s = (3m+1) + (4m-1)\lfloor \frac{n}{2} \rfloor + (4m+1)(\lceil \frac{n}{2} \rceil - 1) + 2$$ *and* $v = |V(G)|$.

Proof. If $v = |V(G)|$ and $e = |E(G)|$ then $v = 8mn - 2m + 1$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{a_{ir}^{p_{ir}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir}, 1 \leq r \leq 5\} \cup \{c_i : 1 \leq i \leq n\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup$$
$$\{a_{ir}^{p_{ir}} a_{ir}^{p_{ir}+1} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 5\}$$
$$\cup \{a_{ir}^{1} c_i : 1 \leq i \leq n, 1 \leq r \leq 5\}.$$

Now, we define the labeling $\lambda : V(G) \cup E(G) \to \{1, 2, ..., v+e\}$ as in Theorem 1. It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s = (\eta+1)+1, (\eta+1)+2, \cdots, (\eta+1)+e$, with common difference 1. We denote it by $A = \{a_i; 1 \leq i \leq e\}$. Now for G we complete the edge labeling λ for super $(a,1)$-edge antimagic total labeling with values in the arithmetic sequence $v+1$, $v+2$, $\cdots, v+e$ with common difference 1. Let us denote it by $B = \{b_j; 1 \leq j \leq e\}$. Define

$$C = \{a_{2i-1} + b_{e-i+1}; 1 \leq i \leq \frac{e+1}{2}\} \cup$$
$$\{a_{2j} + b_{\frac{e-1}{2}-j+1}; 1 \leq j \leq \frac{e+1}{2} - 1\}.$$

It is easy to see that C constitute an arithmetic sequence with $d = 1$ and $a = s + \frac{3}{2}v = \eta + 2 + \frac{3}{2}(8mn - 2m + 1)$.

Since all vertices receive the smallest labels so λ is a super $(a,1)$-edge antimagic total labeling.

Theorem 3. *For* $m \geq 3$ *,is odd* $n \geq 2,$ $l = 6,$ $\alpha_1 = (m+1, m, m-1, m, 2m, 4m)$ *and* $\alpha_2 = \alpha_3 = ... = \alpha_n = (4m, 4m-1, m, m, 2m, 4m),$ $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n : n, 6)$ *admits super* $(a,0)-$*edge antimagic total labeling with* $a = 2v + s - 1$ *and super* $(a,2)-$*edge antimagic total labeling with* $a = v + s + 1$ *, where*

$$s = (5m+1) + (8m-1)\lfloor \frac{n}{2} \rfloor + (8m+1)(\lceil \frac{n}{2} \rceil - 1) + 2$$ *and* $v = |V(G)|$.

Proof. If $v = |V(G)|$ and $e = |E(G)|$ then $v = 16mn - 6m + 1$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{a_{ir}^{p_{ir}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir}, 1 \leq r \leq 6\} \cup \{c_i : 1 \leq i \leq n\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup$$
$$\{a_{ir}^{p_{ir}} a_{ir}^{p_{ir}+1} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 6\}$$
$$\cup \{a_{ir}^{1} c_i : 1 \leq i \leq n, 1 \leq r \leq 6\}.$$

Now we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows: Throughout the labeling we will consider $\alpha = 16m$ and

$$\eta = (5m+1) + (8m-1)\lfloor \frac{n}{2} \rfloor + (8m+1)(\lceil \frac{n}{2} \rceil - 1).$$

$$\lambda(c_i) = \begin{cases} \eta + (m+1) & \text{for } i = 1, \\ \eta + \frac{\alpha}{2}(i-3) + (17m+1) & \text{for } i \geq 3 \text{ odd}, \\ \frac{\alpha}{2}(i-2) + (9m+1) & \text{for } i = \text{even}. \end{cases}$$

When $i = 1$ and $1 \leq r \leq 6$:
for $p_{1r} = 1, 3, 5, ..., m_{1r}$;

$$\lambda(u) = \begin{cases} \frac{p_{11}+1}{2} & \text{for } u = a_{11}^{p_{11}}, \\ (m+2) - \frac{p_{12}+1}{2} & \text{for } u = a_{12}^{p_{12}}, \\ (m+1) + \frac{p_{13}+1}{2} & \text{for } u = a_{13}^{p_{13}}, \\ 2(m+1) - \frac{p_{14}+1}{2} & \text{for } u = a_{14}^{p_{14}}, \\ (3m+2) - \frac{p_{15}+1}{2} & \text{for } u = a_{15}^{p_{15}}, \\ (5m+2) - \frac{p_{16}+1}{2} & \text{for } u = a_{16}^{p_{16}} \end{cases}$$

and for $p_{1r} = 2,4,6,...,m_{1r} - 1$;

$$\lambda(u) = \begin{cases} \eta + \frac{p_{11}}{2} & \text{for } u = a_{11}^{p_{11}}, \\ \eta + (m+1) - \frac{p_{12}}{2} & \text{for } u = a_{12}^{p_{12}}, \\ \eta + (m+1+)\frac{p_{13}}{2} & \text{for } u = a_{13}^{p_{13}}, \\ \eta + (2m+1) - \frac{p_{14}}{2} & \text{for } u = a_{14}^{p_{14}}, \\ \eta + (3m+1) - \frac{p_{15}}{2} & \text{for } u = a_{15}^{p_{15}}, \\ \eta + (5m+1) - \frac{p_{16}}{2} & \text{for } u = a_{16}^{p_{16}}. \end{cases}$$

When i is even and $1 \le r \le 6$:

for $p_{ir} = 1,3,5,...,m_{ir}$;

$$\lambda(u) = \begin{cases} \eta + \alpha(\frac{i-2}{2}) + 5m + \frac{p_{i1}+1}{2} & \text{for } u = a_{11}^{p_{i1}}, \\ \eta + \alpha(\frac{i-2}{2}) + (9m+1) - \frac{p_{i2}+1}{2} & \text{for } u = a_{12}^{p_{i2}}, \\ \eta + \alpha(\frac{i-2}{2}) + 9m + \frac{p_{i3}+1}{2} & \text{for } u = a_{13}^{p_{i3}}, \\ \eta + \alpha(\frac{i-2}{2}) + (10m+2) - \frac{p_{i4}+1}{2} & \text{for } u = a_{14}^{p_{i4}}, \\ \eta + \alpha(\frac{i-2}{2}) + (11m+2) - \frac{p_{i5}+1}{2} & \text{for } u = a_{15}^{p_{i5}}, \\ \eta + \alpha(\frac{i-2}{2}) + (13m+2) - \frac{p_{i6}+1}{2} & \text{for } u = a_{16}^{p_{i6}}, \end{cases}$$

and for $p_{ir} = 2,4,6,...,m_{ir} - 1$;

$$\lambda(u) = \begin{cases} \alpha(\frac{i-2}{2}) + (5m+1) + \frac{p_{i1}}{2} & \text{for } u = a_{11}^{p_{i1}}, \\ \alpha(\frac{i-2}{2}) + (9m+1) - \frac{p_{i2}}{2} & \text{for } u = a_{12}^{p_{i2}}, \\ \alpha(\frac{i-2}{2}) + (9m+1) + \frac{p_{i3}}{2} & \text{for } u = a_{13}^{p_{i3}}, \\ \alpha(\frac{i-2}{2}) + (10m+1) - \frac{p_{i4}}{2} & \text{for } u = a_{14}^{p_{i4}}, \\ \alpha(\frac{i-2}{2}) + (11m+1) - \frac{p_{i5}}{2} & \text{for } u = a_{15}^{p_{i5}}, \\ \alpha(\frac{i-2}{2}) + (13m+1 - \frac{p_{i6}}{2} & \text{for } u = a_{16}^{p_{i6}}. \end{cases}$$

When $i \ge 3$ odd $1 \le r \le 6$:

for $p_{ir} = 1,3,5,...,m_{ir}$;

$$\lambda(u) = \begin{cases} \alpha(\frac{i-3}{2}) + 13m + \frac{p_{i1}+1}{2} & \text{for } u = a_{11}^{p_{i1}}, \\ \alpha(\frac{i-3}{2}) + (17m+1) - \frac{p_{i2}+1}{2} & \text{for } u = a_{12}^{p_{i2}}, \\ \alpha(\frac{i-3}{2}) + 17m + \frac{p_{i3}+1}{2} & \text{for } u = a_{13}^{p_{i3}}, \\ \alpha(\frac{i-3}{2}) + (18m+2) - \frac{p_{i4}+1}{2} & \text{for } u = a_{14}^{p_{i4}}, \\ \alpha(\frac{i-3}{2}) + (19m+2) - \frac{p_{i5}+1}{2} & \text{for } u = a_{15}^{p_{i5}}, \\ \alpha(\frac{i-3}{2}) + (21m+2) - \frac{p_{i6}+1}{2} & \text{for } u = a_{16}^{p_{i6}}, \end{cases}$$

and for $p_{ir} = 2,4,6,...,m_{ir} - 1$;

$$\lambda(u)=\begin{cases}\eta+\alpha(\frac{i-3}{2})+(13m+1)+\frac{p_{i1}}{2} & \text{for } u=a_{11}^{p_{i1}},\\ \eta+\alpha(\frac{i-3}{2})+(17m+1)-\frac{p_{i2}}{2} & \text{for } u=a_{12}^{p_{i2}},\\ \eta+\alpha(\frac{i-3}{2})+(17m+1)+\frac{p_{i3}}{2} & \text{for } u=a_{13}^{p_{i3}},\\ \eta+\alpha(\frac{i-3}{2})+(18m+1)-\frac{p_{i4}}{2} & \text{for } u=a_{14}^{p_{i4}},\\ \eta+\alpha(\frac{i-3}{2})+(19m+1)-\frac{p_{i5}}{2} & \text{for } u=a_{15}^{p_{i5}},\\ \eta+\alpha(\frac{i-3}{2})+(21m+1)-\frac{p_{i6}}{2} & \text{for } u=a_{16}^{p_{i6}}.\end{cases}$$

The set of all edge-sums generated by the above labeling scheme forms a consecutive integer sequence $s=(\eta+1)+1,(\eta+1)+2,\cdots,(\eta+1)+e$. Therefore, by Lemma 1, λ can be extended to a super (a,0)-edge antimagic total labeling and we obtain the magic constant $a=2v+s-1=\eta+32mn-12m+3$. Similarly, λ can be extended to a super (a,2)-edge antimagic total labeling and we obtain the magic constant $a=v+1+s=\eta+16mn-6m+4$.

Theorem 4. *For* $m\geq 3$ *odd,* $n\geq 2$, $l=6$, $\alpha_1=(m+1,m,m-1,m,2m,4m)$ *and* $\alpha_2=\alpha_3=...=\alpha_n=(4m,4m-1,m,m,2m,4m)$, $G\cong\zeta(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n:n,6)$ *admits super* $(a,1)-$*edge antimagic total labeling with* $a=s+\frac{3}{2}v$ *if* v *is even, where* $s=(5m+1)+(8m-1)\lfloor\frac{n}{2}\rfloor+(8m+1)(\lceil\frac{n}{2}\rceil-1)+2$ *and* $v=|V(G)|$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then $v=16mn-6m-9n+6$ and $e=16mn-6m-9n+5$. We denote the vertex and edge sets of G as follows:

$$V(G)=\{a_{ir}^{p_{ir}}:1\leq i\leq n,1\leq p_{ir}\leq m_{ir},1\leq r\leq 6\}\cup\{c_i:1\leq i\leq n\}$$

$$E(G)=\{c_ic_{i+1}:1\leq i\leq n-1\}$$
$$\cup\{a_{ir}^{p_{ir}}a_{ir}^{p_{ir}+1}:1\leq i\leq n,1\leq p_{ir}\leq m_{ir}-1,1\leq r\leq 6\}$$
$$\cup\{a_{ir}^{1}c_i:1\leq i\leq n,1\leq r\leq 6\}.$$

Now, we define the labeling $\lambda:V(G)\cup E(G)\to\{1,2,...,v+e\}$ as in Theorem 3.

It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s=(\eta+1)+1,(\eta+1)+2,\cdots,(\eta+1)+e$, with common difference 1. We denote it by $A=\{a_i;1\leq i\leq e\}$. Now for G we complete the edge labeling λ for super $(a,1)$-edge antimagic total labeling with values in the arithmetic sequence $v+1$, $v+2$, $\cdots,v+e$ with common difference 1. Let us denote it by $B=\{b_j;1\leq j\leq e\}$. Define $C=\{a_{2i-1}+b_{e-i+1};1\leq i\leq\frac{e+1}{2}\}\cup\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\leq j\leq\frac{e+1}{2}-1\}$. It is easy to see that C constitute an arithmetic sequence with $d=1$ and $a=s+3\frac{v}{2}=\eta+2+\frac{3}{2}(16mn-6m+1)$.

Since all vertices receive the smallest labels so λ is a super $(a,1)$-edge antimagic total labeling.

Theorem 5. *For* $m\geq 3$ *odd,* $n\geq 2$, $l\geq 5$, $\alpha_1=(m+1,m,m-1,m,m_5,...,m_l)$ *and* $\alpha_2=\alpha_3=...=\alpha_n=(m_l,m_l-1,m,m,m_5,...,m_l)$, $G\cong\zeta(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n:n,l)$ *admits super* $(a,0)-$*edge antimagic total labeling with* $a=2v+s-1$ *and super* $(a,2)-$*edge antimagic total labeling with* $a=v+s+1$, *where* $s=(\sum_{p=5}^{l}[m2^{p-5}]+2m+1)+(\sum_{p=5}^{l}[m2^{p-5}]+m-1+m2^{l-4})\lfloor\frac{n}{2}\rfloor+(\sum_{p=5}^{l}[m2^{p-5}]+(m+1)+m2^{l-4})(\lceil\frac{n}{2}\rceil-1)+2$, $m_p=m2^{p-4}$ *for* $5\leq p\leq l$ *and* $v=|V(G)|$.

Proof. Let $v=|V(G)|$, $e=|E(G)|$ then $v=(2mn+2m+1)+m(n-1)2^{l-3}+n\sum_{p=5}^{l}[m2^{p-4}]$ and $e=v-1$. We denote the vertex and edge sets of G as follows:

$$V(G)=\{a_{ir}^{p_{ir}}:1\leq i\leq n,1\leq p_{ir}\leq m_{ir},1\leq r\leq l\}\cup\{c_i:1\leq i\leq n\}.$$

$$E(G)=\{c_ic_{i+1}:1\leq i\leq n-1\}\cup$$
$$\{a_{ir}^{p_{ir}}a_{ir}^{p_{ir}+1}:1\leq i\leq n,1\leq p_{ir}\leq m_{ir}-1,1\leq r\leq l\}$$
$$\cup\{a_{ir}^{1}c_i:1\leq i\leq n,a1\leq r\leq l\}.$$

Now we define the labeling $\lambda:V(G)\to\{1,2,...,v\}$ as follows: Throughout the labeling we will consider

$$a=\sum_{p=5}^{l}[m2^{p-5}]+2m+1,$$
$$b=\sum_{p=5}^{l}[m2^{p-5}]+m-1+m2^{l-4},$$
$$c=\sum_{p=5}^{l}[m2^{p-5}]+m+1+m2^{l-4},$$
$$d=\sum_{p=5}^{l}[m2^{p-5}]+2m,$$

$$\alpha=\sum_{p=5}^{l}[(m-1)2^{p-4}]+m2^{l-3}+2m,$$

$$\eta=a+b\lfloor\frac{n}{2}\rfloor+c(\lceil\frac{n}{2}\rceil-1).$$

$$\lambda(c_i)=\begin{cases}\eta+(m+1) & \text{for } i=1,\\ \eta+\alpha(\frac{i-3}{2})+m2^{l-4}+d+c & \text{for } i\geq 3 \text{ odd},\\ \alpha(\frac{i-2}{2})+m2^{l-4}+a & \text{for } i=even.\end{cases}$$

When $i=1$:
for $p_{1r}=1,3,5,...,m_{1r}$, where $r=1,2,3,4$ and $5\leq r\leq l$, we define

$$\lambda(u)=\begin{cases}\frac{p_{11}+1}{2} & \text{for } u=a_{11}^{p_{11}},\\ (m+2)-\frac{p_{12}+1}{2} & \text{for } u=a_{12}^{p_{12}},\\ (m+1)+\frac{p_{13}+1}{2} & \text{for } u=a_{13}^{p_{13}},\\ 2(m+1)-\frac{p_{14}+1}{2} & \text{for } u=a_{14}^{p_{14}},\end{cases}$$

$\lambda(a_{ir}^{p_{1r}})=2(m+1)+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{1r}+1}{2}$ respectively, and for $p_{1r}=2,4,...,m_{1r}-1$, where $r=1,2,3,4$ and $5\leq r\leq l$,

$$\lambda(u)=\begin{cases}\eta+\frac{p_{11}}{2} & \text{for } u=a_{11}^{p_{11}},\\ \eta+(m+1)-\frac{p_{12}}{2} & \text{for } u=a_{12}^{p_{12}},\\ \eta+(m+1)+\frac{p_{13}}{2} & \text{for } u=a_{13}^{p_{13}},\\ \eta+(2m+1)-\frac{p_{14}}{2} & \text{for } u=a_{14}^{p_{14}},\end{cases}$$

$$\lambda(a_{ir}^{p_{1r}})=\eta+(2m+1)+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{1r}}{2}$$

respectively.
When i is even:
for $p_{ir}=1,3,5,...,m_{ir}$, where $r=1,2,3,4$ and $5\leq r\leq l$,

$$\lambda(u)=\begin{cases}\eta+\alpha(\frac{i-2}{2})+d+\frac{p_{i1}+1}{2}. & \text{for } u=a_{11}^{p_{i1}},\\ \eta+\alpha(\frac{i-2}{2})+d+m2^{l-4}+1-\frac{p_{i2}+1}{2}. & \text{for } u=a_{12}^{p_{i2}},\\ \eta+\alpha(\frac{i-2}{2})+d+m2^{l-4}+\frac{p_{i3}+1}{2}. & \text{for } u=a_{13}^{p_{i3}},\\ \eta+\alpha(\frac{i-2}{2})+d+m+2+m2^{l-4}-\frac{p_{i4}+1}{2}. & \text{for } u=a_{14}^{p_{i4}},\end{cases}$$

$$\lambda(a_{ir}^{p_{ir}})=\eta+\alpha(\frac{i-2}{2})+\sum_{k=5}^{r}[m2^{k-5}]+m2^{l-4}+d+(m+2)-\frac{p_{ir}+1}{2}.$$

respectively,
and for $p_{ir}=2,4,6,...,m_{ir}-1$; where $r=1,2,3,4$ and $5\leq r\leq l$,

$$\lambda(u)=\begin{cases}a+\alpha(\frac{i-2}{2})+\frac{p_{i1}}{2} & for\ u=a_{11}^{p_{i1}},\\ c+\alpha(\frac{i-2}{2})+m-\frac{p_{i2}}{2} & for\ u=a_{12}^{p_{i2}},\\ c+\alpha(\frac{i-2}{2})+m+\frac{p_{i3}}{2} & for\ u=a_{13}^{p_{i3}},\\ c+\alpha(\frac{i-2}{2})+2m-\frac{p_{i4}}{2} & for\ u=a_{14}^{p_{i4}},\end{cases}$$

$$\lambda(a_{ir}^{p_{ir}}) = c+\alpha(\frac{i-2}{2})+2m+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{ir}}{2}$$

respectively,

When $i \geq 3$ odd

for a $p_{ir} = 1,3,5,...,m_{ir}$; where $r = 1,2,3,4$ and $5 \leq r \leq l$,

$$\lambda(u) = \begin{cases} (a+b)+\alpha(\frac{i-3}{2}) + \frac{p_{i1}+1}{2}. & \text{for } u = a_{11}^{p_{i1}}, \\ (a+b)+\alpha(\frac{i-3}{2})+ 1+m2^{l-4}-\frac{p_{i2}+1}{2}. & \text{for } u = a_{12}^{p_{i2}}, \\ (a+b)+\alpha(\frac{i-3}{2})+ m2^{l-4}+\frac{p_{i3}+1}{2}. & \text{for } u = a_{13}^{p_{i3}}, \\ (a+b)+\alpha(\frac{i-3}{2})+ (m+2)+m2^{l-4}-\frac{p_{i4}+1}{2}. & \text{for } u = a_{14}^{p_{i4}}, \end{cases}$$

$$\lambda(a_{ir}^{p_{ir}}) = (a+b)+\alpha(\frac{i-3}{2})+\sum_{k=5}^{r}[m2^{k-5}]+ m2^{l-4}+m+2-\frac{p_{ir}+1}{2}.$$ respectively,

and for $p_{ir} = 2,4,6,...,m_{ir}-1$; where $r = 1,2,3,$ and $5 \leq r \leq l$, we define

$$\lambda(u) = \begin{cases} (c+d)+\eta+\alpha(\frac{i-3}{2})+\frac{p_{i1}}{2}, & \text{for } u = a_{11}^{p_{i1}}, \\ (c+d)+\eta+\alpha(\frac{i-3}{2}) +m2^{l-4}+1-\frac{p_{i2}}{2}. & \text{for } u = a_{12}^{p_{i2}}, \\ (c+d)+\eta+\alpha(\frac{i-3}{2}) +m2^{l-4}+\frac{p_{i3}}{2}. & \text{for } u = a_{13}^{p_{i3}}, \\ (c+d)+\eta+\alpha(\frac{i-3}{2}) +m2^{l-4}+m-\frac{p_{i4}}{2}. & \text{for } u = a_{14}^{p_{i4}}, \end{cases}$$

$$\lambda(a_{ir}^{p_{ir}}) = (c+d)+\eta+\alpha(\frac{i-3}{2})+m2^{l-4}+m +\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{ir}}{2}.$$

respectively. The set of all edge-sums generated by the above labeling scheme forms a consecutive integer sequence $s = (\eta+1)+1, (\eta+1)+2, \cdots, (\eta+1)+e$. Therefore, by Lemma 1, λ can be extended to a super (a,0)-edge antimagic total labeling and we obtain the magic constant $a = 2v+s-1 = \eta+1+2(2mn+2m+1)+ m(n-1)2^{l-3}+n\sum_{p=5}^{l}[m2^{p-4}]$.

Similarly, λ can be extended to a super (a,2)-edge antimagic total labeling and we obtain the magic constant $a = v+1+s = \eta+3+(2mn+2m+1)+ m(n-1)2^{l-3}+n\sum_{p=5}^{l}[m2^{p-4}]$.

Theorem 6. *For* $m \geq 3$ *odd,* $n \geq 2$, $l \geq 5$, $\alpha_1 = (m+1, m, m-1, m, m_5, ..., m_l)$ *and* $\alpha_2 = \alpha_3 = ... = \alpha_n = (m_l, m_l-1, m, m, m_5, ..., m_l)$, $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n : n, l)$ *admits super* $(a,1)-$*edge antimagic total labeling with* $a = s+\frac{3}{2}v$ *if* v *is even, where* $s = (\sum_{p=5}^{l}[m2^{p-5}]+2m+1)+(\sum_{p=5}^{l}[m2^{p-5}]+ m-1+m2^{l-4})\lfloor\frac{n}{2}\rfloor+(\sum_{p=5}^{l}[m2^{p-5}]+ (m+1)+m2^{l-4})(\lceil\frac{n}{2}\rceil-1)+2.$, $m_p = m2^{p-4}$ *for* $5 \leq p \leq l$ *and* $v = |V(G)|$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then $v=(2mn+2m+1)+m(n-1)2^{l-3}+n\sum_{p=5}^{l}[m2^{p-4}]$ and $e=v-1$. We denote the vertex and edge sets of G as follows:

$V(G)=\{a_{ir}^{p_{ir}}:1\le i\le n,1\le p_{ir}\le m_{ir},1\le r\le l\}$
$\cup\{c_i:1\le i\le n\}.$

$E(G)=\{c_ic_{i+1}:1\le i\le n-1\}\cup$
$\{a_{ir}^{p_{ir}}a_{ir}^{p_{ir}+1}:1\le i\le n,1\le p_{ir}\le m_{ir}-1,1\le r\le l\}$
$\cup\{a_{ir}^{1}c_i:1\le i\le n,1\le r\le l\}.$

Now, we define the labeling $\lambda:V(G)\cup E(G)\to\{1,2,...,v+e\}$ as in Theorem 5:

It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s=(\eta+1)+1,(\eta+1)+2,\cdots,(\eta+1)+e$, with common difference 1. We denote it by $A=\{a_i;1\le i\le e\}$. Now for G we complete the edge labeling λ for super $(a,1)$-edge antimagic total labeling with values in the arithmetic sequence $v+1$, $v+2$, $\cdots,v+e$ with common difference 1. Let us denote it by $B=\{b_j;1\le j\le e\}$. Define $C=\{a_{2i-1}+b_{e-i+1};1\le i\le \frac{e+1}{2}\}\cup\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\le j\le \frac{e+1}{2}-1\}$. It is easy to see that C constitute an arithmetic sequence with $d=1$ and $a=s+\frac{3}{2}v=\eta+2+\frac{3}{2}((2mn+2m+1)+m(n-1)2^{l-3}$ $+n\sum_{p=5}^{l}[m2^{p-4}])$.

Since all vertices receive the smallest labels so λ is a super $(a,1)$-edge antimagic total labeling.

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