HARMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVE IN ABSOLUTE VALUE ARE PRE-INVEX

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ABSTRACT: In this work we aim to find new Hermite-Hadamard inequalities for the functions whose first derivative in absolute value are pre-invex functions, related to the difference between $\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx$ and $\frac{1}{2}\{f\left(\frac{4a+\eta(b,a)}{4}\right)+f\left(\frac{4a+3\eta(b,a)}{4}\right)\}$. Some interesting consequences of our results are observed as a special cases.

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1. INTRODUCTION

Recently many inequalities have been studied for convex functions, but Hermite-Hadamard inequality is the fundamental event due to its geometrical importance and applications in Mathematical tools. Hanson [6] introduced the invex functions which is the generalization of convex functions. Noor in [9] studied the basic properties of pre-invex function and its significance in Mathematics. The Hermite-Hadamard inequalities and their variant forms for convex function was also investigated by Wier and Mond [15]. Ben-Israel and Mond [7] introduced the pre-invex functions which is the special case of invexity. In present paper we investigate some new Hermite-Hadamard type inequalities for functions whose first derivative are pre-invex function.

2. PRELIMINARIES

Let $f : K \to \mathrm{R}$ and $\eta(.,.): K\times K \to \mathrm{R}$ be continuous functions, where $K \subset \mathrm{R}^n$ is a none empty closed set. We require the following well known concepts and results which are essential in our investigations.

Definition 1[10, 15]. Let $u \in K$. Then the set K is said to be invex at $u \in K$ with respect to $\eta(.,.)$, if

$$u+t\eta(v,u)\in K, \qquad \forall u,v \in K, t\in[0,1].$$

K is said to be invex set with respect to $\eta(.,.)$ if it is invex at every $u \in K.$ The invex set K is also called a η connected set.

Geometrically the above definition says that there is a path starting from the point u which is contained in K. The point v should not be one of the end points of the path in general, see [1] . This observation plays a key role in our study. If we require that v should be an end point of the path for every pair of points $u,v\in K$, then $\eta(v,u)=v-u$, and consequently invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v,u)=v-u$, but the converse is not necessarily true, see [15, 16] and the references therein. For the sake of simplicity, we assume that $K=[a,a+\eta(b,a)]$, unless otherwise specified.

Definition 2 [15]. The function f on the invex set K is said to be pre-invex with respect to η, if

$$f(u+t\eta(v,u)) \le (1-t)f(u)+tf(v), \forall u,v\in K, t\in[0,1].$$

Note that every convex function is pre-invex but the converse is not true. For example, the function $f(u)=-|u|$ is not a convex function, but it is a pre-invex function with respect to η, where

$$\eta(v,u)=\begin{cases} v-u, \text{ if } v\le 0, \quad u\le 0 \quad \text{or} \quad v\ge 0, u\ge 0, \\ \\ u-v, \qquad \text{otherwise.}\end{cases}$$

The concepts of the invex and pre-invex functions have played very important roles in the development of generalized convex programming.

Definition 3 [3, 14]. The well-known Hermite-Hadamard inequality for a convex function defined on the interval $[a,b]$ is given by

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Noor [11] proved the following Hermite-Hadamard inequality for the pre-invex functions.

Let $f : K=[a,a+\eta(b,a)]\to(0,\infty)$ be a pre-invex function on the interval of real numbers K^0 (interior of K) and $a,b\in K^0$ with $a<a+\eta(b,a)$. Then the following inequalities holds:

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\le \frac{1}{\eta(b,a)}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Recently M. A. Latif and S. S. Dragomir [8] proved the following inequalities for the functions whose first derivative are convex functions.

(i). Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$ where $a,b \in I$ with $a<b.$ If $|f'|$ is convex function then the following inequality hold:

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \frac{b-a}{96}\left[\begin{array}{c}|f'(a)|+4|f'(\frac{3a+b}{4})|+2|f'(\frac{a+b}{2})| \\ +4|f'(\frac{a+3b}{4})|+|f'(b)|\end{array}\right]$$

(ii) Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$ where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex for some $q>1$, then the following inequality hold:

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \left(\frac{1}{1+p}\right)^{\frac{1}{p}}(\frac{1}{2})^{\frac{1}{q}}\frac{b-a}{16}\times$$

$$\left\{\left(\left|f'\frac{3a+b}{4}\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|f'(\frac{a+b}{2})\right|^q+\left|f'\frac{3a+b}{4}\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\left|f'\frac{a+3b}{4}\right|^q+\left|f'(\frac{a+b}{2})\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\left|f'\frac{a+3b}{4}\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}$$

(iii) Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$ where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex for some $q>1$, then the following inequality hold:

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le (\frac{1}{2})(\frac{1}{3})^{\frac{1}{q}}\frac{b-a}{16}\times\left\{\left(|f'(a)|^q+2\left|f'\frac{3a+b}{4}\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|f(\frac{a+b}{2})\right|^q+2|f'(3a+b|^q\right)^{\frac{1}{q}}$$

$$+\left(\left|f'(\frac{a+b}{2})\right|^q+2\left|\frac{f'(a+3b}{4}\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(2\left|f'(\frac{a+3b}{4})\right|^q+|f'(b)|^{\frac{1}{q}}\right)^{\frac{1}{q}}\right\}$$

(iv) Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$ where $a,b \in I$ with $a<b$. If $|f'|^q$ is concave for some $q>1$, then the following inequality hold:

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\times\left(\frac{b-a}{16}\right)$$

$$\times\{|f(\tfrac{7a+b}{8})|+|f(\tfrac{5a+3b}{8})|+|f(\tfrac{3a+5b}{8})|+|f(\tfrac{a+7b}{8})|\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Motivated from the above works, we aim to find new Hermite-Hadamard inequalities for the functions whose first derivative in absolute value are pre-invex functions, related to the difference between $\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx$ and $\frac{1}{2}\{f\left(\frac{4a+\eta(b,a)}{4}\right)+f\left(\frac{4a+3\eta(b,a)}{4}\right)\}$.

3. MAIN RESULTS

Lemma 1: Let $f : R \to R$ be a differentiable function on I^0, the interior of I where $a.a+\eta(b,a)\in I$ with $a<a+\eta(b,a),$ then the following equality hold:

$$\frac{f\left(\frac{4a+\eta(b,a)}{4}\right)+f\left(\frac{4a+3\eta(b,a)}{4}\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx$$

$$=\frac{\eta(b,a)}{16}\int_0^1 tf'\left(t\frac{4a+\eta(b,a)}{4}+(1-t)a\right)dt$$

$$+\int_0^1 (t-1)f'(t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4})dt$$

$$+\int_0^1 tf'\left(t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4}\right)dt$$

$$+\int_0^1 (1-t)f'(t(a+\eta(b,a))+(1-t)\frac{2a+3\eta(b,a)}{4})dt$$

Proof: Using by parts formula and substitution $x = t\frac{4a+\eta(b,a)}{4}+(1-t)a$, we have

$$\frac{\eta(b,a)}{16}\int_0^1\left\{tf'(t\frac{4a+\eta(b,a)}{4}+(1-t)a)\right\}dt$$

$$=\frac{\eta(b,a)}{16}[4tft\frac{4a+\eta(b,a)}{4}\Big|_0^1$$

$$-\frac{4}{\eta(b,a)}\int_0^1 f(t\frac{4a+\eta(b,a)}{4}+(1-t)a)dt$$

$$=\frac{1}{4}f(\frac{4a+\eta(b,a)}{4})$$

$$-\frac{1}{\eta(b,a)}\int_a^{\frac{4a+\eta(b,a)}{4}} f(x)dx \qquad (1)$$

Similarly,

$$\frac{\eta(b,a)}{16}\int_0^1 (t-1)f'(t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4})dt$$

$$=\frac{1}{4}f(\frac{4a+\eta(b,a)}{4})-\frac{1}{\eta(b,a)}\int_{\frac{4a+\eta(b,a)}{4}}^{\frac{2a+\eta(b,a)}{2}} f(x)dx\,\frac{\eta(b,a)}{16}$$

$$\int_0^1 tf'(\frac{4a+\eta(b,a)}{4}+(1-t)\frac{2a+\eta(b,a)}{2})dt \qquad (2)$$

$$=\frac{1}{4}f(\frac{4a+3\eta(b,a)}{4})-\frac{1}{\eta(b,a)}\int_{\frac{2a+\eta(b,a)}{2}}^{\frac{4a+3\eta(b,a)}{4}} f(x)dx$$

$$\frac{\eta(b,a)}{16}\int_0^1 (1-t)f'(t(a+\eta(b,a))+(1-t)\frac{2a+3\eta(b,a)}{4})dt \qquad (3)$$

$$=\frac{1}{4}f(\frac{4a+3\eta(b,a)}{4})-\frac{1}{\eta(b,a)}\int_{\frac{4a+3\eta(b,a)}{4}}^{a+\eta(b,a)} f(x)dx \qquad (4)$$

adding (1) to (4). we get the desired equality.

Theorem 2: Let $f:R\to R$ be a differentiable function on I^0 such that $f'\in[a,a+\eta(b,a)]$ where $a,\ a+\eta(b,a)\in I$ with $a<a+\eta(b,a)$. If $|f'|$ is pre-invex function , then the following inequality hold:

$$\left|\frac{f(\frac{4a+\eta(b,a)}{4})+f(\frac{4a+3\eta(b,a)}{4})}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq\frac{\eta(b,a)}{96}\begin{bmatrix}|f'(a)|+4\left|f'(\frac{4a+\eta(b,a)}{4})\right|\\ +2\left|f'(\frac{2a+\eta(b,a)}{2})\right|\\ +4\left|f'(\frac{4a+3\eta(b,a)}{4})\right|\\ +|f'(a+\eta(b,a))|\end{bmatrix}$$

Proof: Using Lemma 1 and modulus, we have

$$\left|\frac{f\left(\frac{4a+\eta(b,a)}{4}\right)+f\left(\frac{4a+3\eta(b,a)}{4}\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq\frac{\eta(b,a)}{16}\left\{\int_0^1 t\left|f'(t\frac{4a+\eta(b,a)}{4}+(1-t)a\right|dt\right.$$

$$+\int_0^1(1-t)\left|f'(t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4}\right|dt$$

$$+\int_0^1 t\left|f\,t\frac{4a+3\eta(b,a)}{4}+(1-t)\frac{2a+\eta(b,a)}{2}\right|dt$$

$$\left.+\int_0^1(1-t)\left|f'(t(a+\eta(b,a)\eta(b,a)))+(1-t)\frac{4a+3\eta(b,a)}{4}\right|dt\right\}. \qquad (5)$$

Now using the definition of pre-invex of $|f'|$ on $[a,a+\eta(b,a)]$ we have,

$$\int_0^1 t\left|f'(t\frac{4a+\eta(b,f'a)}{4}+(1-t)a\right|dt$$

$$\leq\left|f'\frac{4a+\eta(b,a)}{4}\right|\int_0^1 t^2dt+|f'(a)|\int_0^1 t(1-t)dt$$

$$=\frac{1}{3}\left|f'\frac{4a+\eta(b,a)}{4}\right|+\frac{1}{6}|f'(a)|. \qquad (6)$$

In similar way the following inequalities holds:

$$\int_0^1 (1-t)\left| f t\frac{2a+\eta(b,a)}{2} + (1-t)\frac{4a+\eta(b,a)}{4}\right| dt$$

$$\le \frac{1}{6}\left| f'\frac{2a+\eta(b,a)}{2}\right| + \frac{1}{3}\left| f'\frac{4a+\eta(b,a)}{4}\right|$$

$$\int_0^1 t\left| f t\frac{4a+3\eta(b,a)}{4} + (1-t)\frac{2a+\eta(b,a)}{2}\right| dt \quad (7)$$

$$\le \frac{1}{3}\left| f'\frac{4a+3\eta(b,a)}{4}\right| + \frac{1}{6}\left| f'\frac{2a+\eta(b,a)}{2}\right|$$

$$\int_0^1 (1-t)\left| f t(a+\eta(b,a)\eta(b,a)) + (1-t)\frac{4a+3\eta(b,a)}{4}\right| dt \quad (8)$$

$$\le \frac{1}{6}\left| f'(a+\eta(b,a)\right| + \frac{1}{3}\left| f'\frac{(4a+3\eta(b,a)}{4}\right| \quad (9)$$

us ing inequalities (6) to (9) in (5), we get the desired inequality.

Theorem 3: Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a, a+\eta(b,a)]$ where $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $|f'|^q$ is pre-invex for some q ⚇1 , then the following inequality hold:

$$\left|\frac{f(\frac{4a+\eta(b,a)}{4}) + f(\frac{4a+3\eta(b,a)}{4})}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \left(\frac{1}{1+p}\right)^{\frac{1}{p}} (\frac{1}{2})^q \frac{\eta(b,a)}{16}$$

$$\times\left\{\left[\left| f'\left(\frac{4a+\eta(b,a)}{4}\right)\right|^q + |f'(a)|^q\right]^{\frac{1}{q}}\right.$$

$$+\left[\left| f'\left(\frac{(2a+\eta(b,a)}{2}\right)\right|^q + \left| f'\left(\frac{4a+\eta(b,a)}{4}\right)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\left| f'\left(\frac{4a+3\eta(b,a)}{4}\right)\right|^q \left| f'\left(\frac{4a+3\eta(b,a)}{4}\right)\right|^q\right.$$

$$\left.+\left| f'\left(\frac{(2a+\eta(b,a)}{2}\right)\right|^q\right]^{\frac{1}{q}}$$

$$\left.+\left[\left| f'\left(\frac{4a+3\eta(b,a)}{4}\right)\right|^q + |f'(a+\eta(b,a)|^q\right]^{\frac{1}{q}}\right\}$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Using Holder's integral inequality and Lemma 1 we have,

$$\left|\frac{f(\frac{4a+\eta(b,a)}{4}) + f(\frac{4a+3\eta(b,a)}{4)})}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{16}\left\{\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left| t\frac{4a+\eta(b,a)}{4} + (1-t)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 (1-t)^p\right)^{\frac{1}{p}}\left(\int_0^1\left| t\frac{2a+\eta(b,a)}{2} + (1-t)\frac{4a+\eta(b,a)}{4}\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left| t\frac{4a+3\eta(b,a)}{4} + (1-t)\frac{2a+\eta(b,a)}{2}\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 (1-t)^p\right)^{\frac{1}{p}}\left(\int_0^1\left| t(a+\eta(b,a) + (1-t)\frac{4a+3\eta(b,a)}{4}\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (10)$$

Since $|f'|^q$ is pre-invex, then

$$\int_0^1\left| t\frac{4a+\eta(b,a)}{4} + (1-t)\right|^q dt$$

$$\le \left| f'\frac{(4a+\eta(b,a)}{4}\right|^q \int_0^1 t dt$$

$$+|f'(a)|^q \int_0^1 (1-t)dt$$

$$= \frac{1}{2}\left| f'\frac{(4a+\eta(b,a))}{4}\right|^q + \frac{1}{2}|f'(a)|^q. \quad (11)$$

Similarly

$$\int_0^1\left| t\frac{2a+\eta(b,a)}{2} + (1-t)\frac{4a+\eta(b,a)}{4}\right|^q dt$$

$$\le \frac{1}{2}\left| f'\left(\frac{(2a+\eta(b,a)}{2}\right)\right|^q \frac{1}{2}\left| f'\left(\frac{(3a+\eta(b,a)}{4}\right)\right|^q$$

$$\int_0^1\left| t\frac{4a+3\eta(b,a)}{4} + (1-t)\frac{2a+\eta(b,a)}{2}\right|^q dt \quad (12)$$

$$\le \frac{1}{2}\left| f'\left(\frac{4a+3\eta(b,a)}{4}\right)\right|^q + \left| f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q$$

$$\int_0^1\left| t(a+\eta(b,a) + (1-t)\frac{4a+3\eta(b,a)}{4}\right|^q dt \quad (13)$$

$$\le \frac{1}{2}\left| f'\left(\frac{4a+\eta(b,a)}{4}\right)\right|^q + \frac{1}{2}\left| f'\left(\frac{a+\eta(b,a)}{2}\right)\right|^q \quad (14)$$

using inequalities (11) to (14) in (10) we get the desired inequality.

Theorem 4: Let $R \to R$ be a differentiable function on I^0 such that $f' \in L[a, a+\eta(b,a)]$ where $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $|f'|^q$ is pre-invex for some $q>1$, then the following inequality hold:

$$\left|\frac{f(\frac{4a+\eta(b,a}{4}) + f(\frac{4a+\eta(b,a)}{4})}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq (\frac{1}{2})(\frac{1}{3})^{\frac{1}{q}} \frac{\eta(b,a)}{16} \times \left\{ \left(|f'(a)|^q + 2\left|f'(\frac{4a+\eta(b,a)}{4})\right|^q \right)^{\frac{1}{q}} \right.$$

$$\left(\left|f(\frac{2a+\eta(b,a)}{2})\right|^q + 2|f'(4a+\eta(b,a)|^q \right)^{\frac{1}{q}}$$

$$\left(+ \left|f'(\frac{2a+\eta(b,a)}{2})\right|^q + 2\left|\frac{f'(4a+3\eta(b,a)}{4}\right|^q \right)^{\frac{1}{q}}$$

$$\left. + \left(2\left|f'(\frac{4a+\eta(b,a)}{4})\right|^q + |f'(a+\eta(b,a)|^{\frac{1}{q}} \right)^{\frac{1}{q}} \right\}$$

Proof: Suppose that $q \geq 1.$ Using power mean inequality and Lemma 1, we have

$$\left|\frac{f(\frac{4a+\eta(b,a}{4}) + f(\frac{4a+\eta(b,a)}{4})}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{16}\left\{ \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t \left| f'\left(t\frac{4a+\eta(b,a)}{4} + (1-t)a \right) \right|^q dt \right)^{\frac{1}{q}} \right.$$

$$+ \left(\int_0^1 (1-t) dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) \left| f'\left(t\frac{2a+\eta(b,a)}{2} + (1-t)\frac{4a+c}{4} \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t \left| f'\left(\frac{4a+3\eta(b,a)}{4} + (1-t)\frac{2a+\eta(b,a)}{2} \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$\left. + \left(\int_0^1 (1-t) dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) \left| f t(a+\eta(b,a)+(1-t)\frac{4a+3\eta(b,a)}{4} \right|^q dt \right)^{\frac{1}{q}} \right\}. \quad (15)$$

Since $|f'|^q$ is pre-invex on $[a, a+\eta(b,a)]$, therefore

$$\int_0^1 t \left| f t\frac{4a+\eta(b,a)}{4} + (1-t)a \right|^q dt$$

$$\leq \left| f'(\frac{4a+\eta(b,a)}{4}) \right|^q \int_0^1 t^2 dt + |f'(a)|^q \int_0^1 t(1-t)dt$$

$$= \frac{1}{3}\left| f'(\frac{4a+\eta(b,a)}{4}) \right|^q + \frac{1}{6}|f'(a)|^q. \qquad (16)$$

Similarly we have the following inequalities

$$\int_0^1 (1-t) \left| f'\left(t\frac{2a+\eta(b,a)}{2} + (1-t)\frac{4a+c}{4} \right) \right|^q dt$$

$$\leq \frac{1}{6}|f'(2a+\eta(b,a))|^q + \frac{1}{3}\left| f'\left(\frac{4a+\eta(b,a)}{4} \right) \right|^q \qquad (17)$$

$$\int_0^1 t \left| f'\left(\frac{4a+3\eta(b,a)}{4} + (1-t)\frac{2a+\eta(b,a)}{2} \right) \right|^q dt$$

$$\leq \frac{1}{3}\left| f'\left(\frac{4a+3\eta(b,a)}{4} \right) \right|^q + \frac{1}{6}|f'(a+\eta(b,a)|^q \qquad (18)$$

$$\int_0^1 (1-t) \left| f'\left(t(a+\eta(b,a)+(1-t)\frac{4a+3\eta(b,a)}{4} \right) \right|^q dt$$

$$\leq \frac{1}{3}|f'(4a+3\eta(b,a)|^q + \frac{1}{6}|f'(a+\eta(b,a)|^q \qquad (19)$$

using (16) to (19) in (15), we get the desired inequality.

Theorem 5: Let $f : R \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$ where $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $|f'|^q$ is preconcave for some $q>1$, then the following inequality hold:

$$\left|\frac{f\ \frac{4a+\eta(b,a)}{4} + f\ \frac{4a+3\eta(b,a)}{4}}{2} - \frac{1}{\eta(b,a)}\int_a^b f(x)dx\right|$$

$$\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{\eta(b,a)}{16}\right)$$

$$\left\{ \begin{array}{l} \left|f'(\frac{8a+\eta(b,a)}{8})\right| + \left|f'(\frac{8a+3\eta(b,a)}{8})\right| \\ + \left|f'(\frac{8a+5\eta(b,a)}{8})\right| + \left|f'(\frac{8a+7\eta(b,a)}{8})\right| \end{array} \right\}$$

Proof: Using the well-known Holder's inequality for $q>1$, Lemma 1 and $p=\frac{q}{q-1}$, we have

$$\left|\frac{f(\frac{4a+\eta(b,a)}{4})+f(\frac{4a+\eta(b,a)}{4})}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$

$$\leq\frac{\eta(b,a)}{16}\left\{\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|t\frac{4a+\eta(b,a)}{4}+(1-t)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1(1-t)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\int_0^1\left|t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4}\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|t\frac{4a+3\eta(b,a)}{4}+(1-t)\frac{2a+\eta(b,a)}{2}\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1(1-t)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\int_0^1\left|t(a+\eta(b,a)+(1-t)\frac{4a+3\eta(b,a)}{4}\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (20)$$

Since $|f'|^q$ is preconcave therefore

$$\int_0^1\left|t\frac{4a+\eta(b,a)}{4}+(1-t)\right|^q dt$$

$$\leq\left|f'\left(\frac{\frac{4a+\eta(b,a)}{4}+a}{2}\right)\right|^q=\left|f'(\frac{8a+\eta(b,a)}{8})\right|^q. \quad (21)$$

Similarly the following inequalities holds

$$\int_0^1\left|t\frac{2a+\eta(b,a)}{2}+(1-t)\frac{4a+\eta(b,a)}{4}\right|^q dt$$

$$\leq\left|f'\left(\frac{8a+7\eta(b,a)}{8}\right)\right|^q \quad (22)$$

$$\int_0^1\left|t\frac{4a+3\eta(b,a)}{4}+(1-t)\frac{2a+\eta(b,a)}{2}\right|^q dt$$

$$\leq\left|f'\left(\frac{8a+7\eta(b,a)}{8}\right)\right|^q \quad (23)$$

$$\int_0^1\left|t(a+\eta(b,a)+(1-t)\frac{4a+3\eta(b,a)}{4}\right|^q dt$$

$$\leq\left|f'\left(\frac{8a+7\eta(b,a)}{8}\right)\right|^q \quad (24)$$

using inequalities from (21) to (24) in (20), we get the desired inequality.

REFERENCES

1. T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473--1484.
2. T. Antczak, r-pre-invexity and r-invexity in mathematical programming, J. Comput. Math. Appl,. 50 (2005), 551-566.
3. S.S. Dragomir and C. E. M. Pearce, Selected topic on Hermite-Hardard inequalities and applications, Melbourne and Adelaide, December 2000.
4. S. S. Dragomir and S Fitzpatrick, The Hermite-Hadamard type inequalities for S convex function in the second seance, Demonst. Math., 32 (4) (1999) 687-696.
5. S. S. Dragomir and B. Mond, Integral inequilities of Hermite-Hadamard type for log-convex functions, Demonst. Math., 31 (1998), 354-364.
6. M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 80 (1981), 545--550.
7. B. Israel and B. Mond, what is invexity, J. Aus Math. Soc.(B), 28 (1986), 1-9.
8. M. A. Latif and S. S.Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications., Acta Uni. Matt. Belli series Math., 21(2013), 24-39.
9. M. A. Noor., Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463--475.
10. M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323--330.
11. M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2 (2007),126--131.
12. M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory , 2 (2007),126--131.
13. B.G. Pachpatte, A note on integral inequalities involving two log-convex functions, Math. Inequal. Appl., 7 (2004), 511--515.
14. B.G. Pachpatte, Mathematical Inequalities, North-Holland Library, Vol. 67, Elsevier Science, Amsterdam, Holland, 2005.
15. T. Weir and B. Mond, Pre-invex functions in multi objective optimization, J. Math. Anal. Appl., 136 (1988), 29--38.
16. X. M. Yang, X.Q. Yang and K.L. Teo, Generalized invexity and generalized invariant monotonicity, J. Optim. Theory Appl., 117 (2003), 607--625.