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TOTAL VERTEX IRREGULARITY STRENGTH OF GRID-LIKE PLANE GRAPHS

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ABSTRACT. *For a simple graph $G=(V,E)$ with vertex set V and edge set E ,a labeling $\phi: V\cup E\rightarrow\{1,2,\dots,k\}$ is called total $k-$labeling. The associated vertex weight of a vertex $x\in V(G)$ under a total $k-$labeling ϕ is defined as*

$$wt(x)=\phi(x)+\sum_{y\in N(x)}\phi(xy),$$

where $N(x)$ is the set of neighbors of x. A total k-labeling ϕ is defined to be a vertex irregular total labeling of a graph G if for every two different vertices x and y of G, $wt(x)\neq wt(y)$. The minimum k for which a graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of G, $tvs(G)$. In this paper, we study the total vertex irregularity strength for special classes of grid-like plane graphs.

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1 INTRODUCTION AND DEFINITIONS

As a standard notation, assume that $G=G(V,E)$ is a finite, simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex set or the edge set, the labelings are called respectively vertex labelings or edge labelings. If the domain is $V\cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of element. The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. In fact Bloom and Golomb studied applications of graph labelings to other branches of science and it is possible to find part of this work in [10] and [11].

Chartrand *et al*. in [13] introduced edge-labelings of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices in G. Such labelings were called irregular assignments. What is the minimum value of the largest label over all such irregular assignments? This parameter of a graph G is well known as the irregularity strength of the graph G, $s(G)$.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for simple graphs, see [12, 14, 15, 17, 18] and a complete survey [16].

Motivated by this research and by total labelings mentioned in a book of Wallis [23], Bača et al. in [8] recently defined a vertex irregular total labelings of graphs. For a simple graph $G=(V,E)$ with vertex set V and edge set E, a labeling $\phi: V\cup E\rightarrow\{1,2,\dots,k\}$ is called total $k-$*labeling*. The associated vertex weight of a vertex $x\in V(G)$ under a total $k-$labeling ϕ is defined as

$$wt(x)=\phi(x)+\sum_{y\in N(x)}\phi(xy),$$

where $N(x)$ is the set of neighbors of x. A total k-labeling ϕ is defined to be a vertex irregular total labeling of a graph G if for every two different vertices x and y of G,

$$wt(x)\neq wt(y).$$

The minimum k for which a graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of G, $tvs(G)$.

In this paper, we study properties of the vertex irregular total labelings and determine the exact values of the total vertex irregularity strength for four special classes of grid-like plane graphs.

2KNOWN RESULTS

The following theorem proved in [8], establish lower and upper bounds for the total vertex irregularity strength of a (p,q)-graph.

Theorem 1 [8] Let G be a (p,q)-graph with minimum degree δ and maximum degree Δ. Then

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \le tvs(G) \le p+\Delta-2\delta+1. \qquad (1)$$

If G is an r-regular (p,q)-graph then from Theorem 1 it follows:

$$\left\lceil \frac{p+r}{r+1} \right\rceil \le tvs(G) \le p-r+1.$$

For a regular hamiltonian (p,q)-graph G, it was showed in [8] that $tvs(G) \le \left\lceil \frac{p+2}{3} \right\rceil$. Thus for cycle C_p we have that $tvs(C_p) = \left\lceil \frac{p+2}{3} \right\rceil$.

Recently, a much stronger upper bound on the total vertex irregularity strength of graphs has been established in [5]. In [19, 20, 21], Nurdin et al. found the exact values of total vertex irregularity strength of trees, several types of trees and disjoint union of t copies of path. Whereas the total vertex irregularity strengths of cubic graphs, wheel related graphs, Jahangir graphs $J_{n,2}$, for $n \ge 4$, circulant graphs $C_n(1,2)$ for $n \ge 5$, and certain classes of unicyclic graphs have been determined by Ahmad et al. in [1, 3, 4]. Morerecently, the edge irregular total labelings for categorical product of two cycles(two paths) and for generalized prisms have been determined by Ahmad et al.. Wijaya et al. [24, 25] found the exact value of the total vertex irregularity strength of wheels, fans, suns, friendship, and complete bipartite graphs. Slamin et al. [22] determined the total vertex irregularity strength of disjoint union of sun graphs.

3 MAIN RESULTS

We start this section with the result on the total vertex irregularity strength of the plane graph $\mathbb{C}_n$ dened in [8] as below: Let P_1, P_2 and P_3 be paths on vertices $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ and $c_1, c_2, ..., c_n$,respectively.Form the graph $\mathbb{C}_n$ from the disjoint union $P_1 \cup P_2 \cup P_3$ by adjoining the edges $a_i b_{2i-1}$ and $b_{2i} c_i$ for $i = 1, 2, ..., n.$ (Fig. 1)

The vertex set and edge set of $\mathbb{C}_n$ are:

$$V(\mathbb{C}_n) = \{a_i; c_i : 1 \le i \le n\} \cup \{b_i : 1 \le i \le 2n\}$$
$$E(\mathbb{C}_n) = \{a_i a_{i+1}; c_i c_{i+1} : 1 \le i \le n-1\} \cup \{b_i b_{i+1} : 1 \le i \le 2n-1\}$$
$$\cup \{a_i b_{2i-1}; c_i b_{2i} : 1 \le i \le n\}$$

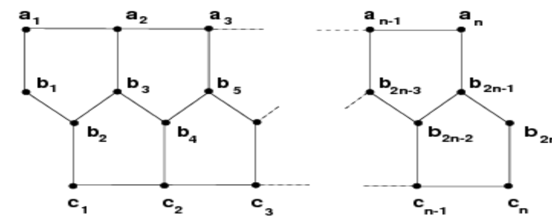

Figure 1: The plane graph $\mathbb{C}_n$

Theorem 2 Let $\mathbb{C}_n$ be the plane graph for $n \ge 2.$ Then $tvs(\mathbb{C}_n) = n+1.$

Proof. Let us consider the vertices of degree 2 and 3. There are $4n$ such vertices and if we want to use only labels $1, 2, \ldots, k$, the lowest and highest weights that we can obtain are respectively 3 and $4k$, what implies that $4k-3+1 \ge 4n$ and thus

$$tvs(\mathbb{C}_n) \ge \left\lceil \frac{4n+2}{4} \right\rceil = n+1.$$

To show that $tvs(\mathbb{C}_n) \le n+1$, we define a labeling $\phi : V(\mathbb{C}_n) \cup E(\mathbb{C}_n) \to \{1, 2, \ldots, n+1\}$ as follow:

$$\phi(a_i) = \begin{cases} 1, & if\ i = 1, n \\ 3, & if\ 2 \le i \le n-1. \end{cases}$$

$$\phi(b_i) = \begin{cases} 2, & if\ i = 1 \\ n, & if\ i = 2, 2n-1 \\ \left\lceil \frac{i-1}{2} \right\rceil, & if\ 3 \le i \le 2n-2 \\ 3, & if\ i = 2n \end{cases}$$

$$\phi(c_i) = \begin{cases} 2, & if\ i = 1 \\ 1, & if\ i = n \\ n-1, & if\ 2 \le i \le n-1 \end{cases}$$

$\phi(a_n b_{2n-1}) = 1; \phi(a_1 b_1) = \phi(b_{2n} c_n) = \phi(b_2 c_1) = 2;$

$\phi(b_1 b_2) = \phi(b_{2n-1} b_{2n}) = 3$

$\phi(a_i b_{2i-1}) = \phi(b_{2i} c_i) = i+2$ for $2 \le i \le n-1$

$\phi(b_i b_{i+1}) = n+1$ for $2 \le i \le 2n-2$

For $1 \le i \le n-1$

$\phi(a_i a_{i+1}) = 1; \phi(c_i c_{i+1}) = 2$

This labeling gives weight of the vertices as follows:

$$wt(a_i)=\begin{cases}3, & if\ i=n\\ 4, & if\ i=1\\ i+7, & if\ 2\le i\le n-1\end{cases}$$

$$wt(b_i)=\begin{cases}7, & if\ i=1\\ 8, & if\ i=2n\\ 2n+5, & if\ i=2n-1\\ 2n+4+i, & if\ 2\le i\le 2n-2\end{cases}$$

$$wt(c_i)=\begin{cases}5, & if\ i=n\\ 6, & if\ i=1\\ n+5+i, & if\ 2\le i\le n-1\end{cases}$$

It is easy to check that the weights of the vertices are distinct, that is $3,4,\ldots,4n+2$. This labeling construction shows that

$$tvs(C_n)\le n+1.$$

Combining with the lower bound, we conclude that

$$tvs(C_n)=n+1.$$

Which complete the proof.

For $n\ge 2$, A_n be the plane graph consisting of 3-sided faces, 4-sided faces, 5-sided faces and one external infinite face. The plane graph A_n is defined in [7] as shown in Figure 2.

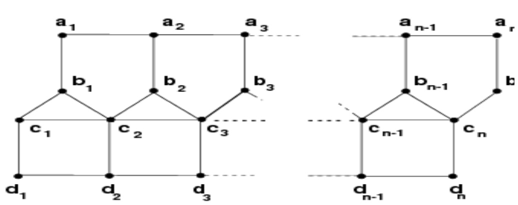

Figure 2: The plane graph A_n

Theorem 3 Let A_n be the plane graph for $n\ge 5$. Then

$$tvs(A_n)=\lceil\frac{3n+3}{4}\rceil.$$

Proof. Recall that the vertex set and the edge set of A_n are

$$V(A_n)=\{a_i;b_i;c_i;d_i:1\le i\le n\}$$

$$E(A_n)=\{a_ia_{i+1};c_ic_{i+1};d_id_{i+1};b_ic_{i+1}:1\le i\le n-1\}\cup\{a_ib_i;b_ic_i;c_id_i:1\le i\le n\}$$

Thus the plane graph A_n has 5 vertices of degree 2, $3n-4$ vertices of degree 3; 1 vertex of degree 4 and $n-2$ vertices of degree 5. The smallest weight of A_n is at least 3, so the largest weight of vertices of degree 2 is at least 7. Moreover, the largest weight of $3n-4,1,n-2$ vertices of degree 3,4,5 is at least $3n+3,3n+4,4n+2$, respectively. Consequently, the largest label of one of vertices or edges of A_n is at least $\max\{\lceil\frac{7}{3}\rceil,\lceil\frac{3n+3}{4}\rceil,\lceil\frac{3n+4}{5}\rceil,\lceil\frac{4n+2}{6}\rceil\}=\lceil\frac{3n+3}{4}\rceil$ for $n\ge 2$. Thus

$$tvs(A_n)\ge\lceil\frac{3n+3}{4}\rceil.$$

Let $k=\lceil\frac{3n+3}{4}\rceil$

We now prove the upper bound by providing labelling construction for A_n as follow:

$$\phi(a_i)=\begin{cases}1, & if\ i=1;n\\ 5, & if\ 2\le i\le k+1\\ i-k+4 & if\ k+2\le i\le n-1\end{cases}$$

$$\phi(b_i)=\begin{cases}n-k+3, & if\ i=1\\ 1, & if\ i=n\\ 2n+5-2k, & if\ 2\le i\le k+1\\ i+2n+4-3k & if\ k+2\le i\le n-1\end{cases}$$

$$\phi(c_i)=\begin{cases}2, & if\ i=1\\ 3n-3k+2, & if\ 2\le i\le n\end{cases}$$

$$\phi(d_i)=\begin{cases}3, & if\ i=1\\ n-k+4, & if\ 2\le i\le n-2\\ 2n+4-2k-\lceil\frac{n-1}{2}\rceil, & if\ i=n-1\\ 2n+5-2k, & if\ i=n\end{cases}$$

$$\phi(a_ib_i)=\begin{cases}1, & if\ i=1\\ 2, & if\ i=n\\ i-1, & if\ 2\le i\le k+1\\ k, & if\ k+2\le i\le n-1\end{cases}$$

$$\phi(b_ic_i)=\begin{cases}2, & if\ i=1;n\\ k, & if\ 2\le i\le n-1\end{cases}$$

$$\phi(c_i d_i) = \begin{cases} 2, & if\ i = 1 \\ k, & if\ 2 \le i \le n \end{cases}$$

$$\phi(c_i c_{i+1}) = \phi(d_i d_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil, & if\ 1 \le i \le n-2 \\ k, & if\ i = n-1 \end{cases}$$

For $1 \le i \le n-1$

$$\phi(a_i a_{i+1}) = 1; \phi(b_i c_{i+1}) = k$$

This labeling gives weight of the vertices as follows:

$$wt(a_i) = \begin{cases} 3, & if\ i = 1 \\ 4, & if\ i = n \\ 6+i, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(b_i) = \begin{cases} n+6, & if\ i = 1 \\ 5, & if\ i = n \\ 2n+4+i, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(c_i) = \begin{cases} 7, & if\ i = 1 \\ 3n+4, & if\ i = n \\ 3n+3+i, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(d_i) = \begin{cases} 6, & if\ i = 1 \\ n+5+i, & if\ 2 \le i \le n \end{cases}$$

It is easy to check that the weight of the vertices are different. This labelling construction shows that

$$tvs(\mathrm{A}_n) \le \lceil \frac{3n+3}{4} \rceil.$$

Combining with the lower bound, we conclude that

$$tvs(\mathrm{A}_n) = \lceil \frac{3n+3}{4} \rceil.$$

Which completes the proof.

In the next two theorems, we show that the lower bound of Theorem 1 is tight. In the sequel we shall investigate the total vertex irregularity strength of the plane graph B_n, defined in [7] which consists of 3-sided faces, 4-sided faces and one external infinite face (Figure 3).

Theorem 4 Let B_n be the plane graph for $n \ge 5.$ Then

$$tvs(\mathrm{B}_n) = \lceil \frac{2n+1}{3} \rceil.$$

Proof. Denote the vertex set and the edge set of B_n are

$$V(\mathrm{B}_n) = \{a_i; b_i; c_i; d_i : 1 \le i \le n\}$$

$$E(\mathrm{B}_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1} : 1 \le i \le n-1\}$$

$$\cup \{a_i b_i; b_i c_i; c_i d_i : 1 \le i \le n\} \cup \{b_i a_{i+1}; c_i d_{i+1} : 1 \le j \le n-1\}$$

Thus the plane graph B_n has $4n$ vertices and maximum degree is 5. By Theorem 1,

$$tvs(\mathrm{B}_n) \ge \lceil \frac{4n+2}{6} \rceil = \lceil \frac{2n+1}{3} \rceil \text{ for } n \ge 5.$$

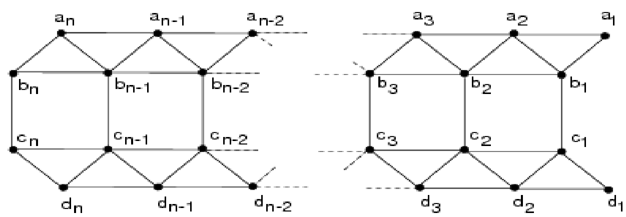

Figure 3: The plane graph B_n

Let $k = \lceil \frac{2n+1}{3} \rceil.$

We now prove the upper bound by providing labelling construction for B_n as follow:

$$\phi(a_i) = \phi(d_i) = \begin{cases} 1, & if\ i = 1 \\ 2, & if\ i = 2,3 \\ i - \lfloor \frac{i-1}{3} \rfloor, & if\ 4 \le i \le n-2 \\ k, & if\ i = n. \end{cases}$$

If $n \equiv 1,2 (\mathrm{mod}\, 3)$, then $\phi(a_{n-1}) = k-1$ and $\phi(d_{n-1}) = k-2.$

If $n \equiv 0 (\mathrm{mod}\, 3)$, then $\phi(a_{n-1}) = k-2$ and $\phi(d_{n-1}) = k-3.$

$$\phi(b_i) = \phi(c_i) = \begin{cases} 1, & if\ i = 1 \\ k, & if\ 2 \le i \le n-1. \end{cases}$$

Also $\phi(b_n) = k-1$ and $\phi(c_n) = k$.

$\phi(a_i a_{i+1}) = \phi(d_i d_{i+1}) = 1$ if $1 \le i \le n-2.$

If $n \equiv 0,1 (\mathrm{mod}\, 3)$, then $\phi(a_{n-1} a_n) = 2$ and $\phi(d_{n-1} d_n) = 3.$

If $n \equiv 2 (\mathrm{mod}\, 3)$, then $\phi(a_{n-1} a_n) = 1$ and $\phi(d_{n-1} d_n) = 2.$

$$\phi(b_ib_{i+1}) = \phi(c_ic_{i+1}) = \begin{cases} 1, & if\ i=1 \\ k, & if\ 2 \le i \le n-1 \end{cases}$$

$$\phi(a_ib_i) = \begin{cases} 1, & if\ i=1 \\ i-\lfloor \frac{i}{3} \rfloor, & if\ 2 \le i \le n-1 \\ k, & if\ i=n \end{cases}$$

$$\phi(a_ib_{i-1}) = \begin{cases} 1, & if\ i=2 \\ 3, & if\ i=3 \\ i-\lfloor \frac{i+1}{3} \rfloor, & if\ 4 \le i \le n-1 \\ k, & if\ i=n \end{cases}$$

$$\phi(b_ic_i) = \begin{cases} 1, & if\ i=1 \\ k, & if\ i=2 \\ i-\lfloor \frac{i-2}{3} \rfloor, & if\ 3 \le i \le n-1 \\ 5, & if\ i=n\ \ and\ n \equiv 0,1 \bmod 3 \\ 4, & if\ i=n\ \ and\ n \equiv 2 \bmod 3 \end{cases}$$

$$\phi(c_id_i) = \begin{cases} 2, & if\ i=1 \\ 3, & if\ i=2 \\ i-\lfloor \frac{i-3}{3} \rfloor, & if\ 3 \le i \le n-1 \\ k, & if\ i=n \end{cases}$$

$$\phi(c_id_{i+1}) = \begin{cases} 1, & if\ i=1 \\ 3, & if\ i=2 \\ i-\lfloor \frac{i-1}{3} \rfloor, & if\ 3 \le i \le n-2 \\ k, & if\ i=n-1 \end{cases}$$

This labeling gives weight of the vertices as follows:

$wt(a_1)=3,\ wt(b_1)=5,\ wt(c_1)=6\ and\ wt(d_1)=4$.

If $n \equiv 0 (\bmod 3)$

$$wt(a_i) = \begin{cases} 2i+3, & if\ 2 \le i \le n-1 \\ 2i+5, & if\ i=n \end{cases}$$

$$wt(b_i) = \begin{cases} 2n+2i+5, & if\ 2 \le i \le n-2 \\ 4(n+1) & if\ i=n-1 \\ 2n+7 & if\ i=n \end{cases}$$

$$wt(c_i) = \begin{cases} 2n+2i+6, & if\ 2 \le i \le n-2 \\ 4n+5 & if\ i=n-1 \\ 2n+8 & if\ i=n \end{cases}$$

$$wt(d_i) = \begin{cases} 2i+4, & if\ 2 \le i \le n-1 \\ 2n+6, & if\ i=n \end{cases}$$

If $n \equiv 1 (\bmod 3)$

$wt(a_i) = 2i+3, \text{if}\ 2 \le i \le n,$

$$wt(b_i) = \begin{cases} 2n+2i+3, & if\ 2 \le i \le n-1 \\ 2n+5 & if\ i=n \end{cases}$$

$$wt(c_i) = \begin{cases} 2n+2i+4, & if\ 2 \le i \le n-1 \\ 2n+6 & if\ i=n \end{cases}$$

$wt(d_i) = 2i+4, if\ 2 \le i \le n.$

If $n \equiv 2 (\bmod 3)$

$wt(a_i) = 2i+3, if\ 2 \le i \le n,$

$$wt(b_i) = \begin{cases} 2n+2i+4, & if\ 2 \le i \le n-2 \\ 4n+3 & if\ i=n-1 \\ 2n+5 & if\ i=n \end{cases}$$

$$wt(c_i) = \begin{cases} 2n+2i+5, & if\ 2 \le i \le n-2 \\ 4(n+1) & if\ i=n-1 \\ 2n+6 & if\ i=n \end{cases}$$

$wt(d_i) = 2i+4, \text{if}\ 2 \le i \le n.$

It is easy to check that the weight of the vertices are distinct. This labelling construction shows that

$$tvs(B_n) \le \lceil \frac{2n+1}{3} \rceil.$$

Combining with the lower bound, we conclude that

$$tvs(\mathrm{B}_n) = \lceil \frac{2n+1}{3} \rceil.$$

Which completes the proof.

In [6] M. Bača defined the plane graph D_n as below: Let P_1, P_2, P_3 and P_4 be paths on n vertices. Let a_i, b_i, c_i, d_i where $1 \le i \le n$ are the vertices of the paths P_1, P_2, P_3 and P_4 respectively. Form the graph D_n, from the disjoint union $P_1 \cup P_2 \cup P_3 \cup P_4$ by adjoining the edges a_ib_i, b_ic_i, c_id_i for $1 \le i \le n$ and c_ib_{i+1} for $i = 1, 2, \ldots, n-1$. (Figure 4).

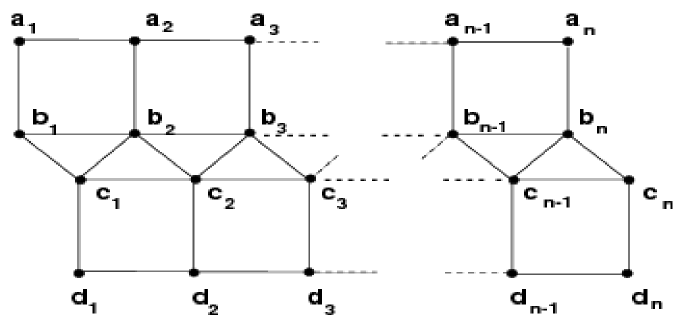

Figure 4: The plane graph D_n

Theorem 5 Let D_n be the plane graph for $n \ge 5$. Then $tvs(\mathrm{D}_n) = \lceil \frac{2n+1}{3} \rceil$.

Proof. Denote the vertex set and edge set of D_n are

$$V(\mathrm{D}_n) = \{a_i; b_i; c_i; d_i : 1 \le i \le n\}$$

$$E(\mathrm{D}_n) = \{a_ia_{i+1}; b_ib_{i+1}; c_ic_{i+1}; d_id_{i+1} : 1 \le i \le n-1\}$$
$$\cup \{a_ib_i; b_ic_i; c_id_i : 1 \le i \le n\} \cup \{c_ib_{i+1} : 1 \le i \le n-1\}$$

Thus the plane graph D_n has $4n$ vertices and maximum degree is 5. By Theorem 1 $tvs(\mathrm{D}_n) \ge \lceil \frac{4n+2}{6} \rceil = \lceil \frac{2n+1}{3} \rceil$ for $n \ge 5$. Let $k = \lceil \frac{2n+1}{3} \rceil$.

We now prove the upper bound by providing labelling construction for D_n as follow:

$$\phi(a_i) = \begin{cases} 1, & \text{if } 1 \le i \le n-2 \\ n - \lceil \frac{n}{2} \rceil, & \text{if } i = n-1 \\ 2, & \text{if } i = n \end{cases}$$

$$\phi(b_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2n+1-2k, & \text{if } 2 \le i \le n-2 \\ 3n+1-3k-\lceil \frac{n-1}{2} \rceil, & \text{if } i = n-1 \\ 2n+4-3k, & \text{if } i = n \end{cases}$$

$$\phi(d_i) = \begin{cases} n+3-k, & \text{if } 1 \le i \le n-2 \\ n+4+\lceil \frac{n-3}{2} \rceil - 2k, & \text{if } i = n-1 \\ 2n+1-2k, & \text{if } i = n \end{cases}$$

$$\phi(a_ia_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n-2 \\ 1, & \text{if } i = n-1 \end{cases}$$

$$\phi(a_ib_i) = \begin{cases} 1, & \text{if } i = 1 \\ 3, & \text{if } 2 \le i \le n-1 \\ 2, & \text{if } i = n \end{cases}$$

$$\phi(d_id_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n-3 \\ n-1-\lceil \frac{n-3}{2} \rceil, & \text{if } i = n-2 \\ k, & \text{if } i = n-1 \end{cases}$$

$$\phi(b_ib_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil, & \text{if } 1 \le i \le n-2 \\ k, & \text{if } i = n-1 \end{cases}$$

$$\phi(b_ic_i) = \begin{cases} 1, & \text{if } i = 1 \\ k, & \text{if } 2 \le i \le n \end{cases}$$

For $1 \le i \le n-1;\ \phi(c_ib_{i+1}) = k.$

For $1 \le i \le n;\ \phi(c_id_i) = k$

If $k \equiv 0 (\mathrm{mod}\, 2)$, then

$$\phi(c_ic_{i+1}) = \begin{cases} 3n+4-4k, & if\ 1 \le i \le k-1 even \\ k, & otherwise \end{cases}$$

$$\phi(c_i) = \begin{cases} 2n+4-3k, & if\ i = 1, n \\ i-1, & if\ 2 \le i \le k-1 \\ i+3n+3-5k, & if\ k \le i \le n-1 \end{cases}$$

If $k \equiv 1 (\bmod 2)$, then

$$\phi(c_ic_{i+1}) = \begin{cases} 3n+4-4k, & if\ 1 \le i \le k \text{ even} \\ k, & otherwise \end{cases}$$

$$\phi(c_i) = \begin{cases} 2n+4-3k, & if\ i = 1, n \\ i-1, & if\ 2 \le i \le k \\ i+3n+3-5k, & if\ k+1 \le i \le n-1 \end{cases}$$

This labeling gives weights of the vertices as follows:

$$wt(a_i) = \begin{cases} 3, & if\ i = 1 \\ 5, & if\ i = n \\ i+4, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(b_i) = \begin{cases} 4, & if\ i = 1 \\ 2n+6, & if\ i = n \\ 2n+5+i, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(c_i) = \begin{cases} 2n+5, & if\ i = 1 \\ 2n+4, & if\ i = n \\ 3n+3+i, & if\ 2 \le i \le n-1 \end{cases}$$

$$wt(d_i) = \begin{cases} n+3+i, & if\ 1 \le i \le n-3 \\ n+4+i, & if\ n-2 \le i \le n-1 \\ 2n+1, & if\ i = n \end{cases}$$

It is easy to check that the weight of the vertices are distinct, that is $3, 4, \ldots, 4n+2$. This labelling construction shows that

$$tvs(D_n) \le \lceil \frac{2n+1}{3} \rceil.$$

Combining with the lower bound, we conclude that

$$tvs(D_n) = \lceil \frac{2n+1}{3} \rceil.$$

Which completes the proof.

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