ON STOCHASTIC COMPARISONS OF PARALLEL AND SERIES SYSTEMS WITH HALF-LOGISTIC COMPONENTS

Ali Dolati[1], Mina Towhidi[2], Marzieh Shekari[*]
[1]Department of Statistics, Faculty of mathematics, University of Yazd, Yazd, Iran.
[2]Department of Statistics, Faculty of sciences, Shiraz University, Shiraz, Iran.
[*3]Department of statistics, Faculty of Sciences, Shiraz University, Shiraz, Iran.
Corresponding Author: shekari.su@hotmail.com

ABSTRACT:. *Let $Y_1,\dots,Y_n$ be independent random variables with Y_k having half-logistic distribution with the scale parameter λ_k, for $k=1,\dots,n$. Let $X_1,\dots,X_n$ be a random sample of size n from a half-logistic distribution with the common scale parameter $\lambda=\sum_{k=1}^{n}\lambda_k/n$. The purpose of this paper is to study stochastic comparisons including usual stochastic ordering, hazard rate ordering and the dispersive ordering between the smallest and largest order statistics from these two samples.*
MSC: Primary 62H05; Secondary 62H20.
Keywords:Half-logistic distribution, order statistics, stochastic ordering,hazard rate ordering,dispersive ordering.

INTRODUCTION

The random variable X has the half-logistic distribution if it has the cumulative distribution function

$$F(x)=\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}},\quad x>0, \tag{1}$$

for any arbitrary scale parameter λ. The probability density function corresponding to (1) is

$$f(x)=\frac{2\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2},\quad x>0. \tag{2}$$

The half-logistic distribution with the scale parameter λ will be denoted as $HL(\lambda)$. Use of this distribution as a possible life-time model has been suggested by Balakrishnan [1]. For more detail on this distribution we refer to the chapter 5 of the book by Johnson et. al [7].

Order statistics and statistics based on them play an important role in reliability and statistics. Let $X_{1:n}\le X_{2:n}\le\dots\le X_{n:n}$ denote the order statistics of independent and identically distributed (i.i.d.) random variables $X_1,X_2,\dots,X_n$ with common distribution function F. As is well known, the lifetimes of parallel and series systems are correspond to the $X_{n:n}$ and $X_{1:n}$, respectively. However, in practice, observations are independent but non-identically distributed. Its of general interest to study the impact of heterogeneity among components on the characteristics of a stochastic system. There is an extensive literature on this topic for the well-known distributions such as the exponential, geometric, gamma and Weibull model; see, e.g., [3-5, 8-11, 13-18]. However, there is no results on stochastic comparisons of order statistics from heterogenous half-logistic random variables. This paper, focuses on stochastic comparisons of the extreme order statistics when the underlying random variables follow the half-logistic family of distributions.

For ease of reference, let us first recall some stochastic orders from Shaked and Shanthikumar [15] which will be used in the sequel. Let X and Y be univariate random variables with the respective survival functions $\overline{F}=1-F$, $\overline{G}=1-G$ and the density functions f, g.

Definition 1 *The random variable X said to smaller than the random variable Y in the*

(i) usual stochastic order (denoted by $X\prec_{st}Y$) if $\overline{F}(x)\le\overline{G}(x)$ for all x;

(ii) hazard rate order (denoted by $X\prec_{hr}Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;

(iii) reverse hazard rate order (denoted by $X\prec_{rh}Y$) if $G(x)/F(x)$ is increasing in x;

(iv) likelihood ratio order (denoted by $X\prec_{lr}Y$) if $g(x)/f(x)$ is increasing in x;

(v) dispersive order (denoted by $X\prec_{Disp}Y$) if $F^{-1}(v)-F^{-1}(u)\le G^{-1}(v)-G^{-1}(u)$, for all $0<u<v<1$.

MAIN RESULTS

Let $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$ be two sets of mutually independent random variables. Assume that for $k\in\{1,\dots,n\}$, $Y_k\sim HL(\lambda_k)$ and $X_k\sim HL(\lambda)$, where $\lambda=\sum_{k=1}^{n}\lambda_k/n$. Let $F_{(n)}$ and $G_{(n)}$ denote the distribution functions of $X_{n:n}$ and $Y_{n:n}$ with the corresponding densities $f_{(n)}$ and $g_{(n)}$ and let $F_{(1)}$ and $G_{(1)}$ denote the distribution functions of $X_{1:n}$ and $Y_{1:n}$, with the corresponding densities $f_{(1)}$ and $g_{(1)}$. Then for $x>0$,

$$F_{(n)}(x)=\left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^n,\quad G_{(n)}(x)=\prod_{k=1}^{n}\frac{1-e^{-\lambda_k x}}{1+e^{-\lambda_k x}},$$

$$f_{(n)}(x)=F_{(n)}(x)\frac{2n\lambda e^{-\lambda x}}{1-e^{-2\lambda x}},\quad g_{(n)}(x)=G_{(n)}(x)\sum_{k=1}^{n}\frac{2\lambda_k e^{-\lambda_k x}}{1-e^{-2\lambda_k x}},$$

and

$$F_{(1)}(x)=1-\frac{2^n e^{-n\lambda x}}{(1+e^{-\lambda x})^n},\quad G_{(1)}(x)=1-\frac{2^n e^{-n\lambda x}}{\prod_{k=1}^{n}(1+e^{-\lambda_k x})},$$

$$f_{(1)}(x)=\bar{F}_{(1)}(x)\frac{n\lambda}{1+e^{-\lambda x}},\quad g_{(1)}(x)=\bar{G}_{(1)}(x)\sum_{k=1}^{n}\frac{\lambda_k}{1+e^{-\lambda_k x}}.$$

The following proposition provides a result for the usual stochastic order of the extreme order statistics from the half-logistic distribution.

Proposition 1 *Let $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$ be two sets of mutually independent random variables. Assume that for $k\in\{1,\dots,n\}$, $Y_k\sim HL(\lambda_k)$ and $X_k\sim HL(\lambda)$, where $\lambda=\sum_{k=1}^{n}\lambda_k/n$. Then*

(i) $Y_{1:n}\prec_{st}X_{1:n}$;

(ii) $X_{n:n}\prec_{st}Y_{n:n}$.

Proof 1 *From the definition,* $Y_{1:n} \prec_{st} X_{1:n}$ *if and only if,*

$$\frac{1}{\prod_{k=1}^{n}(1+e^{-\lambda_k x})} \le \frac{1}{(1+e^{-\lambda x})^n}.$$

By applying the arithmetic-geometric mean inequality we have

$$\frac{1+e^{-\lambda x}}{\prod_{k=1}^{n}(1+e^{-\lambda_k x})^{\frac{1}{n}}} = \prod_{k=1}^{n}\left(\frac{1}{1+e^{-\lambda_k x}}\right)^{\frac{1}{n}} + \prod_{k=1}^{n}\left(\frac{e^{-\lambda_k x}}{1+e^{-\lambda_k x}}\right)^{\frac{1}{n}}$$
$$\le \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+e^{-\lambda_k x}} + \sum_{k=1}^{n}\frac{e^{-\lambda_k x}}{1+e^{-\lambda_k x}} = 1,$$

which is the required result. For part (ii), note that $X_{n:n} \prec_{st} Y_{n:n}$ if and only if

$$\prod_{k=1}^{n}\frac{1-e^{-\lambda_k x}}{1+e^{-\lambda_k x}} \le \left(\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right)^n.$$

Using (3) and

$$\prod_{k=1}^{n}(1-e^{-\lambda_k x})^{\frac{1}{n}} \le 1-\frac{1}{n}\sum_{k=1}^{n}e^{-\lambda_k x} \le 1-\prod_{k=1}^{n}(e^{-\lambda_k x})^{\frac{1}{n}} = 1-e^{-\lambda x}, \tag{4}$$

we have the result.

The following result compares the extreme order statistics form the half logistic distribution in terms of the dispersive ordering. To prove the result we need the following technical lemma from p. 73 of [12].

Lemma 2 *Let* h *be a nondecreasing function such that* $h(0) > 0$, *and let* $a_i \ge 0$, $i = 1, \dots, n$. *Then*

$$\prod_{k=1}^{n} a_k \sum_{k=1}^{n} h(a_k) \le \sum_{k=1}^{n} a_k^n h(a_k).$$

Proposition 3 *Let* $X_1, \dots, X_n$ *and* $Y_1, \dots, Y_n$ *be two sets of mutually independent random variables. Assume that for* $k \in \{1, \dots, n\}$, $Y_k \sim HL(\lambda_k)$ *and* $X_k \sim HL(\lambda)$, *where* $\lambda = \sum_{k=1}^{n}\lambda_k/n$. *Then*

(i) $X_{n:n} \prec_{\text{disp}} Y_{n:n}$;

(ii) $Y_{1:n} \prec_{\text{disp}} X_{1:n}$.

Proof 2 *For part (i) note that*

$$X_{n:n} \prec_{\text{disp}} Y_{n:n} \Leftrightarrow g_{(n)}(x) \le f_{(n)}\{F_{(n)}^{-1}(G_{(n)}(x))\}. \tag{5}$$

After some simplifications (5) is equivalent to

$$\sum_{k=1}^{n}\frac{4\lambda_k e^{-\lambda_k x}}{1-e^{-2\lambda_k x}} \le \left(\sum_{k=1}^{n}\lambda_k\right)\prod_{k=1}^{n}\left(\frac{1+e^{-\lambda_k x}}{1-e^{-\lambda_k x}}\right)^{\frac{1}{n}} - \left(\sum_{k=1}^{n}\lambda_k\right)\prod_{k=1}^{n}\left(\frac{1-e^{-\lambda_k x}}{1+e^{-\lambda_k x}}\right)^{\frac{1}{n}}.$$

Multiplying both sides by $x(>0)$, it is sufficient to prove

$$\sum_{k=1}^{n}\frac{4y_k e^{-y_k}}{1-e^{-2y_k}} \le \left(\sum_{k=1}^{n}y_k\right)\prod_{k=1}^{n}\left(\frac{1+e^{-y_k}}{1-e^{-y_k}}\right)^{\frac{1}{n}} - \left(\sum_{k=1}^{n}y_k\right)\prod_{k=1}^{n}\left(\frac{1-e^{-y_k}}{1+e^{-y_k}}\right)^{\frac{1}{n}}, \tag{6}$$

For all $y_1, \dots, y_k(>0)$. In lemma 2.1, let $a_k = \left(\frac{1-e^{-y_k}}{1+e^{-y_k}}\right)^{\frac{1}{n}}$ and consider the increasing function $h(t) = -\ln\left(\frac{1-t^n}{1+t^n}\right)/t^n$, so that $h(a_k) = \frac{y_k(1+e^{-y_k})}{1-e^{-y_k}}$. Then we have

$$\prod_{k=1}^{n}\left(\frac{1-e^{-y_k}}{1+e^{-y_k}}\right)^{\frac{1}{n}}\sum_{k=1}^{n}\frac{y_k(1+e^{-y_k})}{1-e^{-y_k}} \le \sum_{k=1}^{n}y_k. \tag{7}$$

Similarly, if we let $h(t) = -\ln\left(\frac{1-t^n}{1+t^n}\right)$, then $h(a_k) = y_k$, and thus

$$\left(\sum_{k=1}^{n}y_k\right)\prod_{k=1}^{n}\left(\frac{1-e^{-y_k}}{1+e^{-y_k}}\right)^{\frac{1}{n}} \le \sum_{k=1}^{n}\frac{y_k(1+e^{-y_k})}{1-e^{-y_k}}. \tag{8}$$

From (7) and (8), the right-hand side of (6) denoted by $\psi(y_1, \dots, y_n)$ satisfies

$$\psi(y_1, \dots, y_n) \ge \sum_{k=1}^{n}\frac{y_k(1+e^{-y_k})}{1-e^{-y_k}} - \sum_{k=1}^{n}\frac{y_k(1-e^{-y_k})}{1+e^{-y_k}} \tag{3}$$
$$= \sum_{k=1}^{n}\frac{4y_k e^{-y_k}}{1-e^{-2y_k}},$$

which completes the proof. For part (ii), note that

$$f_{(1)}\{F_{(1)}^{-1}(G_{(1)}(x)\} = \frac{n\lambda}{2}\bar{G}_{(1)}(x)\{2-\bar{G}_{(1)}^{\frac{1}{n}}(x).$$

After some simplifications, $Y_{1:n} \prec_{\text{disp}} X_{1:n}$ if, and only if,

$$\left(\sum_{k=1}^{n}\lambda_k\right)\left[1-\prod_{k=1}^{n}\left(\frac{e^{-\lambda_k x}}{1+e^{-\lambda_k x}}\right)^{\frac{1}{n}}\right] \le \sum_{k=1}^{n}\frac{\lambda_k}{1+e^{-\lambda_k x}}$$
$$= \sum_{k=1}^{n}\lambda_k\left(1-\frac{e^{-\lambda_k}}{1+e^{-\lambda_k x}}\right),$$

or equivalently,

$$\sum_{k=1}^{n}\frac{\lambda_k x e^{-\lambda_k}}{1+e^{-\lambda_k x}}\prod_{k=1}^{n}\left(\frac{1+e^{-\lambda_k x}}{e^{-\lambda_k x}}\right)^{\frac{1}{n}} \le \left(\sum_{k=1}^{n}\lambda_k x\right). \tag{9}$$

Applying Lemma 2, with $a_k = \left(\frac{1+e^{-\lambda_k x}}{e^{-\lambda_k x}}\right)^{\frac{1}{n}}$ and $h(t) = \ln(t^n - 1)/t^n, t > 1$, yields the required result.

Remark 1 *The above result gives a lower bound on the variance of a parallel system and an upper bound on the variance of a series system of heterogeneous half logistic components in terms of that of a system of identically distributed half logistic random variables. The explicit expressions for the moments of order statistics from the half-logistic distribution is given in [6].*

Proposition 4 *Let* $X_1, \dots, X_n$ *and* $Y_1, \dots, Y_n$ *be two sets of mutually independent random variables. Assume that for* $k \in \{1, \dots, n\}$, $Y_k \sim HL(\lambda_k)$ *and* $X_k \sim HL(\lambda)$, *where* $\lambda = \sum_{k=1}^{n}\lambda_k/n$. *Then*

(i) $X_{n:n} \prec_{\text{hr}} Y_{n:n}$;

(ii) $Y_{1:n} \prec_{\text{hr}} X_{1:n}$.

Proof 3 *From Theorem 2.B.20 of [15] if* $X \prec_{disp} Y$ *and either* X *or* Y *has IFR (increasing failure rate) distribution, then* $X \prec_{hr} Y$. *Since half logistic distribution has IFR it follows from Theorem 5..8 of [2] that* $X_{n:n}$ *and* $X_{1:n}$ *have IFR. Using this and Proposition 3 we get the required result.*

In view of the chain of implications

$$X \prec_{\mathrm{lr}} Y \Rightarrow \begin{matrix} X \prec_{\mathrm{rh}} Y \\ X \prec_{\mathrm{hr}} Y \end{matrix} \Rightarrow X \prec_{\mathrm{st}} Y, \quad (10)$$

one naturally wonders whether the result of Proposition 4 can be strengthened to the likelihood ratio order. The following counterexample gives a negative answer to this question.

Example 1 *Let Y_1, Y_2 and Y_3 be independent half logistic random variables with the scale parameters $\lambda_1 = 0.5, \lambda_2 = 0.2$ and $\lambda_3 = 0.9$ and let X_1, X_2 and X_3 be a random sample from half logistic distribution with the scale parameter $\lambda = (0.5 + 0.2 + 0.9)/3 = 0.5333$. Then it holds that*

$$\frac{F_{(1)}(0.5)}{G_{(1)}(0.5)} \approx 0.9860 > 0.9812 \approx \frac{F_{(1)}(1)}{G_{(1)}(1)},$$

and

$$\frac{G_{(n)}(1)}{F_{(n)}(1)} \approx 0.5824 > 0.5523 \approx \frac{G_{(n)}(3)}{F_{(n)}(3)}.$$

As a result $F_{(1)}(x)/G_{(1)}(x)$ and $G_{(n)}(x)/F_{(n)}(x)$ are not increasing in $x > 0$ and, hence $Y_{1:n} \nprec_{\mathrm{rh}} X_{1:n}$ and $X_{n:n} \nprec_{\mathrm{rh}} Y_{n:n}$, which in turn imply that $Y_{1:n} \nprec_{\mathrm{lr}} X_{1:n}$ and $X_{n:n} \nprec_{\mathrm{lr}} Y_{n:n}$.

REFERENCES

[1] Balakrishnan, N.” Order statistics from the half logistic distribution”, *J. Stat. Comput. Simul.*, **20**, 287–309. 1985.

[2] Barlow, R.E. and Proschan, F. “*Statistical Theory of Reliability and Life Testing”*, To Begin With, Silver Spring, Maryland, 1981.

[3] Dolati, A., Genest, C., and Kochar, S. C. “On the dependence between the extreme order statistics in the proportional hazards model”, *J. Mult. Analysis*, **99**, 777–786, 2008.

[4] Dolati, A., Towhidi, M. and Shekari, M. “Stochastic and Dependence Comparisons Between Extreme Order Statistics in the Case of Proportional Reversed Hazard Model”. *JIRSS*, **10**, 29–43, 2011.

[5] Dykstra, R., Kochar, S. C. and Rojo, J. “Stochastic comparisons of parallel systems of heterogenous exponential components”, *J. Statist. Planning and inference*, **65**, 203–211, 1997

[6] Jodraa, P. and Jimenez-Gamero, M. D. “On a logarithmic integral and themoments of orderstatistics from the Weibull-geometric and half-logistic families of distributions”, *Journal of Mathematical Analysis and Applications*, **410**, 882–890, 2014.

[7] Johnson, N., Kotz, S. and Balakrishnan, N. “*Continuous univariate distributions”*, Vol. 2, Wiley, New York, 1995.

[8] Khaledi, B. and Kochar, S. C. “ Some new results on stochastic comparisons of parallel systems”. *J. Applied Probability*, **37**, 1123–1128. 2000.

[9] Kochar, S. C. and Rojo, J. “Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions”. *J. Mult. Analysis*, **59**, 272–281, 1996.

[10] Kochar, S.C. and Xu, M. “Some Recent Results on Stochastic Comparisons and Dependence among Order Statistics in the Case of PHR Model”, *JIRSS*, **2**, 125–140, 2007.

[11] Mao, T. and Hu, T. “Equivalent characterizations on orderings of order statistics and sample ranges”, *Probability in the Engineering and Informational Sciences*, **24**, 245–262, 2010.

[12] Mitrinocov, D.S., Pecaric, J.E. and Fink, A.M. “Classical and new inequalities in analysis”. Kluwer Academic Publishers. Dordrecht. Netherlands, 1993

[13] Müller, A. and Stoyan, D. “*Comparison Methods for Stochastic Models and Risks”*, Wiley, New York, 2002.

[14] Proschan, F. and Sethuraman, J. “Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability”. *J. Mult. Analysis*, **6**, 608–616, 1976.

[15] Shaked, M. and Shanthikumar, J.G. “*Stochastic Orders”*. NewYork: Springer, 2007.

[16] Zhao, P., Li, X., and Balakrishnan, N. “Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables”, *Journal of Multivariate Analysis*, **100**, 952-962, 2009.

[17] Zhao, P., Balakrishnan, N. “New results on comparisons of parallel systems with heterogeneous gamma components”. *Statistics and Probability Letters*, **81**, 36–44, 2011.

[18] Zhao. P. and Su, F. “On maximum order statistics from heterogeneous geometric variables”. *Ann Oper Res*, **212**, 215–223, 2014.