VARYING FORWARD BACKWARD SWEEP METHOD USING RUNGE-KUTTA, EULER AND TRAPEZOIDAL SCHEME AS APPLIED TO OPTIMAL CONTROL PROBLEMS

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ABSTRACT— *In this paper the basic optimal control problems (OCPs) are solved using Forward Backward Sweep method (FBSM). The results obtained from the test problems already being solved by Runge-kutta based FBSM are compared with Euler and trapezoidal based FBSM.*

Keywords— Forward-Backward Sweep Method (FBSM), Optimal Control Problems.

1. INTRODUCTION

Take a dynamical system, with constraints, based on differential equations (DEs), the process of adjusting (minimizing or maximizing) its variables, to get the best outcome (cost functional), is called Optimal Control.

It is frequently difficult to find solutions of optimal control problems (OCPs). Some problems have closed form solutions and for the problems without closed form solutions the numerical approximation is generated with the help of a simple numerical scheme.

These numerical techniques either belong to direct or indirect methods. In direct method the DE and integral is discretized by transforming the problem into nonlinear programming problem. The indirect method numerically solves the boundary value problem (BVP) which is generated with the help of Maximum Principle [2].

The FBSM is described as indirect method for the solution of OCPs. FBSM solves the state equation (first equation) forwardly and the other adjoint equation (second equation) backwardly while being updated by the first one. Mitter used a technique depicting forward backward essence [3]. Muir and Enright also used the Runge- Kutta (R-K) methods (explicit, implicit and an average of them) for two point BVPs solution having reflection of the FBSM [4].

2.MATERIALS AND METHODS

The Basic OCP:

Take a differential equation (D.E):

$$x'(t) = g(t, x(t), u(t)) \quad (1)$$

The D.E, depending on control variable u(t), is satisfied by the state variable x(t). Let the state variable (x) is continuously differentiable and control (u) be piecewise continuous. The basic OCP is to locate u(t) and associated x(t) so that the objective functional is maximized.

$$max \int_{t_0}^{t_f} f(t, x(t), u(t))dt$$

The objective functional is a function of x and u subject to the D.E (1) along with the initial conditions (IC)

$$x(t_0) = x_0 \, and \, x(t_f) \; free$$

Hamiltonian is obtained by adding the integrand and adjoint times the right hand side of the D.E (1).

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Now maximization of Hamiltonian with respect to u at u*(optimal control), we get optimality equation, adjoint equation and transversality equation respectively as:

$$\frac{\partial H}{\partial u} = 0 \; at \; u^* \Rightarrow f_u + \lambda g_u = 0$$

$$\lambda(t_f) = 0$$

$$\lambda' = -\frac{\partial H}{\partial x} \Rightarrow \lambda' = -(f_x + \lambda g_x)$$

Pontryagin's Maximum principle:

Let x*(t) and u *(t) be the optimal pair then there exist adjoint variable (λ (t)) s.t.

$$H(t, x^*(t), u(t), \lambda(t)) \le H(t, x^*(t), u^*(t), \lambda(t))$$

For all t, where Hamiltonian is defined by

$$H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda g(t, x(t), u(t))$$

$$\lambda(T) = 0$$

$$\lambda'(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$$

Algorithm of FBSM

Let's take vector approximations

$$\vec{x} = (x_1, x_2, ..., x_{M+1}) \, , \vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_{M+1})$$

be state and adjoint approximations respectively. An outline of the algorithm is:

- ❖ Let initial guess $u_0 = b$ for the control over $t = [t_0, t_f]$
- ❖ Using $x(t_0) = x_0$ and $\vec{u}$ value solve the D.E for $\vec{x}$ forward in time.
- ❖ Using $\lambda_{M+1} = \lambda(t_f) = 0$, $\vec{x}$ and $\vec{u}$ values, solve the D.E for $\vec{\lambda}$ in backward direction.
- ❖ By placing new $\vec{x}$ and $\vec{\lambda}$ into old $\vec{u}$ gives new $\vec{u}$ and thus sweeping the old $\vec{u}$. Use the new values to characterize the OCP.
- ❖ The negligible variation between the new and old iteration provide the solution or else the two equations are solved all over again

So, our resulting optimality system (2) is

$$\left.\begin{array}{l} x'(t) = g(t, x(t), u(t)), x(t_0) = x_0 \\ \lambda'(t) = -(f_x + \lambda g_x), \lambda(t_f) = 0 \end{array}\right\} \quad (2)$$

The FBSM is based on R-K4 scheme in order to solve the state equation forwardly and the adjoint equation backwardly in time. [1].In this paper the optimality system (2) is evaluated using FBSM based on the following numerical routines:

(i) The Euler's Routine
(ii) The Trapezoidal Routine

The Euler's Routine:

The system (2), the two point BVP, is calculated approximately by Euler's method on a uniform line mesh.

Take the step size $k= T/M$ and select the mesh points $t_p=(p-1)k$; where $p=1,2,\ldots,M+1$. Using the Euler method for system (2), starting at $t_0=0$ for x and at $t_{M+1}=t_f=T$ for λ results in the following non-linear system of equations.

$$x_{p+1} = x_p + k\{g(t_{p+1}, x_{p+1}, u_{p+1})\}, x_1 = x_0;\; p = 1,2,\ldots,M \quad \ldots \quad \text{(Eq3)}$$

$$\lambda_p = \lambda_{p+1} - k\left(f_{x_p} + \lambda_p g_{x_p}\right),\; \lambda_{M+1} = 0;\;\; p = 1,2,\ldots,M \quad \ldots \text{(Eq4)}$$

This is a system of equations for the 2M m vector

$$x_p \approx x(t_p), \quad p = 2,3,\ldots,M+1$$

$$\lambda_p \approx \lambda(t_p), \quad p = 1,2,\ldots,M$$

Initializing at the end condition $\lambda(T)=0$, the Euler routine for the adjoint variable is represented by Eq4. Let

$$r = col(x_2 x_3, \ldots, x_{M+1}, \lambda_1, \lambda_2, \ldots, \lambda_M)\epsilon\mathbb{R}^{2Mm}$$

and define the functions $F_1, F_2, \ldots, F_{2M}$ by

$$F_p(r) = x_p - x_{p+1} + k\{g(t_{p+1}, x_{p+1}, u_{p+1})\};\; p = 1,2,\ldots,M$$

$$F_p(r) = \lambda_{P+1} - \lambda_p - k\left(f_{x_p} + \lambda_p g_{x_p}\right);\; p = M+1,\ldots,2M$$

Maximizing the cost functional gives the solution of equations (3) and (4)

$$J(r) = \sum_{p=1}^{2M} F_p^{\,2}(r) \ldots (Eq5)$$

Trapezoidal routine:

Using the Trapezoidal scheme for system (2), the resulting system of equations are:

$$x_{p+1} = x_p + \frac{k}{2}\left\{g\left(t_{p+1}, x_{p+1}, u_{p+1}\right) + g\left(t_p, x_p, u_{p+1}\right)\right\}, x_1 = x_0;$$

where p=1,2,…,M (Eq6)

$$\lambda_p = \lambda_{p+1} - \frac{k}{2}\{(f_{x_p} + \lambda_p g_{x_p}) + (f_{x_{p+1}} + \lambda_{p+1} g_{x_{p+1}})\},\; \lambda_{M+1} = 0;$$

where p=1,2,…,M (Eq7)

The functions are given by

$$F_p(r) = x_p - x_{p+1} + \frac{k}{2}\left\{g\left(t_{p+1}, x_{p+1}, u_{p+1}\right) + g\left(t_p, x_p, u_{p+1}\right)\right\};\; p = 1,2,\ldots,M$$

$$F_p(r) = \lambda_{p+1} - \lambda_p - \frac{k}{2}\left\{\left(f_{x_p} + \lambda_p g_{x_p}\right) + \left(f_{x_{p+1}} + \lambda_{p+1} g_{x_{p+1}}\right)\right\};\; p = M+1,\ldots,2M$$

The solutions of (6) and (7) are produced by the maximization of cost functional (5).

3.RESULTS

Example 1:

Maximize the following objective functional

$$max \int_0^1 Ax(t) - Bu(t)^2 \; dt$$

Subject to the D.E

$$x'(t) = Cu(t) - \frac{1}{2}x(t)^2$$

Where A, B and C are the parameters

The initial condition is

$$x(0) = x_0, -2 < x(0)$$

$$0 \le A, 0 < B$$

It is maximization problem as

Finding Hamiltonian

$$H = Ax(t) - Bu^2(t) + \lambda Cu(t) - \frac{1}{2}\lambda x(t)^2$$

Using optimality condition

$$\frac{\partial H}{\partial u} = C\lambda - 2Bu = 0$$

By adjoint equation

$$\lambda' = -(-\lambda x + A)$$

$$\lambda'(t) = -A + \lambda x$$

A is a constant, by transversality condition, as T=1 so $\lambda(1)=0$

Optimality system comprises of state equation and initial condition:

$$x'(t) = Cu(t) - \frac{1}{2}x(t)^2;\, x(0) = x_0$$

adjoint equation and transversality Condition:

$$\lambda'(t) = -A + \lambda x;\, \lambda(1) = 0$$

and optimal control:

$$u^* = \frac{C\lambda}{2B}$$

Taking A=1, B=1, C=4 and x_0=1 and plotting together FBSM based on Euler, R-K and Trapezoidal routines respectively the graph obtained is:

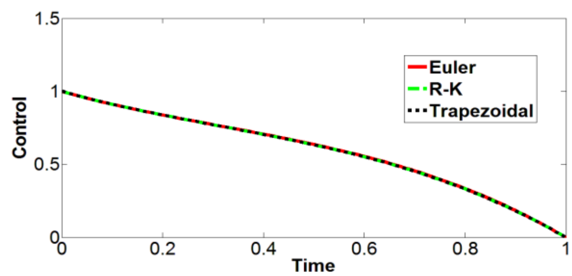

Fig.1 State & Time graph applying FBSM (Euler, RK & Trapezoidal scheme) for Example 1, k=0.01 or M=100

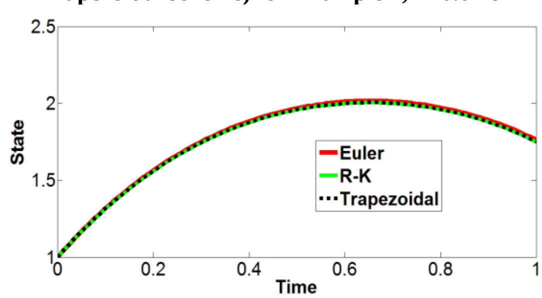

Fig.2 Control & Time graph applying FBSM (Euler, RK &Trapezoidal scheme), Example 1, k=0.01

Table 1: Some values of State & Time from example 1 applying FBSM (RK, Euler & Trapezoidal routines), k=0.01

Time	R-K based FBSM	Euler based FBSM	Trapezoidal based FBSM	Absolute Difference between R-K and Euler based FBSM	Absolute Difference between R-K and Trapezoidal based FBSM
0	1.0000	1.0000	1.0000	0.0000	0.0000
0.1000	1.3138	1.3167	1.3138	0.0029	0.0000
0.2000	1.5594	1.5648	1.5593	0.0054	0.0001
0.3000	1.7439	1.7513	1.7439	0.0074	0.0000
0.4000	1.8749	1.8838	1.8748	0.0089	0.0001
0.5000	1.9586	1.9686	1.9585	0.0100	0.0001
0.6000	1.9996	2.0105	1.9995	0.0109	0.0001
0.7000	2.0003	2.0120	2.0002	0.0117	0.0001
0.8000	1.9609	1.9733	1.9609	0.0124	0.0001
0.9000	1.8793	1.8922	1.8792	0.0129	0.0001
1.0000	1.7512	1.7645	1.7511	0.0133	0.0001

Table 2: Some values of Control & Time from example 1 applying FBSM (RK, Euler& Trapezoidal routine), k=0.01

Time	R-K based FBSM	Euler based FBSM	Trapezoidal based FBSM	Absolute Difference between R-K and Euler based FBSM	Absolute Difference between R-K and Trapezoidal based FBSM
0	1.0011	0.9993	1.0011	0.0018	0.0000
0.1000	0.9125	0.9116	0.9125	0.0009	0.0000
0.2000	0.8386	0.8387	0.8386	0.0001	0.0000
0.3000	0.7719	0.7727	0.7719	0.0008	0.0000
0.4000	0.7057	0.7073	0.7057	0.0016	0.0000
0.5000	0.6344	0.6365	0.6344	0.0021	0.0000
0.6000	0.5521	0.5547	0.5521	0.0026	0.0000
0.7000	0.4532	0.4558	0.4532	0.0026	0.0000
0.8000	0.3317	0.3340	0.3317	0.0023	0.0000
0.9000	0.1821	0.1835	0.1821	0.0014	0.0000
1.0000	0	0	0	0	0

The graphs (Fig1 and Fig 2) and the tables (Table 1 and Table 2) shows that the FBSM based on RK and Trapezoidal routine produces almost identical results with absolute difference of 0.0001 in some of its iterations.

Example 2:

$$\min \frac{1}{2}\int_0^T Ax^2(t) + Bu^2(t)\ dt$$

Subject to D.E

$$x'(t) = x(t) + u(t)$$

With conditions $x(0)= x_0$ fixed , x (1) free

Forming Hamiltonian

$$H = \frac{A}{2}x^2 + \frac{B}{2}u^2 + \lambda x + \lambda u$$

By optimality condition

$$\frac{\partial H}{\partial u} = Bu + \lambda = 0$$

By adjoint equation

$$\lambda' = -(Ax + \lambda)$$

$$\lambda'(t) = -Ax(t) - \lambda(t)$$

By transversality condition:

$$\lambda(1) = 0$$

Optimality system comprises of state equation and initial condition:

$$x'(t) = x(t) + u(t); x(0) = x_0$$

adjoint equation and transversality Condition:

$$\lambda'(t) = -Ax(t) - \lambda(t); \lambda(1) = 0$$

and optimal control:

$$u^* = -\frac{\lambda}{B}$$

Now plotting FBSM based on Euler, RK and Trapezoidal routines together using the values A=1, B=1 & x_0=1, the following graphs are obtained

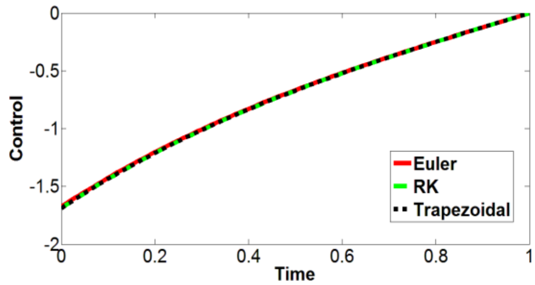

Fig.3 State & Time applying FBSM (Euler, RK & Trapezoidal routine), Example 2, k=0.01

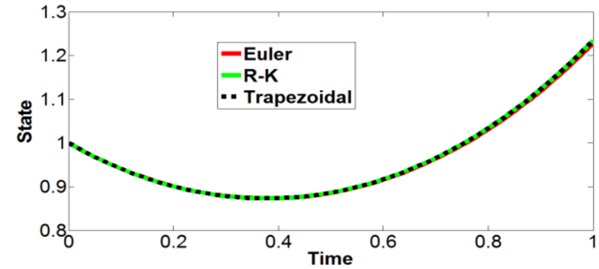

Fig.4 Control & Time applying FBSM(Euler,RK & Trapezoidal routine), Example 2, k=0.01

Table 3: Some values of State & Time from example 2 applying FBSM (RK, Euler & Trapezoidal routine), k=0.01

Time	R-K based FBSM	Euler based FBSM	Trapezoidal based FBSM	Absolute Difference between R-K and Euler based FBSM	Absolute Difference between R-K and Trapezoidal based FBSM
0	1.0000	1.0000	1.0000	0.0000	0.0000
0.1000	0.9410	0.9413	0.9410	0.0003	0.0000
0.2000	0.9008	0.9011	0.9008	0.0003	0.0000
0.3000	0.8785	0.8787	0.8785	0.0002	0.0000
0.4000	0.8738	0.8737	0.8738	0.0001	0.0000
0.5000	0.8865	0.8860	0.8865	0.0005	0.0000
0.6000	0.9170	0.9159	0.9170	0.0011	0.0000
0.7000	0.9659	0.9639	0.9659	0.0020	0.0000
0.7900	1.0264	1.0234	1.0263	0.0030	0.0001
0.8000	1.0341	1.0310	1.0341	0.0031	0.0000
0.9000	1.1230	1.1186	1.1230	0.0044	0.0000
0.9100	1.1332	1.1286	1.1331	0.0046	0.0001
1.0000	1.2345	1.2285	1.2345	0.006	0.0000

Table 4: Some values of Control & Time from example 2 applying FBSM (RK, Euler & Trapezoidal routine), k=0.01

Time	R-K based FBSM	Euler based FBSM	Trapezoidal based FBSM	Absolute Difference between R-K and Euler based FBSM	Absolute Difference between R-K and Trapezoidal based FBSM
0.1000	-1.6616	-1.6486	-1.6616	0.0130	0.0000
0.2000	-1.4125	-1.4022	-1.4125	0.0103	0.0000
0.3000	-1.1914	-1.1833	-1.1914	0.0081	0.0000
0.4000	-0.9941	-0.9877	-0.9940	0.0064	0.0001
0.5000	-0.8164	-0.8115	-0.8164	0.0049	0.0000
0.6000	-0.6551	-0.6513	-0.6551	0.0038	0.0000
0.7000	-0.5068	-0.5040	-0.5067	0.0028	0.0001
0.8000	-0.3685	-0.3666	-0.3685	0.0019	0.0000
0.9000	-0.2377	-0.2365	-0.2377	0.0012	0.0000
1.0000	-0.1115	-0.1110	-0.1115	0.0005	0.0000

4.CONCLUSION:

The results from the problems already being solved by FBSM based on RK routine are compared with the results from FBSM based on Euler and Trapezoidal routines for the OCPs. The above approximation tables and graphs shows that while solving OCPs via FBSM the Euler and Trapezoidal schemes can be used instead of Runge Kutta scheme. The results obtained from Trapezoidal scheme produces the approximations almost similar to Runge-Kutta scheme with absolute difference of 0.0001 while taking step size 0.01. As there is slight error in some of its iterations implying that the FBSM can be implemented using Trapezoidal scheme, for which less computational labour is required.

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