PERFORMANCE OF RUNGE-KUTTA-NYSTRÖM 6(4) FORMULAE WITH ROUND-OFF ERROR CONTROL TECHNIQUE

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ABSTRACT: *N-body simulations of the Sun, the planets, and small celestial bodies are commonly used to model the evolution of the Solar System. Enormous numerical integrators for performing such simulations have been developed; see, for example, [1, 2]. The primary objective of this paper is to analyze the error growth for an embedded Runge-Kutta-Nyström (ERKN) integrator. The experiments are performed with round-off error control technique for ERKN64 integrator applied to the Jovian problem over a long interval of duration, as long as one million years, with the local error tolerances ranging from 10^{-16} to 10^{-8}. Error is estimated in terms of global error in the position and velocity, and the relative error in energy and angular momentum.*

Keywords: N-Body Simulations, Jovian Problem, Numerical Integrator, Round-off error.

1. INTRODUCTION

To study the dynamics of the Solar System, computational astronomers make use of long term N-body simulations. These simulations are performed by developing the mathematical model of the problem. This model takes the form of first- and second-order initial value problem (IVP). The problem we are discussing is of the form,

$$y''(t)=f(t,y(t)),\ y(t_0)=y_0\ ,y'(t_0)=y'_0\ , \quad (1)$$

where $y_0, \acute{y}_0 \in R^n$ denote the initial positions and velocities, n is the dimension of the IVP, and f is sufficiently smooth function. Numerical approximation techniques are used to solve these problems, because complexity of the differential equations make analytic solution very hard to find. Different integrators are used to find the numerical approximations of these differential equations, for example, Runge-Kutta [3, 4], linear multistep [5], and Runge-Kutta-Nyström [6].

1.2. Jovian problem

The Jovian problem [7] predicts the orbital motion of the Sun and outer four Gas-giants, Jupiter, Saturn, Uranus and Neptune interacting with each other gravitationally. In Jovian problem, the bodies are ordered from Sun to Neptune and three dimensional Cartesian coordinate system having origin at the barycenter of the bodies is used. Let $r_i=[x_i\,,y_i\,,z_i\,]$ be the position vector of the i^{th} body then the equations of motion for the i^{th} body can be written as

$$r_i''(t)=\sum_{j=1,\,j\neq i}^{5}\frac{\mu_j\,(r_j(t)-r_i(t)\ )}{\left\|r_j(t)-r_i(t)\right\|_2^3},\quad i=1,...,5\ , \quad (2)$$

Where $\|.\|$ denotes the L_2-norm and μ_j is gravitational constant G times the mass m_j of j^{th} body $i.e, \mu_j = Gm_j$. As a whole, we have a system of fifteen second-order ordinary differential equations. The distance is measured in astronomical units (1AU = 150,000,000 km), independent variable time in Earth days and mass as Solar mass.

2. MATERIAL AND METHODS

Explicit Runge-Kutta-Nyström methods (ERKN) were introduced by E.J Nyström in 1925 [6]. These methods directly solve a system of second-order ordinary differential equations. For efficiency purpose we have used adaptive step-size technique. In this technique the local error is calculated at each single step such that a pair of formulae of different orders have same function evaluations. The numerical solution is often obtained by using higher-order formula and error is calculated by the lower-order formula to gain maximum efficiency [8]. The pair implemented in this fashion is said to be implemented in higher-order mode. The variable step size integrator ERKN64 is used in this paper, which is 6 stage, 6-4 FSAL pair introduced by Dormand and Prince [9].

2.1. Round off error control technique for ERKN64 integrator

In this paper the possibility of reducing the round-off error in ERKN64 integrator is examined by using compensated summation technique [10]. The core theme of this technique is to anticipate the dominant term which contributes to round-off error. Consider the following solution formula

$$y_n = y_{n-1} + hy'_{n-1} + h^2\sum_{i=1}^{s} b_i G_i\ . \quad (3)$$

Equation (3) contains three types of errors; the round off error in the construction of hy'_{n-1} and $h^2\sum_{i=1}^{s} b_i G_i$, the integration error in y_{n-1} from the previous time-step and round off error due to the addition of all the terms on the right hand side of (3). If the step size is small then $hy'_{n-1}+h^2\sum_{i=1}^{s} b_i G_i$ become smaller than y_{n-1}. Addition of these two terms will dominate the contribution to round-off error, which will be estimated in each time-step and solution will be update as follow. First calculate

$$\gamma\ =\ hy'_{n-1}+h^2\sum_{i=1}^{s} b_i G_i - \sigma, \quad (4)$$

where σ is the estimated round-off error calculated on the previous time-step (initially $\sigma = 0$).

As $hy'_{n-1}+h^2\sum_{i=1}^{s} b_i G_i$ and σ are comparatively smaller than y_{n-1}, so the error caused in the formation of γ is negligible. The solution is then updated to

$$H_n\ = y_{n-1}+\gamma \quad (5)$$

And for the next time-step the round off error is estimated as

$$\sigma = H_n - y_{n-1} - \gamma \quad (6)$$

The solution is then updated as $y_n = H_n$. The same concept is used for derivative formula.

3. RESULTS AND DISCUSSION

The performance of the ERKN64 is analyzed with round-off error control (WD) and without round-off error control (WO) technique applied to the Jovian problem. The interval of integration is one million years and error is calculated at each subinterval of 100 years. The tolerances are used in the range from 10^{-16} to 10^{-8}. The smallest tolerance is close to machine precision (2.2×10^{-16})

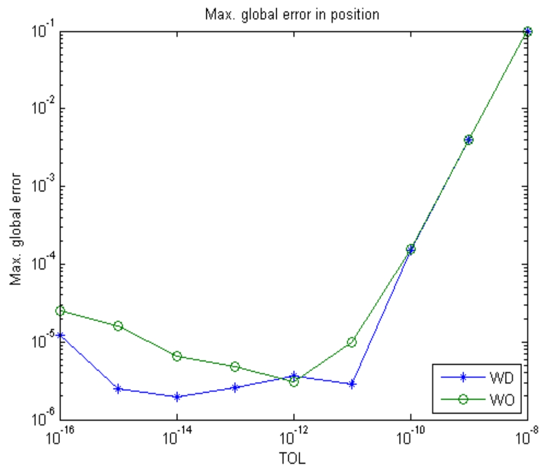

Figure 1. Maximum global error in position using with and without round-off error control in ERKN64 applied to the Jovian problem at tolerances ranging from 10^{-16} to 10^{-8}

The first set of experiments is performed to obtain the maximum global error in the position as shown in Figure 1. The values of maximum global error for WD vary from 1.96×10^{-06} to 9.98×10^{-02}. Whereas for WO, the range is from 3.04×10^{-06} to 9.99×10^{-02}. From Figure 1 we see that the maximum global error has been decreased at a constant rate of $10^{1.5}$ AU and there is no major effect of round-off error. The round-off error effect became significant when the tolerance is decreased to 10^{-16}. The most prominent differences between WO and WD were observed at tolerances 10^{-11}, 10^{-14} and 10^{-15}, where the global error of WO is 247.45%, 230.60% and 533.07% greater than WD, respectively. While at 10^{-12} the global error of WD is 16.65% greater than WO, but the maximum global error values are so small at this tolerance that this small percentage has no significance.

Figure 2 depicts the error growth in position for WD and WO. Error growth with respect to position is calculated using a reference solution obtained in quadruple precision by WD with tolerance 10^{-18}. The integration was performed over 10^6 years using TOL = 10^{-14}, 10^{-15}, 10^{-16}. The selection of these tolerances is due to the fact that they are close to the machine precision.

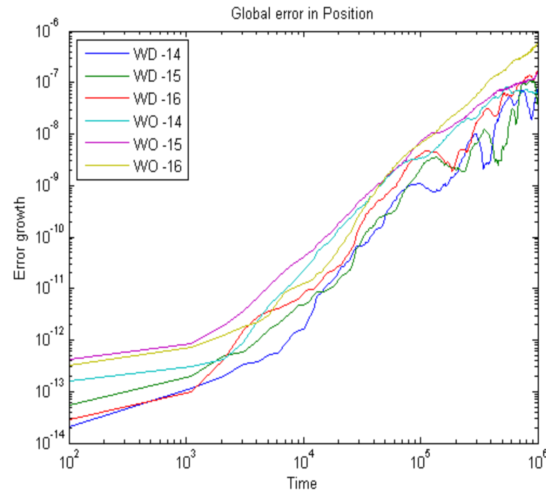

Figure 2. Global error growth in Position using with and without round-off error control in ERKN64 applied to the Jovian problem at selected tolerances 10^{-16}, 10^{-15} and 10^{-14}

The Figure 2 shows further that the error growth for WD at 10^{-14} is minimum for the whole integration expect for some short interval of $10^{5.3}$ to $10^{5.6}$ years. WD at 10^{-16} maintained its error less than WO until at $10^{5.8}$ there was a cross over and WO at 10^{-14}, 10^{-15} became more accurate.

Over all it has been observed that the performance of WD is much better than WO. After 10^3 years WD at 10^{-14} maintains its best accuracy and WD at 10^{-15} became second best up to $10^{5.5}$ afterwards WO at 10^{-14} become more accurate. It is noted that WD and WO at 10^{-15} maintain the difference of approximately 1.5 AU.

Most of the plots of error growth in position have high oscillations. Due to which results deduction was very difficult. In order to smooth the data, Matlab filter command is used with optimal window size. We have performed experiments with window sizes 1, 10, 20 and 50 as shown in Figure 3. Figure 3(a) shows the plots when no filter command is used. There are so many oscillations that it is very difficult to distinguish the plots. When filter command is used with window size 10 the oscillations are reduced but still not feasible to deduce results as shown in Figure 3(b). Figure 3(c) shows that the plots are very much less oscillatory but some points are still over lapping. Here window size of 20 was used. Figure 3(d) shows that with window size of 50 all plots are clear enough to deduce results. Throughout the paper, we have used filter command with window size 50. Let us now consider the accuracy of the integrator ERKN64 in terms of relative error in the angular momentum and energy.

Figure 4 explains the error growth in the angular momentum with and without round-off error control technique using the same tolerances and window size as for the set of experiments performed in Figure 2.

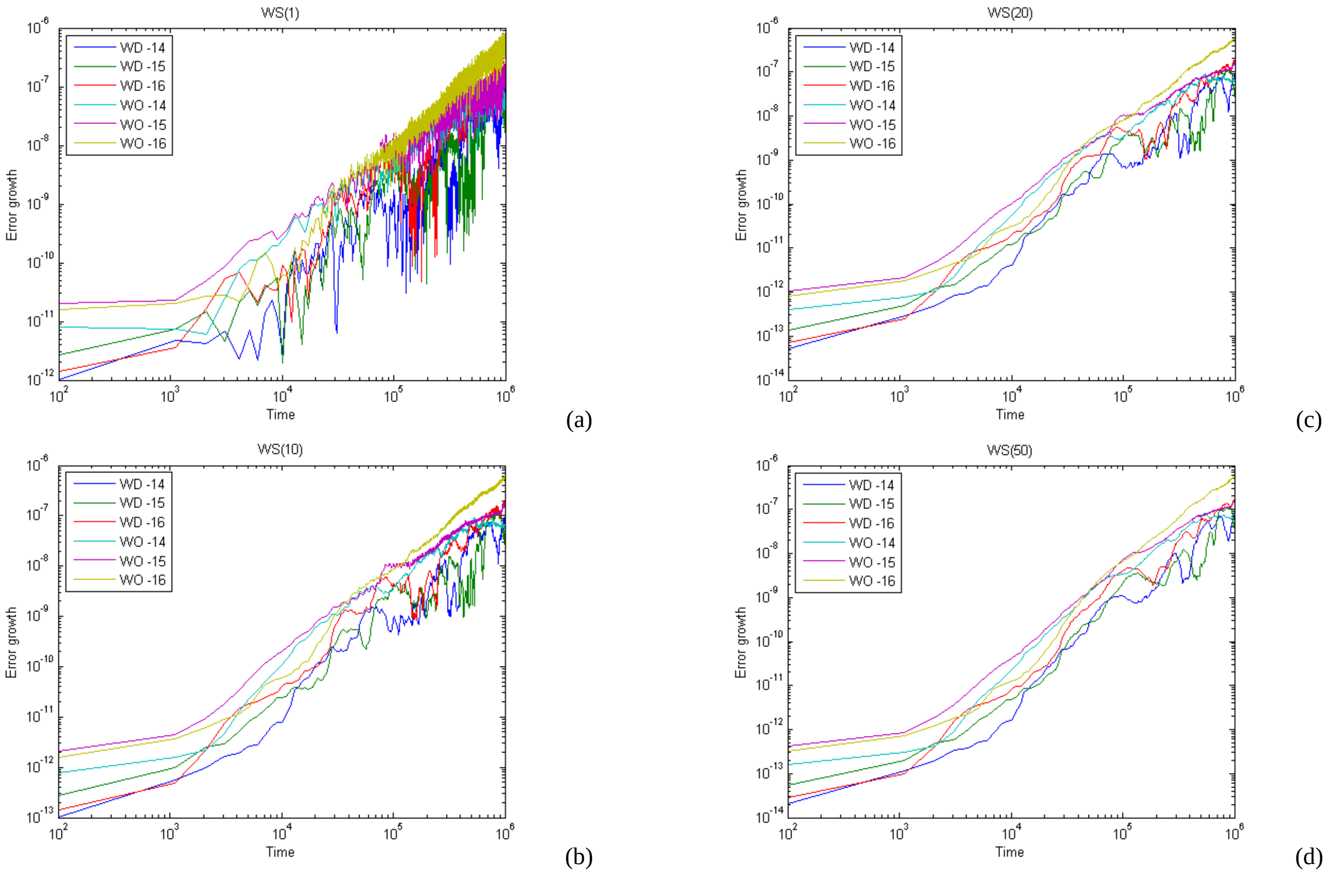

Figure 3. Experiments using Matlab function filter with window sizes 1, 10, 20, and 50.

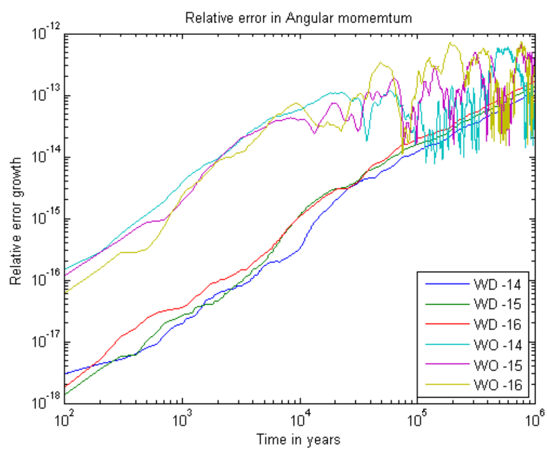

Figure 4. The relative error growth in Angular momentum using with and without round-off error control in ERKN64 applied to the Jovian problem at selected tolerances 10^{-16}, 10^{-15} and 10^{-14}

Figures 4 shows the linear error growth in WD which has more accuracy at TOL $=10^{-14}$ and also attains the best accuracy among all the experiments performed in Figure 4. We observe that WD at 10^{-15} remains the second best in terms of accuracy. The WO at TOL=10^{-16}, 10^{-15}, 10^{-14} shows some oscillations after ten thousand years. This indicates that the round-off error is the possible cause of these oscillations. Approximately linear error growth in energy (not shown here) has also been observed especially with TOL=10^{-14}.

4. CONCLUSION

The main objective of this paper was to investigate the error growth of ERKN64 integrator applied to the Jovian problem with round-off error control technique. Throughout the paper, we examined the error growth of the global error in the positions and velocities of the bodies, and the relative error in the angular momentum and energy. The experiments were performed over 10^6 years with the local error tolerances ranging from 10^{-16} to 10^{-8}. We observed that the round-off error control technique is very effective, especially, with tolerances close to the machine precision. For example, with Tol= 10^{-15} and 10^{-16}, the maximum global error in position obtained without round-off error control technique were approximately *230.60%* and *533.07%* more than those obtained with round-off error control technique.

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