UPPER BOUND OF A THIRD HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. *In this paper a new subclass of analytic functions associated with right half of the lemniscates of Bernoulli* $\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=0$ *has been introduced. An upper bound of the third Hankel determinant is determined for this class.*

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1. INTRODUCTION

Let A denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathfrak{A}=\left\{z\in\mathbb{C}:\left|z\right|<1\right\}$ and normalized by the conditions $f\left(0\right)=f'(0)-1=0.$ Therefore, for $f(z)\in A,$ one has

$$f\left(z\right)=z+\sum_{n=2}^{\infty}a_n\,z^n,\ z\in\mathfrak{A}, \tag{1.1}$$

while the class of normalized univalent functions will be denoted by $S.$ For two functions $f(z)$ and $g(z)$ analytic in $\mathfrak{A}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z)\prec g(z),$ if there is an analytic function $w(z)$ with $\left|w(z)\right|\le\left|z\right|$ such that $f\left(z\right)=g\left(w\left(z\right)\right).$ If $g(z)$ is univalent in $\mathfrak{A}$, then $f(z)\prec g(z),$ if and only if $f\left(0\right)=g\left(0\right)$ and

$$f\left(\mathfrak{A}\right)\subseteq g\left(\mathfrak{A}\right).$$

In 1996, Sokol and Stankiewicz [1] intoduced a class $\mathcal{SL}$ of Sokol-Stankiewicz starlike functions defined by

$$\left|\left(\frac{zf'\left(z\right)}{f\left(z\right)}\right)^2-1\right|<1,\ z\in\mathfrak{A}.$$

Geometrically, a function $f\left(z\right)\in\mathcal{SL}$ if $zf'\left(z\right)/f\left(z\right)$ is in the interior of the right half of the lemniscates of Bernoulli $\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=0.$ Alternatively, we can write $f\left(z\right)\in\mathcal{SL}$ if and only if

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec\sqrt{1+z},\quad z\in\mathfrak{A}.$$

For detail of this class, we refer the work of Sokol [2-4], Raza and Malik [5], Halim and Omar [6] and Ali etal.[7].

Utilizing the above concepts we now introduce a subclass $\mathcal{RS}\left(\alpha,\beta\right),$ $\alpha,\beta>0$ with $0<\alpha+\beta\le1,$ of analytic functions as

$$\mathcal{RS}\left(\alpha,\beta\right)=\left\{f\left(z\right)\in A:\left|\left(\frac{\alpha}{\alpha+\beta}\frac{f\left(z\right)}{z}+\frac{\beta}{\alpha+\beta}f'\left(z\right)\right)^2-1\right|<1,\ z\in\mathfrak{A}\right\}.$$

The geometrical interpretation of a function $f(z)$ belongs to $\mathcal{RS}\left(\alpha,\beta\right)$ is such that $\left(\alpha\,f\left(z\right)/z+\beta f'\left(z\right)\right)/\alpha+\beta$ lies in the region bounded by the right half of the lemniscates of Bernouli given by the relation $\left|w^2-1\right|<1.$ It can easily be seen that $f(z)\in\mathcal{RS}\left(\alpha,\beta\right)$ if it satisfies

$$\frac{\alpha}{\alpha+\beta}\frac{f\left(z\right)}{z}+\frac{\beta}{\alpha+\beta}f'\left(z\right)\prec\sqrt{1+z},\quad z\in\mathfrak{A}. \tag{1.2}$$

The qth Hankel determinant $H_q\left(n\right),$ $q\ge1,$ $n\ge1,$ for a function $f(z)\in A$ is studied by Noonan and Thomas [8] as:

$$H_q\left(n\right)=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}.$$

In literature many authors have studied the determinant $H_q\left(n\right)$. For example, Arif *et al.* [9,10] studied the qth Hankel determinant for some subclasses of analytic functions. Hankel determinant of exponential ploynomials is obtained by Ehrenborg in [11]. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [12]. It is well known that the Fekete-Szego functional $|a_3-a_2^2|$ is $H_2\left(1\right).$ Fekete-Szego then further generalized the estimate $|a_3-\lambda a_2^2|$ with λ real and $f(z)\in S.$ Moreover, we also know that the functional $|a_2a_4-a_3^2|$ is $H_2(2).$ Janteng, Halim and Darus [13] have considered the functional $|a_2a_4-a_3^2|$ and found a sharp bound for the function $f(z)$ in the subclass $\mathcal{RT}$ of $S,$ consisting of functions whose derivative has a positive real part studied by Mac Gregor [14]. In their work, they have shown that if

$f(z) \in \mathcal{RT}$, then $|a_2a_4 - a_3^2| \le 4/9$. The sharp upper bound of the second Hankel determinant for the familiar classes of starlike and covex functions were studied in [15], that is, for $f(z) \in S^*$ and $f(z) \in C$, the authors obtained $|a_2a_4 - a_3^2| \le 1$. and $|a_2a_4 - a_3^2| \le 1/8$. respectively. Recently in 2010, Babalola [16] considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of bounded-turning, starlike and convex functions. In the present investigation, we study the upper bound of $H_3(1)$ for a subclass $\mathcal{RS}(\alpha,\beta)$ of analytic functions associated with lemniscate of Bernouli by using Toeplitz determinants.

For our main results we need the following lemmas.

Lemma 1.1 [17] . If $p(z) = 1 + c_1z + c_2z^2 + \cdots \in P$, then

$$\left|c_2 - \nu a_1^2\right| \le \begin{cases} -4\nu + 2 & (\nu \le 0) \\ 2 & (0 \le \nu \le 1) \\ 4\nu - 2 & (\nu \ge 1). \end{cases}$$

when $\nu < 0$ or $\nu > 0$, equality holds if and only if $p(z)$ is $(1+z)(1-z)^{-1}$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $p(z)$ is $(1+z^2)(1-z^2)^{-1}$ or one of its rotations. If $\nu = 0$, then equality holds if and only if

$$p(z) = \left(\frac{1+\xi}{2}\right)\left(\frac{1+z}{1-z}\right) - \left(\frac{1-\xi}{2}\right)\left(\frac{1-z}{1+z}\right), \quad 0 \le \xi \le 1,$$

or one of its rotations. While for $\nu = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$. Although the above upper bound is sharp, it can be improved as follows when $0 < \nu < 1$,

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \le 2, \quad 0 < \nu \le \frac{1}{2},$$

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2, \quad \frac{1}{2} < \nu \le 1.$$

Lemma 1.2 [17]. If $p(z) = 1 + c_1z + c_2z^2 + \cdots \in P$, then for complex number ν,

$$\left|c_2 - \nu c_1^2\right| \le 2 \max\left\{1; \left|2\nu - 1\right|\right\}$$

and the result is sharp for the functions given by

$$p(z) = (1+z^2)(1-z^2)^{-1} \text{ or } p(z) = (1+z)(1-z)^{-1}.$$

Lemma 1.3 [18]. If $p(z) = 1 + c_1z + c_2z^2 + \cdots \in P$, then

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right),$$

$$4c_3 = c_1^3 + 2\left(4-c_1^2\right)c_1x - \left(4-c_1^2\right)c_1x^2 + 2\left(4-c_1^2\right)\left(1-|x|^2\right)z$$

for some x, z such that $|x| \le 1$, $|z| \le 1$.

2. MAIN RESULTS

Theorem 2.1. Let $f(z) \in \mathcal{RS}(\alpha,\beta)$ and of the form (1.1) . Then

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} -\frac{(\alpha+\beta)}{8}\left[\frac{1}{(\alpha+3\beta)} + \frac{2\lambda(\alpha+\beta)}{(\alpha+2\beta)^2}\right]; & \text{if } \lambda < \frac{-5(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}, \\ \frac{(\alpha+\beta)}{2(\alpha+3\beta)}; & \text{if } -\frac{5(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)} \le \lambda \le \frac{3(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}, \\ \frac{(\alpha+\beta)}{8}\left[\frac{1}{(\alpha+3\beta)} + \frac{2\lambda(\alpha+\beta)}{(\alpha+2\beta)^2}\right]; & \text{if } \lambda > \frac{3(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}. \end{cases}$$

Furthermore

$$\left|a_3 - \lambda a_2^2\right| + \left(\lambda + \frac{5(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}\right)\left|a_2\right|^2 \le \frac{(\alpha+\beta)}{2(\alpha+3\beta)}$$

for

$$-\frac{5(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)} < \lambda \le -\frac{(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}$$

and

$$\left|a_3 - \lambda a_2^2\right| + \left(\frac{3(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)} - \lambda\right)\left|a_2\right|^2 \le \frac{(\alpha+\beta)}{2(\alpha+3\beta)}$$

for

$$-\frac{(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)} < \lambda \le \frac{3(\alpha+2\beta)^2}{2(\alpha+3\beta)(\alpha+\beta)}.$$

These results are sharp.

Proof. If $f(z) \in \mathcal{RS}(\alpha,\beta)$, then, using (1.2), it follows that

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z} + \frac{\beta}{\alpha+\beta}f'(z) \prec \Phi(z), \quad z \in \mathfrak{A}, \qquad (2.1)$$

where $\Phi(z) = \sqrt{1+z}$. Let us define a function

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots.$$

It is clear that $p(z) \in P$. This implies that

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

From (2.1), we have

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z} + \frac{\beta}{\alpha+\beta}f'(z) = \Phi(w(z)),$$

with

$$\Phi(w(z)) = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

Now

$$\sqrt{\frac{2p(z)}{p(z)+1}}=1+\frac{1}{4}c_1z+\left(\frac{1}{4}c_2-\frac{5}{32}c_1^2\right)z^2+\left(\frac{1}{4}c_3-\frac{5}{16}c_1c_2+\frac{13}{128}c_1^3\right)z^3+\cdots.$$

Similarly,

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z}+\frac{\beta}{\alpha+\beta}f'(z)=1+\frac{(\alpha+2\beta)a_2}{\alpha+\beta}z+\frac{(\alpha+3\beta)a_3}{\alpha+\beta}z^2+\frac{(\alpha+4\beta)a_4}{\alpha+\beta}z^3+\cdots.$$

Equating the coefficients of z, z^2, z^3 and z^4, we obtain

$$a_2=\frac{\alpha+\beta}{4(\alpha+2\beta)}c_1 \qquad (2.2)$$

$$a_3=\frac{\alpha+\beta}{4(\alpha+3\beta)}c_2-\frac{5(\alpha+\beta)}{32(\alpha+3\beta)}c_1^2 \qquad (2.3)$$

$$a_4=\frac{\alpha+\beta}{4(\alpha+4\beta)}c_3-\frac{5(\alpha+\beta)}{16(\alpha+4\beta)}c_1c_2+\frac{13(\alpha+\beta)}{128(\alpha+4\beta)}c_1^3 \qquad (2.4)$$

From (2.2) and (2.3), we obtain

$$\left|a_3-\lambda a_2^2\right|=\frac{\alpha+\beta}{4(\alpha+3\beta)}\left|c_2-\frac{1}{8}\left(5+\frac{2\lambda(\alpha+3\beta)(\alpha+\beta)}{(\alpha+2\beta)^2}\right)c_1^2\right|.$$

Now by using Lemma 1.1, we ge the required result.

Theorem 2.2. Let $f(z)\in\mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then for $\lambda\in\mathbb{C}$,

$$\left|a_3-\lambda a_2^2\right|\le\frac{(\alpha+\beta)}{2(\alpha+3\beta)}\max\left\{1,\left|\frac{(\alpha+2\beta)^2+2\lambda(\alpha+3\beta)(\alpha+\beta)}{4(\alpha+2\beta)^2}\right|\right\}.$$

This result is sharp.

Proof. Since

$$\left|a_3-\lambda a_2^2\right|=\frac{\alpha+\beta}{4(\alpha+3\beta)}\left|c_2-\frac{1}{8}\left(5+\frac{2\lambda(\alpha+3\beta)(\alpha+\beta)}{(\alpha+2\beta)^2}\right)c_1^2\right|,$$

and therefore by using Lemma 1.2 with

$$\nu=\frac{1}{8}\left(5+\frac{2\lambda(\alpha+3\beta)(\alpha+\beta)}{(\alpha+2\beta)^2}\right)$$

we get the required result.

This result is sharp for the functions

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z}+\frac{\beta}{\alpha+\beta}f'(z)=\sqrt{1+z}$$

or

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z}+\frac{\beta}{\alpha+\beta}f'(z)=\sqrt{1+z^2}.$$

For $\lambda=1$, we have $H_2(1)$.

Corollary 2.1. Let $f(z)\in\mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then

$$\left|a_3-a_2^2\right|\le\frac{\alpha+\beta}{2(\alpha+3\beta)}.$$

This result is sharp.

Theorem 2.3. Let $f(z)\in\mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then

$$\left|a_2a_4-a_3^2\right|\le\frac{(\alpha+\beta)^2}{4(\alpha+3\beta)^2}.$$

This result is sharp.

Proof. From (2.2), (2.3) and (2.4), we can write

$$a_2a_4-a_3^2=\frac{(\alpha+\beta)^2}{1024}\left[-\frac{\left(5c_1^2-8c_2\right)^2}{(\alpha+3\beta)^2}+\frac{2c_2\left(13c_1^3-40c_1c_2+32c_3\right)}{(\alpha+2\beta)(\alpha+4\beta)}\right].$$

Putting the values of c_2 and c_3 from Lemma 1.3, and assuming that $c>0$ where $c_1=c\in[0,2]$, we get

$$\left|a_2a_4-a_3^2\right|\le\frac{(\alpha+\beta)^2}{1024}\left|(8A-B)c^4-8(4A-B)\left(4-c^2\right)c^2x+\left(64Ac^2+16B\left(4-c^2\right)\right)\left(4-c^2\right)x^2+128Ac\left(4-c^2\right)\left(1-|x|^2\right)z\right|,$$

with

$$4A=\frac{1}{(\alpha+2\beta)(\alpha+4\beta)} \text{ and } B=\frac{1}{(\alpha+3\beta)^2}.$$

Using triangle inequality and taking $|x|=\rho$, we have

$$\left|a_2a_4-a_3^2\right|\le\frac{(\alpha+\beta)^2}{1024}\left[(8A-B)c^4+8(4A-B)\left(4-c^2\right)c^2\rho+\left(64Ac^2+16B\left(4-c^2\right)\right)\left(4-c^2\right)\rho^2+128Ac\left(4-c^2\right)\left(1-\rho^2\right)\right]$$
$$=F(c,\rho), \text{ (say)}$$

Differentiating with respect to ρ, we get

$$\frac{\partial F(c,\rho)}{\partial\rho}=\frac{(\alpha+\beta)^2}{1024}\left[8(4A-B)\left(4-c^2\right)c^2-256Ac\left(4-c^2\right)\rho+2\left(64Ac^2+16B\left(4-c^2\right)\right)\left(4-c^2\right)\rho\right].$$

It is clear that $\partial F(c,\rho)/\partial\rho>0$, which shows that $F(c,\rho)$ is an increasing function on the interval $[0,1]$. This implies that maximum value occurs at $\rho = 1$. Therefore $F(c,1) = N(c)$ (say). Now

$$N(c) = \frac{(\alpha+\beta)^2}{1024}\Big[8(4A-B)(4-c^2)c^2 - 256Ac(4-c^2) + 2\left(64Ac^2 + 16B(4-c^2)\right)(4-c^2)\Big]$$

Differentiate again with respect to c, we have

$$N'(c) = \frac{(\alpha+\beta)^2}{1024}\Big[8(4A-B)(8c-4c^2) - 256A(4-3c^2) + +2\left(128Ac+16B(-2c)\right)(4-c^2) - 4\left(64Ac^2+16B(4-c^2)\right)c\Big].$$

Now $N'(c) < 0$ for $c \in [0,2]$, so maximum value occurs at $c = 0$. Thus we obtain

$$\left|a_2a_4 - a_3^2\right| \le \frac{(\alpha+\beta)^2}{4(\alpha+3\beta)^2}.$$

This result is sharp for the functions

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z} + \frac{\beta}{\alpha+\beta}f'(z) = \sqrt{1+z}$$

or

$$\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z} + \frac{\beta}{\alpha+\beta}f'(z) = \sqrt{1+z^2}.$$

Theorem 2.4. Let $f(z) \in \mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then

$$\left|a_2a_3 - a_4\right| \le \frac{(\alpha+\beta)}{2(\alpha+4\beta)}.$$

This result is sharp.

Proof. By similar arguments as used in the last theorem, we get the required result.

Using similar method as in [3], we get the following Lemma.

Theorem 2.5. Let $f(z) \in \mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then

$$|a_2| \le \frac{(\alpha+\beta)}{2(\alpha+2\beta)}, \quad |a_3| \le \frac{5(\alpha+\beta)}{8(\alpha+3\beta)},$$

$$|a_4| \le \frac{13(\alpha+\beta)}{16(\alpha+4\beta)}, \quad |a_5| \le \frac{29(\alpha+\beta)}{32(\alpha+5\beta)}.$$

These estimations are sharp.

Proof. From (1.1), we can write

$$\left(\frac{\alpha}{\alpha+\beta}\frac{f(z)}{z} + \frac{\beta}{\alpha+\beta}f'(z)\right)^2 = 1 + w(z),$$

or, equivalently

$$\left(1 + \sum_{n=2}^{\infty}\frac{(\alpha+n\beta)}{(\alpha+\beta)}a_n z^{n-1}\right)^2 = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

Comparing the coefficients of like powers and then simple computation gives the required result.

Theorem 2.6. Let $f(z) \in \mathcal{RS}(\alpha,\beta)$ and of the form (1.1). Then

$$\left|H_3(1)\right| \le \frac{(\alpha+\beta)^2}{32}\left[\frac{5(\alpha+\beta)}{(\alpha+3\beta)^3} + \frac{13}{(\alpha+4\beta)^3} + \frac{29}{2(\alpha+3\beta)(\alpha+5\beta)}\right].$$

Proof: Since

$$\left|H_3(1)\right| \le |a_3|\left|\left(a_2a_4 - a_3^2\right)\right| + |a_4|\left|a_2a_3 - a_4\right| + |a_5|\left|a_1a_3 - a_2^2\right|.$$

By using Corollary 2.1, Theorem 2.3, Theorem 2.4 and Theorem 2.5, we obtain the required result.

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