LEFT CENSORED NAKAGAMI DISTRIBUTION UNDER BAYESIAN FRAMEWORK

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ABSTRACT: *The failure processes can be modeled using Nakagami distribution with a reasonable flexibility. The data regarding attenuation of wireless signals, traversing multiple paths, deriving unit hydrographs in hydrology, medical imaging studies etc can be modeled under Nakagami distribution. In this paper, the scale parameter of the Nakagami distribution has been estimated under Bayesian inference using left censored samples. As the Bayes estimators cannot be obtained in the closed form; we have used mathematica software to obtain the numerical estimates. It can be concluded that the performance of the PLF and inverse gamma prior is better to estimate the scale parameter of the distribution.*

1. INTRODUCTION

Nakagami distribution can be considered as a flexible life time distribution. It was primarily proposed for modeling the fading of radio signals. Although, the model may also offer a good fit to some failure time data. It has been used to model attenuation of wireless signals traversing multiple paths, fading of radio signals, data regarding communicational engineering, high-frequency seismogram envelopes. The distribution may also be employed to model failure times of a variety of products (and electrical components) such as ball bearing, vacuum tubes, electrical insulation. It is also widely considered in biomedical fields, such as to model the time to the occurrence of tumors and appearance of lung cancer. This distribution is extensively used in reliability theory, reliability engineering and to model the constant hazard rate portion because of its memory less property. Moreover, it is very convenient because it is so simple to add failure rates in a reliability model.

1.1 Probability Density Function of Nakagami Distribution

The probability density function of the distribution is given as:

$$f\left(x;\theta\right)=\frac{2\lambda^{\lambda}x^{2\lambda-1}}{\Gamma\left(\lambda\right)\theta^{\lambda}}\exp\left[\frac{-\lambda x^{2}}{\theta}\right],\qquad \theta>0$$

Where, $\lambda \geq 0.5$ is the shape parameter and $\theta > 0$ is scale parameter. It collapses to Rayleigh distribution when $\lambda = 1$ and half normal distribution $\lambda = 0.5$.

The cumulative distribution function of the distribution is:

$$F\left(x;\theta\right)=P\left(\lambda,\frac{\lambda x^{2}}{\theta}\right)$$

where *P* is the incomplete gamma function (regularized).

The real life applications of the distribution can be seen from the contributions of: [1,2,3,4,5]. The papers considering the classical/frequentist analysis of the distribution include: [6,7, 8, 9, 10, 11,12].

The above discussion suggests that the distribution is still waiting for a significant attention of Bayesian statisticians especially in case of censored samples. We have considered the Bayesian analysis of the scale parameter of the distribution under left censored samples.

The authors considering the Bayesian analysis of the probability distributions include: [13,14,15,16,17, 18,19, 20 , 21,22].

2. MATERIAL AND METHODS

This section covers the material and methods for the study.

2.1 Informative and Uninformative Priors

Following informative and non-informative priors have been used for analysis of the scale parameter of the Nakagami distribution.

2.1.1 Inverse Gamma Prior

The inverse gamma can be presented as:

$$P\left(\theta\right)=\frac{b^{-c}\theta^{-c-1}}{\Gamma\left(c\right)}\exp\left[\frac{-b}{\theta}\right],\qquad \theta>0$$

2.1.2 Uniform Prior

One of the most famous non-informative priors is a uniform prior, it can be given as:

$$P\left(\theta\right)=K$$

2.2 Derivation Of Posterior Estimates

This section contains the derivation of the Bayes estimators and posterior risks under different priors and loss functions.

2.2.1 Likelihood Function under Left Censored Sample

Let $X_{r+1},\ldots,X_{n}$ be last n – r order statistics from a sample of size n from Nakagami distribution. Then the likelihood function for the $X_{r+1},\ldots,X_{n}$ left censored observations is:

$$L\left(\theta|\mathbf{x}\right)\propto\left\{F\left(x_{r+1}\right)\right\}^{r}\prod_{i=r+1}^{n}f\left(x_{i}\right)$$

$$L\left(\theta|\mathbf{x}\right)\propto\left\{P\left(\lambda,\frac{\lambda x_{r+1}^{2}}{\theta}\right)\right\}^{r}\theta^{-\lambda(n-r)}\,\mathrm{e}^{\frac{-\lambda\sum_{i=1}^{r+1}x_{i}^{2}}{\theta}}$$

2.2.2 Posterior Distribution using Uniform Prior

The posterior distribution under the assumption of Uniform prior is:

$$g(\theta|\mathbf{x}) = \frac{\left\{P\left(\lambda, \frac{\lambda x_{r+1}^2}{\theta}\right)\right\}^r \theta^{-\lambda(n-r)} e^{\frac{-\lambda \sum_{i=1}^{r+1} x_i^2}{\theta}}}{\int_0^\infty \left\{P\left(\lambda, \frac{\lambda x_{r+1}^2}{\theta}\right)\right\}^r \theta^{-\lambda(n-r)} e^{\frac{-\lambda \sum_{i=1}^{r+1} x_i^2}{\theta}} d\theta},$$

$\theta > 0$

2.2.3 Posterior Distribution using Inverse Gamma Prior

$$g(\theta|\mathbf{x}) = \frac{\left\{P\left(\lambda, \frac{\lambda x_{r+1}^2}{\theta}\right)\right\}^r \theta^{-\lambda(n-r)-c-1} e^{\frac{-\left(\lambda \sum_{i=1}^{r+1} x_i^2 + b\right)}{\theta}}}{\int_0^\infty \left\{P\left(\lambda, \frac{\lambda x_{r+1}^2}{\theta}\right)\right\}^r \theta^{-\lambda(n-r)-c-1} e^{\frac{-\left(\lambda \sum_{i=1}^{r+1} x_i^2 + b\right)}{\theta}} d\theta}$$

$, \theta > 0$

2.2.3 Bayesian Estimation under Different Loss Functions

In Bayesian analysis the comparisons among different estimators are made on the basis of loss functions. We have used following loss functions for the derivations of Bayes estimates and corresponding posterior risks.

2.2.3.1 Squared Error Loss Function (SELF)

The squared error loss function is defined as: $L(\theta, \theta_{SELF}) = (\theta - \theta_{SELF})^2$. The Bayes estimator under this loss function is: $\theta_{SELF} = \mathrm{E}(\theta)$. The risk associated with this loss function obtained by using the formula: $\rho(\theta_{SELF}) = \mathrm{E}(\theta^2) - \{\mathrm{E}(\theta)\}^2$

2.2.3.2 Precautionary Loss Function (PLF)

The precautionary loss function (PLF) can be presented as:

$$L(\theta_{PLF}, \theta) = \frac{(\theta_{PLF} - \theta)^2}{\theta_{PLF}}$$

The formulas for Bayes estimate and corresponding posterior risk under PLF are as under:

The Bayes estimator under PLF is: $\theta_{PLF} = \left\{\mathrm{E}(\theta^2)\right\}^{\frac{1}{2}}$

The posterior risk of the Bayes estimator under PLF is:

$\rho(\theta_{PLF}) = 2\{\theta_{PLF} - \mathrm{E}(\theta)\}$

The Bayesian estimators under both priors cannot be obtained in a closed form; we have obtained Bayes estimates numerically using mathematica software.

3. RESULTS AND DISCUSSIONS

A simulation study has been conducted to evaluate the behavior and performance of different estimators. A comparison in terms of magnitude of posterior risks is needed to check whether an estimator is inadmissible under some loss function or prior distribution. The samples have been simulated for n = 5, 20, 40 and 100. using $(\lambda, \theta) \in \left\{ \begin{matrix} (1,0.1),(1,1),(1,10),(1,100), \\ (2,0.1),(2,1),(2,10),(2,100) \end{matrix} \right\}$ under 10000 replications. The Bayes estimates have been obtained numerically using mathematica software. The resultant Bayes estimates and posterior risks under different priors and loss functions are presented in the tables below.

3.1 Simulation Results for Bayesian Estimates under Uniform Prior

The Bayesian estimates and their posterior risks under uniform prior are presented in the following tables.

Table 3.1.1: Bayesian estimates under uniform prior using SELF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.1645	1.6397	16.6701	165.8758	0.1249	1.2917	12.5188	126.0779
20	0.1099	1.0985	11.0977	110.2445	0.1046	1.0789	10.7941	106.0958
50	0.1040	1.0378	10.5494	104.5684	0.1019	1.0345	10.4965	103.0914
100	0.1009	1.0070	10.2243	101.4481	0.1005	1.0062	10.1311	101.3590

Table 3.1.2: Bayesian estimates under uniform prior using PLF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.2015	2.0082	20.4165	203.1533	0.1335	1.3808	13.3835	134.7785
20	0.1131	1.1303	11.4195	113.4404	0.1061	1.0934	10.9389	107.5173
50	0.1054	1.0517	10.6909	105.9698	0.1026	1.0412	10.5642	103.7619
100	0.1014	1.0121	10.2773	101.9724	0.1007	1.0088	10.1568	101.6155

Table 3.1.3 : Posterior Risks under uniform prior using SELF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.0192	0.2126	1.9372	14.9916	0.0119	0.1103	3.5453	7.5109
20	0.0036	0.0741	1.0953	14.3666	0.0015	0.0301	0.4639	5.9735
50	0.0015	0.0297	0.4437	5.8021	0.0009	0.0135	0.2085	2.6767
100	0.0007	0.0106	0.1569	2.0516	0.0004	0.0050	0.0770	1.0058

Table 3.1.4 : Posterior Risks under uniform prior using PLF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.0163	0.1614	1.6747	6.5667	0.0085	0.0875	0.8479	8.5393
20	0.0031	0.0312	0.3157	3.1354	0.0014	0.0142	0.1421	1.3963
50	0.0014	0.0137	0.1388	1.3762	0.0007	0.0066	0.0664	0.6553
100	0.0005	0.0050	0.0516	0.5142	0.0003	0.0025	0.0250	0.2515

Table 3.2.2: Bayesian estimates under inverse gamma prior using SELF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.1388	1.1883	11.3773	109.5740	0.1194	1.1356	10.6808	105.8387
20	0.1092	1.0402	10.4113	102.1286	0.1048	1.0605	10.1680	102.4613
50	0.1043	1.0146	10.2096	101.4380	0.1024	1.0250	10.1305	101.7959
100	0.1013	1.0012	10.1258	100.1475	0.1009	1.0049	10.0569	101.0472

Table 3.2.2: Bayesian estimates under inverse gamma prior using PLF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.1552	1.3285	12.7206	122.5090	0.1259	1.1971	11.2586	111.5652
20	0.1120	1.0672	10.6815	104.7801	0.1062	1.0740	10.2974	103.7669
50	0.1056	1.0275	10.3396	102.7335	0.1031	1.0315	10.1942	102.4412
100	0.1018	1.0062	10.1767	100.6516	0.1012	1.0074	10.0824	101.3037

Table **3.2.3: Posterior Risks under Inverse Gamma Prior Using SELF**

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.0156	0.1943	1.3219	10.9892	0.0110	0.1023	1.9701	5.6761
20	0.0032	0.0587	0.8520	10.9135	0.0014	0.0274	0.3881	5.2425
50	0.0014	0.0266	0.3908	5.1266	0.0005	0.0127	0.1877	2.5255
100	0.0005	0.0101	0.1495	1.9407	0.0002	0.0049	0.0717	0.9807

Table 3.2.4: Posterior Risks under Inverse Gamma Prior Using PLF

n	λ= 1, θ = 0.1	λ= 1, θ = 1	λ= 1, θ = 10	λ= 1, θ = 100	λ= 2, θ = 0.1	λ= 2, θ = 1	λ=2, θ = 10	λ= 2, θ = 100
10	0.0124	0.1777	1.2591	5.6078	0.0068	0.0807	0.6743	4.2961
20	0.0028	0.0239	0.2286	2.1978	0.0011	0.0131	0.1133	1.0283
50	0.0012	0.0105	0.1005	0.9628	0.0004	0.0061	0.0530	0.4839
100	0.0004	0.0039	0.0375	0.3579	0.0001	0.0023	0.0197	0.1861

3.2 Simulation Results for Bayesian Estimates under Inverse Gamma Prior

The Bayesian estimates and their posterior risks under inverse gamma prior are presented in the following tables.

4. CONCLUSION

In this paper, the Bayesian analysis of the scale parameter of the Nakagami distribution has been considered under left censored samples. The numerical posterior estimates have been obtained using mathematica software. From the simulation study, it can be assessed that the estimates are consistent in nature as the posterior risk is inversely proportional to the sample size. The performance of inverse gamma prior seems better than that of uniform prior. Similarly, the posterior risks under PLF are smaller than those under SELF. Hence in case of Bayesian estimation of left censored data from Nakagami distribution, the use of PLF and inverse gamma prior can be preferred. The results are useful for the analysts looking to use the Nakagami distribution under left censored samples in any real life situations.

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