INTERVAL LEFT AND RIGHT RETURNS TO SCALES

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ABSTRACT: *One of the discussions of data envelopment analysis (DEA) is the returns to scale. Regarding the fact that in the real life issue we usually face undetermined data, the study of phenomena with determined data does not seem rational. In this regard, considering the data as interval has special importance. Therefore, the purpose of this study is to extend the methods of determining the types of left and right returns to scales in DEA into inexact interval concepts. In this study, we extend the represented models by Eslami and Khoveyni [1] in determining the types of left and right returns to scales at the presence of the Interval Data. First the efficient units are evaluated in pessimistic and optimistic situations. Then we determine the types of left and right returns to scales at the two situations on the efficient units. Finally an example is provided to further explain the method.*

Keywords: Data Envelopment Analysis, Left and Right Returns to Scales, Interval Data, Interval Efficiency

INTRODUCTION

Charles *et al*. [2] extended Farrell's view [3] on the evaluation of performance and represented a model which is able to measure the efficiency of decision making units (DMUs) with multiple inputs and outputs. The presented method became known as data envelopment analysis (DEA). Banker *et al.* [4] presented another model that performed the analysis of units through the assumption of variable returns to scale which is known as the BCC model.

Returns to scale represents the link between changes in inputs and outputs of a system. One of the potentials of the DEA is the use of different models corresponding to different returns to scale and measuring returns to scale of units. Constant returns to scale means that any multiplication of input produces the same multiplied output. CCR model [2] supposes constant returns to scale for DMUs. Therefore the small and big units are compared together. Variable returns to scale, means that any multiplication of input produces the same, more or less multiplied output. Banker et al. model (BCC model) assumes variable returns to scale (upward, stable or downward). Banker [5] estimated returns to scale of a unit using the optimal solution of the CCR model. Banker and Thrall [6] estimated returns to scale of a DMU through solving dual model of BCC and gaining u_p^*. In these methods returns to scale of the extreme efficient units cannot be estimated. Golany and Yu [7] presented a method that estimated left and right returns to scales of efficient units and estimated the returns to scale of inefficient DMU units through picturing them on the efficient frontier.

One of the research topics in returns to scale is left and right return to scale. Right returns to scale is analyzed by the feature of hyperplane that passes the unit which is defined by increasing the level of changes in the inputs and outputs of the unit and left returns to scale is analyzed by the feature of hyperplane that passes the unit which is defined by decreasing the level of changes in the inputs and outputs of the unit. Golany and Yu [7] when studying the theory and analyzing return on scale in DEA presented models through which it is possible to determine the left and right returns to scales. Of course the mentioned research is not always feasible for all units. Jahanshahloo and Soleimani-damaneh [8] presented the basic concepts and methods of calculating returns to scale with BCC model. Khodabakhshi *et al.* [9] proved theorems for determining left and right returns to scales being in the design of models. Eslami and Khoveyni [1] presented a method that measures left and right returns to scales for all efficient units. One of the advantages of the proposed method is to determine the types of left and right returns to scales is that the proposed method is performable for all target DMUs while the method of Golany and Yu [7] is not always performable. Another advantage of Eslami and Khoveyni [1] is that it is possible to measure the left and right returns to scale values of efficient DMUs.

The term inaccurate data in DEA means that the inputs and outputs cannot be precisely obtained due to uncertainty. The only thing we know is that they are all defined within the upper bound and lower bound given by intervals. Copper *et al.* [10] discussed about interval data and considered the combination of ordinal data and interval data as inaccurate data and presented the IDEA (Inaccurate DEA). Thus, the efficiency should be inaccurate. Despotis *et al*. [11] achieved the interval efficiency through interval data. Here with the inspiration from Eslami and Khoveyni [1] who presented a method in determining the types of left and right returns to scales, we will extend their method at the presence of interval data.

This paper is arranged as follows: In section 2 we introduce the methods of determining the types of left and right returns to scale. Next after reviewing the data envelopment analysis of interval data the left and right returns to scales at the presence of interval data is introduced. Finally an example will be provided to explain the proposed method. The final section is related to the conclusion and suggestions for further studies.

TYPES OF LEFT AND RIGHT RETURNS TO SCALES

In order to determine different types of left and right returns to scales BCC efficient units, consider n DMU each $DMU_j\ (j=1,\ldots,n)$ having m input $x_j=\left(x_{1j},\ldots,x_{ij},\ldots,x_{mj}\right)$ and s output $y_j=\left(y_{1j},\ldots,y_{rj},\ldots,y_{sj}\right)$ and $y_j\neq 0$ and $x_j\neq 0$.

Different types of left and right returns to scales

It is assumed that $\left(U^*, V^*, u_p^*\right)$ has an optimal solution in the below BCC model.

$$
\begin{aligned}
& Max \;\; U^t y_p + u_p, \\
& s.t \;\; U^t y_j - V^t x_j + u_p \le 0, \; j = 1, \dots, n, \\
& V^t x_p = 1, \\
& U \ge 0, \\
& V \ge 0, \qquad (1)
\end{aligned}
$$

Definition 1: $DMUp$ is called BCC efficient if $\left(U^*, V^*, u_p^*\right)$ applies to $U^{*t} y_p + u_p^* = 1$.

Theorem 2 [1]: Suppose $DMUp$ as $\left(x_p, y_p\right)$ is a point on the BCC efficient frontier, thus:

(i) Right returns to scale $DMUp$ is increasing if and only if $\xi > 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* < 0$.

(ii) Left returns to scale $DMUp$ is increasing if and only if $0 < \xi < 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* > 0$.

(iii) Right returns to scale $DMUp$ is decreasing if and only if $\xi > 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* > 0$.

(iv) Left returns to scale $DMUp$ is decreasing if and only if $0 < \xi < 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* < 0$

(v) Right returns to scale $DMUp$ is constant if and only if $\xi > 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* = 0$

(vi) Left returns to scale $DMUp$ is constant if and only if $0 < \xi < 1$ as: $U^{*t}\left(\xi y_p\right) - V^{*t}\left(\xi x_p\right) + u_p^* = 0$.

Now with the assumption that $DMUp$ is an efficient BCC unit in order to determine types of right returns to scale, the following model is presented [1] as:

$$
\begin{aligned}
& Min \;\; t_p, \\
& s.t \;\; U^t y_j - V^t x_j + u_p \le 0, \; j = 1, \dots, n, \; j \ne p, \\
& U^t y_p + u_p = 1, \\
& V^t x_p = 1, \\
& U^t\left(\xi y_p\right) - V^t\left(\xi x_p\right) + u_p + t_p = 0, \\
& \xi > 1, \\
& U \ge 0, \\
& V \ge 0, \qquad (2)
\end{aligned}
$$

Theorem 3 [1]: Model (2) is feasible.

Assuming that $\left(U^*, V^*, u_p^*, \xi^*, t_p^*\right)$ is an optimal solution of model (2), based on the theorem (2) the following modes are obtained:

a) If $t_p^* > 0$, then the right returns to scale $DMUp$ is increasing

b) If $t_p^* < 0$, then the right returns to scale $DMUp$ is decreasing

c) If $t_p^* = 0$, then the right returns to scale $DMUp$ is constant.

In order to determine types of left returns to scale $DMUp$, the following model (3) is proposed:

$$
\begin{aligned}
& Min \;\; t_p, \\
& s.t \;\; U^t y_j - V^t x_j + u_p \le 0, \; j = 1, \dots, n, \; j \ne p, \\
& U^t y_p + u_p = 1, \\
& V^t x_p = 1, \\
& U^t\left(\xi y_p\right) - V^t\left(\xi x_p\right) + u_p + t_p = 0, \\
& 0 < \xi < 1, \\
& U \ge 0, \\
& V \ge 0, \qquad (3)
\end{aligned}
$$

Theorem 4 [1]: Model (3) is feasible.

Assuming that $\left(U^*, V^*, u_p^*, \xi^*, t_p^*\right)$ is an optimal solution based on the theorem (2) the following modes are obtained:

d) If $t_p^* > 0$, then the left returns to scale $DMUp$ is increasing

e) If $t_p^* < 0$, then the left returns to scale $DMUp$ is decreasing

f) If $t_p^* = 0$, then the left returns to scale $DMUp$ is constant.

INTERVAL LEFT AND RIGHT RETURNS TO SCALE

Interval DEA

Definition 5: Consider n decision making units with m inputs and s outputs, suppose that the inputs and outputs of $DMU_j\left(j = 1, \dots, n\right)$ are the interval data that $x_j = \left(x_{1j}, \dots, x_{ij}, \dots, x_{mj}\right)$ and $y_j = \left(y_{1j}, \dots, y_{rj}, \dots, y_{sj}\right)$ are represented as follows:

$$
\begin{aligned}
& x_{ij} \in \left[x_{ij}^l, x_{ij}^u\right], i = 1, \dots, m, \; j = 1, \dots, n, \\
& y_{rj} \in \left[y_{rj}^l, y_{rj}^u\right], r = 1, \dots, s, \; j = 1, \dots, n.
\end{aligned}
$$

in which $y_{rj}^l > 0, x_{ij}^l > 0$.

In order to analyze the decision making units with interval data the following views may be considered:

1- In the pessimistic situations the inputs of $DMUp$ are in the upper bound and the outputs are in the lower bound and other DMUs are in the best situation which means the inputs are in the lower bound and the outputs are in the upper bound.

2- In the optimistic situations the inputs of $DMUp$ are in the lower bound and the outputs are in the upper bound and other DMUs are in the worst situation which means the inputs are in the upper bound and the outputs are in the upper lower.

Model (4) in the pessimistic situation analyzes efficiency as:

$$
\begin{aligned}
&Max\ \ U^t y_p^l + u_p ,\\
&s.t.\ \ U^t y_j^u - V^t x_j^l + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^l - V^t x_p^u + u_p \le 0,\\
&V^t x_p^u = 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (4)
\end{aligned}
$$

Model (5) in the optimistic situation analyzes efficiency as:

$$
\begin{aligned}
&Max\ \ U^t y_p^u + u_p ,\\
&s.t.\ \ U^t y_j^l - V^t x_j^u + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^u - V^t x_p^l + u_p \le 0,\\
&V^t x_p^l = 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (5)
\end{aligned}
$$

Interval left and right returns to scales

Model (2) in the pessimistic situation analyzes the right returns to scale as follows:

$$
\begin{aligned}
&Min\ \ t_p^l ,\\
&s.t.\ \ U^t y_j^u - V^t x_j^l + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^l + u_p = 1,\\
&V^t x_p^u = 1,\\
&U^t\left(\xi y_p^l\right) - V^t\left(\xi x_p^u\right) + u_p + t_p^l = 0,\\
&\xi > 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (6)
\end{aligned}
$$

Suppose that $\left(U^*, V^*, u_p^*, \xi^*, t_p^{l*}\right)$ is an optimal solution of the pessimistic model (6). Then, based on the Theorem 2 we have:

a') If $t_p^{l*} > 0$, then the right returns to scale $DMUp$ is increasing in the pessimistic mode.

b') If $t_p^{l*} < 0$, then the right return to scale $DMUp$ is decreasing in the pessimistic mode.

c') If $t_p^{l*} = 0$, then the right return to scale $DMUp$ is constant in the pessimistic mode.

Model (2) in the optimistic situation that analyzes the right return to scale at best situation is as follows:

$$
\begin{aligned}
&Min\ \ t_p^u ,\\
&s.t.\ \ U^t y_j^l - V^t x_j^u + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^u + u_p = 1,\\
&V^t x_p^l = 1,\\
&U^t\left(\xi y_p^u\right) - V^t\left(\xi x_p^l\right) + u_p + t_p^u = 0,\\
&\xi > 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (7)
\end{aligned}
$$

Suppose that $\left(U^*, V^*, u_p^*, \xi^*, t_p^{u*}\right)$ is an optimal solution of the pessimistic model (7). Then based on the Theorem 2 we have:

a") If $t_p^{u*} > 0$, then the right returns to scale $DMUp$ is increasing in the optimistic mode.

b") If $t_p^{u*} < 0$, then the right returns to scale $DMUp$ is decreasing in the optimistic mode.

c") If $t_p^{u*} = 0$, then the right returns to scale $DMUp$ is constant in the optimistic mode.

Definition 6:

1- Right returns to scale $DMUp$ with interval data is increasing if the right return to scale in the pessimistic and optimistic modes is increasing.

2- Right returns to scale $DMUp$ with interval data is decreasing if the right return to scale in the pessimistic and optimistic modes is decreasing.

3- Right returns to scale $DMUp$ with interval data is constant if the right returns to scale in the pessimistic and optimistic modes is constant.

Model (3) in the pessimistic situation that analyzes the left returns to scale is extended as follows:

$$
\begin{aligned}
&Min\ \ t_p^l ,\\
&s.t.\ \ U^t y_j^u - V^t x_j^l + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^l + u_p = 1,\\
&V^t x_p^u = 1,\\
&U^t\left(\xi y_p^l\right) - V^t\left(\xi x_p^u\right) + u_p + t_p^l = 0,\\
&0 < \xi < 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (8)
\end{aligned}
$$

If $\left(U^*, V^*, u_p^*, \xi^*, t_p^{l*}\right)$ is an optimal solution of the pessimistic model (8). Then based on the theorem 2 we have:

d') If $t_p^{l*} > 0$, then the left returns to scale $DMUp$ is decreasing in pessimistic mode.

e') If $t_p^{l*} < 0$ then the left returns to scale $DMUp$ is increasing in pessimistic mode.

f') If $t_p^{l*} = 0$ then the left returns to scale $DMUp$ is constant in pessimistic mode.

Model (3) in the optimistic situation that analyzes the left returns to scale at best situation is extended as follows:

$$
\begin{aligned}
&Min\ \ t_p^u,\\
&s.t.\ \ U^t y_j^l - V^t x_j^u + u_p \le 0, j = 1,\dots,n,\ j \ne p,\\
&U^t y_p^u + u_p = 1,\\
&V^t x_p^l = 1,\\
&U^t\left(\xi y_p^u\right) - V^t\left(\xi x_p^l\right) + u_p + t_p^u = 0,\\
&0 < \xi < 1,\\
&U \ge 0,\\
&V \ge 0, \qquad\qquad (9)
\end{aligned}
$$

If $\left(U^*, V^*, u_p^*, \xi^*, t_p^{u*}\right)$ is an optimal solution of the optimistic model (9). Then based on the Theorem 2 we have:

d") If $t_p^{u*} > 0$, then the left returns to scale $DMUp$ is decreasing in optimistic mode.

e") If $t_p^{u*} < 0$, then the left returns to scale $DMUp$ is increasing in optimistic mode.

f") If $t_p^{u*} = 0$, then the left returns to scale $DMUp$ is constant in optimistic mode.

Definition 7:

1- Left returns to scale $DMUp$ with interval data is increasing if left returns to scale in pessimistic and optimistic modes are increasing.

2- Left return to scale $DMUp$ with interval data is decreasing if left returns to scale in pessimistic and optimistic modes are decreasing.

3- Left return to scale $DMUp$ with interval data is constant if left returns to scale in pessimistic and optimistic modes are constant.

EXAMPLE

In this example we suppose 5 decision making units with 2 inputs and 2 outputs.

Based on the interval data the efficiency of the interval is by obtained models (4) and (5) listed in Table (1). In order to determine the left and right returns to scales in the pessimistic mode, only DMU_5 is analyzed because this decision making unit is efficient in the upper and lower bounds. Also in order to determine the left and right returns to scale in the optimistic mode al DMUs are analyzed except DMU_4. The results are listed in Table (2) and (3).

Based on the data mentioned in Table (2) the lower bound of DMU_5 is increasing to right returns to scale and the upper bound of right returns to scales decreasing. Thus based on definition (6) it is impossible to determine the type of right return to scale for this unit.

Also based on the data in Table (3) the lower bound of DMU_5 is increasing to left returns to scale and the upper bound of left returns to scale is increasing. Based on the definition (7) the left return to scale DMU_5 is increasing.

CONCLUSION

In this article first through considering Death interval data the BCC multiplier model is extended into optimistic and pessimistic modes by the help of which the interval efficiency of the units is obtained. Thus the models of left and right returns to scales by Eslami and Khoveyni [1] are extended into optimistic and pessimistic modes. The results of the example indicate different types of left and right returns to scale at the presence of interval data. The measurement of values of returns to scale with interval data are the future studies of the writers. Also the presented model in this study is analyzable for sequential, stochastic and fuzzy data.

Table 1. Interval data and interval efficiency

DMU	x_1	x_2	y_1	y_2	Efficiency
1	[12,15]	[0.21,0.48]	[138,144]	[21,22]	[0.22,1]
2	[10,17]	[0.10,0.70]	[143,159]	[28,35]	[0.227,1]
3	[4,12]	[0.16,0.35]	[157,198]	[21,29]	[0.823,1]
4	[19,22]	[0.12,0.19]	[158,181]	[21,25]	[0.445,0.907]
5	[14,15]	[0.06,0.09]	[157,161]	[28,40]	[1,1]

Table 2. Interval right returns to scale models (6) and (7)

DMU	$\left[t^{l*},t^{u*}\right]$	$\left[\xi^{l},\xi^{u}\right]$	Optimistic RTS	Pessimistic RTS	Interval Right RTS
1	$[-,-2.406E+9]$	$[-,1.001]$	-	DRS	-
2	$[-,-1.00E+10]$	$[-,1.001]$	-	DRS	-
3	$[-,-1.00E+10]$	$[-,1.001]$	-	DRS	-
4	$[-,-]$	$[-,-]$	-	-	-
5	$[0.168,-1.00E+10]$	$[1.001,1.001]$	IRS	DRS	-

Table 3. Interval left returns to scale Models (8) and (9)

DMU	$\left[t^{l*},t^{u*}\right]$	$\left[\xi^{l},\xi^{u}\right]$	**Optimistic RTS**	**Pessimistic RTS**	**Interval Left RTS**
1	$[-,-1.000]$	$[-,-1.0000E-6]$	-	IRS	-
2	$[-,-1.001]$	$[-,-1.0000E-6]$	-	IRS	-
3	$[-,-1.001]$	$[-,-1.0000E-6]$	-	IRS	-
4	$[-,-]$	$[-,-]$	-	-	-
5	$[-0.733,-1.000]$	$[-1.0000E-6,-1.0000E-6]$	IRS	IRS	IRS

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