QUASI-s-TOPOLOGICAL GROUPS

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ABSTRACT**:** *In this paper, we will introduce and study the concept of quasi s-topological groups.*

quasi s-topological groups are the generalization of quasi topological groups. quasi s-topological group results when the continuity property of left (right) translation and inversion mapping is weakened by semi continuity in lieu of continuity. It is shown that in quasi s-topological group $(G,\circ,\tau)$*, if* $A \in SO(G)$*, then* $A^{-1} \in SO(G)$*. It is proved that if* $(G,\circ,\tau)$*, is quasi s-topological group then* $(G,\circ,\ \tau^{-1})$ *is also quasi s-topological group. Further, quasi s-homeomorphism is defined and showed that left and right translations in quasi s-topological groups are quasi s-homeomorphism.*

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1. INTRPDUCTION

A quasi topological group is one in which left translation $L_x : G \to G$ (right translation: $R_x : G \to G$) and inverse mappings $G \to G : x \to x^{-1}$ are continuous. For a subset A of a topological space X, the symbols $Int(A)$ and $Cl(A)$ are used to denote the interior of A and the closure of A respectively. If $f: X \to Y$ is a mapping between topological spaces X and Y, and B is a subsets of Y, then $f^{\leftarrow}(B)$ denotes the pre-image of B. Our other topological notations and terminology will be standard. If $(G,\circ)$ is a group, then e denotes its identity element, and for a given $x \in G$, $L_x : G \to G$, $y \to x\circ y$, and $R_x : G \to G$, $y \to y\circ x$, denote the left and the right translation by x, respectively. The operation $\circ$ is called the multiplication mapping $m: G \times G \to G$, and the inverse operation $x \to x^{-1}$ is denoted by i. With the invent of semi open sets [1] and semi continuity [1] in topological spaces, many mathematicians generalized concepts on topological spaces. Recall that a subset A of a topological space X is said to be semi-open if there exists an open set U in X such that $U \subseteq A \subseteq Cl(A)$, or equivalently if $A \subseteq cl(Int(A))$. $SO(X)$ denotes the collection of all semi-open sets in X, whereas $SO(X,x)$ is the collection of all semi open sets in X containing $x \in X$. The complement of a semi-open set is said to be semi-closed; the semi-closure [2] of $A \subseteq X$, denoted by $sCl(A)$, is the intersection of all semi-closed subsets of X containing A. $x \in sCl(A)$ if and only if for any semi-open set U containing x, $U \cap A \neq \emptyset$. Clearly, every open (closed) set is semi-open (semi-closed). It is known that the union of any collection of semi-open sets is again a semi-open set, while the intersection of two semi-open sets need not be semi-open. The intersection of an open set and a semi-open set is semi-open. If $A \subseteq X$ and $B \subseteq Y$ are semi-open in spaces X and Y respectively, then $A \times B$ is semi-open in the product space $X \times Y$. Basic properties of semi-open sets are given in [1, 3, 4] and the references therein, and of semi-closed sets and the semi-closure in [2, 4]. A set $U \subseteq X$ is a semi neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that $x \in A \subseteq U$. A set $A \subseteq X$ is semi-open in X if and only if A is a semi neighbourhood of each of its points. If a semi neighbourhood U of a point x is a semi-open set, we say that U is a semi open neighbourhood of x. Semi continuity, irresoluteness, s-open, semi open and pre semi open mappings were defined and studied as a consequence of semi open sets in topological spaces. In present work we will define quasi s-topological groups by introducing semi-open sets and semi-continuity of left (right) translations and semi continuity of inversion mappings.

A mapping $f : X \to Y$ between topological spaces X and Y is called:

(1) semi continuous [1] (resp. irresolute [4]) if for each open (resp. semi open) set $V \subseteq Y$ the set $f^{-1}(V)$ is semi open in X. Equivalently, the mapping f is semi continuous (irresolute) if for each $x \in X$ and for each open (semi open) neighbourhood V of $f(x)$, there exists a semi open neighbourhood U of x such that $f(U) \subseteq V$. Every irresolute mapping is semi continuous; (2) pre semi open [4] if for every semi open set A of X, $f(A)$ is semi open in Y;

(3) semi open [4] if for every open set A of X, $f(A)$ is semi open in Y;

(4) semi homeomorphism [4] if f is bijective, irresolute and pre semi open;

(5) S-homeomorphism [5] if f is bijective, semi-continuous and pre semi open. Every semi homeomorphism is an S- homeomorphism.

2. quasi s-topological groups

In this section, we will define quasi s-topological group and investigate its properties. The concept of quasi s-topological groups is the generalization of the concept of quasi-topological groups. quasi s-topological group results when the continuity properties of inverse mapping and left, right translation mapping in quasi topological groups are weakened by semi continuity in lieu of continuity. We will show that every quasi topological group is quasi s-topological group, whereas quasi s-topological group may not be a quasi topological group. This means that the notion of quasi s-topological group is weaker than that of quasi topological group. For quasi topological groups, we refer two beautiful monographs [6, 7]. Note that one kind of generalization of topologized groups by using semi-open sets and semi continuity was initiated by Bohn and Lee [8], and further it has been recently extended by Siddique et. al. [5] and Cao et al [9]. The other kind of generalization is a generalized topological group which has been initiated by Murad et al. [10-13].

2.1 Definition

$(G, \circ, \tau)$ is quasi s-topological group if $(G, \circ)$ is a group, (G, τ) is a topological space and left translation $L_x : G \to G$ for all $x \in G$ and right translation $R_x : G \to G$ for all $x \in G$ are semi-continuous and the mapping of inversion $i : G \to G$ defined by $i(x) = x^{-1}$ is semi-continuous on G.

Since every continuous function is semi-continuous, therefore every quasi topological group is a quasi s-topological group. Thus the notion of quasi s-topological group is a generalization of quasi topological group.

2.2 Example

Any group with the discrete topology, or indiscrete topology, is a topological group, hence quasi s-topological group.

2.3 Example

The set $G = \{-1,1\}$ is a group under usual multiplication. Let topology on G be $\tau = \{\emptyset, G, \{1\}\}$. Then $(G, \circ, \tau)$ is neither quasi topological group nor quasi s-topological group.

2.4 Theorem

Let $(G, \circ, \tau)$ be a quasi s-topological group and β_e be the base at identity element e of G. Then:

(i) for every $U \in \beta_e$, there is an element $V \in SO(G,e)$ such that $V^{-1} \subseteq U$.

(ii) For every $U \in \beta_e$, there is an element $V \in SO(G,e)$ such that $V \circ x \subseteq U$, and $x \circ V \subseteq U$, for each $x \in U$.

Proof

(i) Since $(G, \circ, \tau)$ is a quasi s-topological group. Therefore, for every $U \in \beta_e$ there exists $V \in SO(G,e)$ such that $i(V) = V^{-1} \subseteq U$ because the inverse mapping $i: G \to G$ is semi continuous.

(ii) Since $(G, \circ, \tau)$ is a quasi s-topological group. Thus, for each open set U containing x, there exists $V \in SO(G,e)$ such that $R_x(V) = V \circ x \subseteq U$. Similarly, $L_x(V) = x \circ V \subseteq U$.

2.5 Lemma

Let A be a subset of a quasi s-topological group $(G, \circ, \tau)$. Then $sCl(A)^{-1} \subseteq Cl(A^{-1})$.

Proof

Let $x \in \left(sCl(A)\right)^{-1}$ and let U be an open neighbourhood of x. Then, U^{-1} is a semi open neighborhood of x^{-1}. Since $x^{-1} \in sCl(A)$, therefore, $U^{-1} \cap A \neq \emptyset$. This implies that $U \cap A^{-1} \neq \emptyset$. That is $x \in Cl(A^{-1})$ and so $(sCl(A))^{-1} \subseteq Cl(A^{-1})$.

2.6 Remark

If $(G, \circ, \tau)$ is a quasi topological group, then τ^{-1} is also a topology on G, called conjugate topology of τ. In [14], Josefa et al. pointed out that: If (G, τ) is a quasi topological group, then so is (G, τ^{-1}).

Note that: $\tau^{-1} = \{A \subseteq G : A^{-1} \in \tau\}$. We give the following important result:

2.7 Theorem

Let $(G,\ \tau)$ be a quasi s-topological group. If U is semi open set in $(G,\ \tau)$, then U^{-1} is semi open in $(G,\ \tau^{-1})$.

Proof

As we know that if τ is a topology then τ^{-1} is also a topology. Let $U \in SO\,(G,\tau)$ then, there exists, an open set $O \in \tau$ such that $O \subseteq U \subseteq Cl\,(O)$ or $O^{-1} \subseteq U^{-1} \subseteq (Cl\,(O))^{-1}$. This gives $O^{-1} \subseteq U^{-1} \subseteq (Cl(O)^{-1})$. Now $O^{-1} \in \tau^{-1}$ implies that $U^{-1} \in SO\,(G,\ \tau^{-1})$. Hence U^{-1} is semi open in $(G,\ \tau^{-1})$.

2.8 Theorem

If $(G,\ \circ,\ \tau)$ is quasi s-topological group, then $(G,\ \circ,\ \tau^{-1})$ is also a quasi s-topological group.

Proof

Since $(G,\ \circ)$ is a group, $(G,\ \tau)$ is a topological space. Therefore, $(G,\ \tau^{-1})$ is again a topological space. We need to prove that:

$i\colon(G,\tau^{-1}) \rightarrow (G,\tau^{-1})$, $L_x : (G,\tau^{-1}) \rightarrow (G,\tau^{-1})$ and $R_x : (G,\tau^{-1}) \rightarrow (G,\tau^{-1})$ are semi-continuous mappings.

First, we show that L_x is semi continuous. For this, let V be an open set in (G,τ^{-1}). Then $V^{-1} = U$ is open in $(G,\ \tau)$. Since $(G,\ \tau)$ is quasi s-topological group, therefore, left (right) translation is semi-continuous. Hence $r_x^{\leftarrow}(U) \in SO(G,\ \tau)$. Or $(U \circ x^{-1})^{-1} \in SO(G,\ \tau^{-1})$. Or $U \circ x^{-1} = V^{-1} \circ x^{-1} = (x \circ V)^{-1} = L_x^{\leftarrow}(V) \in SO(G,\tau^{-1})$. This proves that $L_x : (G,\tau^{-1}) \rightarrow (G,\tau^{-1})$ is semi-continuous for every $x \in G$. Similarly we can prove that right translation $R_x : (G,\tau^{-1}) \rightarrow (G,\tau^{-1})$ is semi-continuous. Trivially $i\colon(G,\tau^{-1}) \rightarrow (G,\tau^{-1})$ is continuous and hence semi continuous. Hence $(G,\ \circ,\ \tau^{-1})$ is also a quasi s-topological group.

2.9 Theorem

If H is a discrete subgroup of a quasi s-topological group $(G,\ \circ,\ \tau)$, then $sCl(H)$ is a subgroup of G.

Proof

Let $x, y \in sCl(H)$. If U and V are respective open neighbourhoods of x and y, then $L_{x^{-1}}(U) = x^{-1} \circ U$ and $L_{y^{-1}}(V) = y^{-1} \circ V$ are semi open neighbourhoods of e. Since H is a discrete subgroup of a quasi s-topological group G, therefore $x^{-1} \circ U \cap H \neq \emptyset$ and $y^{-1} \circ V \cap H \neq \emptyset$. Therefore,

$(x \circ y^{-1} \circ x^{-1} \circ U \cap x \circ y^{-1} \circ H) \cup (x \circ y^{-1} \circ y^{-1} \circ V \cap x \circ y^{-1} \circ H) \neq \emptyset$

and distributivity implies,
$(x \circ y^{-1} \circ x^{-1} \circ U \cup x \circ y^{-1} \circ y^{-1} \circ V) \cap x \circ y^{-1} \circ H = \emptyset$.

That is, $W \cap x \circ y^{-1} \circ H \neq \emptyset$, where
$W = x \circ y^{-1} \circ x^{-1} \circ U \cup x \circ y^{-1} \circ y^{-1} \circ V$

is a semi open neighbourhood of $x \circ y^{-1}$. Thus for each $x, y \in sCl(H)$ implies that $x \circ y^{-1} \in sCl(H)$. Hence $sCl(H)$ is a subgroup of G.

2.10 Corollary

If H is a discrete subgroup of a quasi s-topological group $(G,\ \circ,\ \tau)$, then $Cl(H)$ is a subgroup of G.

2.11 Theorem

Let $(G,\ \circ,\ \tau)$ be a quasi s-topological group. If A is open in G, then $A \circ B$ and $B \circ A$ are semi-open in $(G,\ \circ,\ \tau)$ for any subset B of G.

Proof

Let $x \in B$ and $z \in A \circ x$ we show that z is semi-interior point of $A \circ x$. Let $z = y \circ x$ for some $y \in A = A \circ x \circ x^{-1}$. This implies that $y = z \circ x^{-1}$. Now $R_{x^{-1}} : G \rightarrow G$ is semi-continuous, i.e for every open set containing $R_{x^{-1}}(z) = z \circ x^{-1} = y$, there exists a semi-open set M_z containing z such that $R_{x^{-1}}(M_z) \subseteq A$. This implies $M_z \circ x^{-1} \subseteq A$ or $M_z \subseteq A \circ x$. This implies z is semi-interior point of $A \circ x$. Thus $A \circ x$ is semi-open. This implies $A \circ B = \bigcup_{x \in B} A \circ x$ is semi-open in $(G,\ \circ,\ \tau)$. Similarly we can prove that for every open set A of G and arbitrary subset B of G, $B \circ A$ is semi-open in a quasi s-topological group $(G,\ \circ,\ \tau)$.

3. quasi s-homeomorphism

3.1 Definition

A bijective mapping $f : (X,\ \tau_X) \rightarrow (Y,\ \tau_Y)$, is called quasi s-homeomorphism if it is semi-continuous and semi-open.

3.2 Theorem

Let $(G,\ \circ,\ \tau)$ be a quasi s-topological group. Then each left (right) translation $L_x : G \rightarrow G$, $(R_x : G \rightarrow G)$ is a quasi s-homeomorphism.

Proof

Since $(G, \circ, \tau)$ is quasi s-topological group.Therefore $L_x : G \rightarrow G$ is semi-continuous by definition , so it is enough to show that $L_x : G \rightarrow G$ is semi-open.

Let V be an open set in G.Then by Theorem 2.11,

$L_g(V) = g \circ V \in SO(G)$. Hence $L_x : G \rightarrow G$ is a semi-open mapping. This completes the proof.

3.3 Theorem

Suppose that a subgroup H of a quasi s-topological group $(G, \circ, \tau)$ contains a non-empty open subset of G.Then H is semi-open in G.

Proof

By Theorem 3.2, for every $g \in H$, $R_g : G \rightarrow G$ is quasi s-homeomorphism. Let U be a non-empty open subset of G, with $U \subseteq H$, then for every $g \in H$, the set $R_g(U) = U \circ g$ is semi-open in $(G, \circ, \tau)$. Now $H = \cup\{U \circ g : g \in H\}$ is semi-open in G being union of semi-open sets of G. This completes the proof.

4. quasi s-homogeneous

4.1 Definition

A topological space (G, τ) is said to be quasi s-homogeneous if for all $x, y \in G$, there is a quasi s-homeomorphism f of the space G onto itself such that $f(x) = y$.

4.2 Theorem

If $(G, \circ, \tau)$ is a quasi s-topological group, then every open subgroup of G is also semi-closed.

Proof.

Since $(G, \circ, \tau)$ is a quasi s-topological group and H is an open subgroup of G. Then any left or right translation

$$x \circ H \quad \text{or} \quad H \circ x$$

is semi-open for each $x \in G$. So $Y = \{x \circ H : x \in G\}$ of all left cosets of H in G forms a partition of G. Thus Y is a semi open covering of G by disjoint semi open sets of G. This gives $G - H$ is union of semi open sets and hence semi open. This proves that H is semi-closed.

4.3 Corollary

Every quasi s-topological group is a quasi s- homogeneous space.

Proof

Let us take elements x and y in $(G, \circ, \tau)$ and put $z = x^{-1} \circ y$. Since $R_x : G \rightarrow G$ is a quasi s-homeomorphism of $(G, \circ, \tau)$ and $R_z(x) = x \circ z = x \circ (x^{-1} \circ y) = e \circ y = y$. Hence $(G, \circ, \tau)$ is quasi-s homogeneous space.

4.4 Lemma [4]

If $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is semi-continuous and H is open subset of X, then $f/_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$ is semi-continuous.

4.5 Theorem

Every open subgroup H of a quasi s-topological group $(G, \circ, \tau)$ is also a quasi s-topological group (called quasi s-topological subgroup) of G.

Proof

Let $(G, \circ, \tau)$ be a quasi s-topological group and H be an open subgroup of G. We need to prove that $(H, \circ, \tau_H)$ is a quasi s-topological group. For this, we show that $i : H \rightarrow H$, $L_x : H \rightarrow H$ and $R_x : H \rightarrow H$ are semi-continuous with respect to the relative topology. Since H is an open subgroup of G.Then by Lemma 4.4, $i_H : (H, \circ, \tau_H) \rightarrow (H, \circ, \tau_H)$, $L_H : (H, \circ, \tau_H) \rightarrow (H, \circ, \tau_H)$ and $R_H : (H, \circ, \tau_H) \rightarrow (H, \circ, \tau_H)$ are semi-continuous. This proves that $(H, \circ, \tau_H)$ is a quasi s-topological group.

4.6 Theorem

Let $f : (G, \circ, \tau_G) \rightarrow (H, \circ, \tau_H)$ be a homomorphism of quasi s-topological groups. If f is irresolute at the neutral (identity) element e_G, then f is semi continuous on G.

Proof

Let $x \in G$ be an arbitrary element. Suppose that W is an open neighbourhood of $y = f(x) \in H$. Since the left translation in H is a semi continuous mapping, there is a semi open neighbourhood V of the neutral element e_H of H such that $L_y(V) = y \circ V \subseteq W$. Since f is irresolute at e_G, therefore, $f(U) \subseteq V$ for some semi open neighbourhood U of e_G, in G. Since $f(U) \subseteq V$, now $y \circ f(U) \subseteq y \circ V \subseteq W$. This implies $(x \circ U) \subseteq W$. Since $(G, \circ, \tau_G)$ is a quasi s-topological group therefore, $x \circ U$ is semi open in G. This proves that f is semi continuous at x.

Since x was the arbitrary element of G. This completes the proof.

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