

# HADAMARD’S TYPE INEQUALITIES FOR FUNCTIONS WHOSE ABSOLUTE VALUES OF FIRST DERIVATIVE ARE<sub>s</sub>-CONVEX

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**ABSTRACT:** In this paper, we find new inequalities for functions whose absolute values of first derivative are *s*-convex and deduce some relations related to special mean.

**Keywords:** Integral Inequalities; Hermite-Hadamard Inequality; *s*-convex function; Beta and incomplete Beta Functions; Holder’s Inequality; Means for real numbers.

## 1. INTRODUCTION

The term “convex” stems from a result obtained by Hermite in 1881 and published in 1883 as a short note in *Mathesis*, a journal of elementary mathematics. In a letter sent on November 22, 1881, to the journal *Mathesis* (and published there two years later), Ch. Hermite noted that every convex function  $f : [a, b] \rightarrow \mathbf{R}$  satisfies the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

The left-hand side inequality was rediscovered ten years later by J. Hadamard. Nowadays, the double inequality (1.1) is called *the Hermite-Hadamard inequality*. The interested reader can find its complete story in [2,3,7,8]. Number of articles are published on generalization, refinements and extensions of inequality (1.1).

We include some basic definitions which will be necessary in the forthcoming theorems.

**Definition 1.1[4]** Let  $X$  be a convex set in a real vector space. A function  $f : X \rightarrow \mathbb{R}$  is said to be convex if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

$\forall x_1, x_2 \in X$  and  $t \in [0,1]$ .

**Definition 1.2[4]** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is said to be *s*-convex (in second sense) if,

$$f(tx_1 + (1-t)x_2) \leq t^s f(x_1) + (1-t)^s f(x_2)$$

holds  $\forall x_1, x_2 \in [0, \infty)$ ,  $t \in [0,1]$  and some fixed  $s \in (0,1)$ .

The class of *s*-convex (in second sense) functions are usually denoted by  $k_s^2$ . It can be easily seen that for  $s = 1$ , *s*-convexity reduces to ordinary convexity.

Dragomir [1] has extended inequality (1.1) as:

**Theorem 1.1.** Let  $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function on  $I^0$ , and  $a, b \in I^0$  with  $b > a$ . If  $|f'(x)|$  is convex on  $[a, b]$ , then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|) \tag{1.2}$$

Another result has been published in [1] as:

**Theorem 1.2.** Let  $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function on  $I^0$ ,  $a, b \in I^0$  with  $b > a$  and  $p > 1$ . If mapping

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{q}}} \left( |f'(x)|^{\frac{p}{p-1}} + |f'(x)|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \tag{1.3}$$

U. S. Kirmaci in [5] refine inequality (1.1) as:

**Theorem 1.3.** Let  $f : I \subset [0, \infty)$  be a differentiable mapping on  $I^0$  and  $a, b \in I$  with  $b > a$ , then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[ \frac{2+(\frac{1}{2})^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \tag{1.4}$$

And also in [5]:

**Theorem 1.4.** Let  $f : I \subset [0, \infty)$  be a differentiable mapping on  $I^0$  and  $a, b \in I$  with  $b > a$ . If  $|f'(x)|^q$  is *s*-convex on  $[a, b]$  for some fixed  $s \in (0,1)$  and  $q > 1$ , then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left[ \frac{1}{s+1} \right]^{\frac{1}{q}} \left\{ \left[ |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{\frac{1}{q}} + \left[ |f'\left(\frac{a+b}{2}\right)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right\} \tag{1.5}$$

In this paper, we find new inequalities for those functions whose absolute values of first derivative are *s*-convex and apply these inequalities to discover inequalities for special mean.

## 2. MAIN RESULT

We use the following lemma, to introduce our main results.

**Lemma 2.1 ([6]):** Let  $I$  be an interval and  $f: I \rightarrow \mathbb{R}$  be a differentiable on  $I^0$ ,  $a, b \in I$  with  $a < b$  and  $\lambda, \mu \in \mathbb{R}$ . If  $f' \in L^1[a, b]$  then the following inequality holds:

$$(1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx = (b - a) \left\{ \int_0^{1/2} (\lambda - t)f'(ta + (1 - t)b) dt + \int_{1/2}^1 (\mu - t)f'(ta + (1 - t)b) dt \right\} \tag{2.1}$$

**Theorem 2.1:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a, 1 \geq \mu \geq 1/2 \geq \lambda \geq 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|$  is  $s$ -convex on  $[a, b]$ , then the following inequality holds.

$$\left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)} \left[ \left\{ \frac{2\lambda^{s+2}}{(s+2)} - \frac{\lambda}{2^{s+1}} + \frac{2\mu^{s+2}}{(s+2)} - \mu \left( \frac{1}{2^{s+1}} + 1 \right) + \frac{s+1}{s+2} \left( \frac{1}{2^{s+1}} + 1 \right) \right\} |f'(a)| + \left\{ \frac{2(1-\lambda)^{s+2}}{s+2} + \lambda \left( 1 + \frac{1}{2^{s+1}} \right) - \frac{1}{2^{s+2}} \left( \frac{1}{s+2} + 1 \right) - \frac{1}{s+2} + \frac{2(1-\mu)^{s+2}}{s+1} - \frac{1}{2^{s+2}} \left( 1 + \frac{1}{s+2} \right) + \frac{\mu}{2^{s+1}} \right\} |f'(b)| \right] \tag{2.2}$$

**Proof:** Let

$$I = \left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

By taking modulus on both sides of (2.1), we get the inequality

$$I \leq (b - a) \left\{ \int_0^{1/2} |\lambda - t| |f'(ta + (1 - t)b)| dt + \int_{1/2}^1 |\mu - t| |f'(ta + (1 - t)b)| dt \right\}$$

As  $|f'(x)|$  is  $s$ -convex, so

$$I \leq$$

$$(b - a) \left\{ \int_0^{1/2} |\lambda - t| (t^s |f'(a)| + (1 - t)^s |f'(b)|) dt + \int_{1/2}^1 |\mu - t| (t^s |f'(a)| + (1 - t)^s |f'(b)|) dt \right\} = (b - a) [I_1 + I_2] \tag{2.3}$$

As

$$I_1 = |f'(a)| \left\{ \frac{2\lambda^{s+2}}{(s+1)(s+2)} - \frac{\lambda}{2^{s+1}(s+1)} + \frac{1}{2^{s+2}(s+2)} \right\} + \frac{|f'(b)|}{s+1} \left\{ \frac{2(1-\lambda)^{s+2}}{s+2} + \lambda \left( 1 + \frac{1}{2^{s+1}} \right) - \frac{1}{2^{s+2}} \left( \frac{1}{s+2} + 1 \right) - \frac{1}{s+2} \right\}$$

$$I_2 = |f'(a)| \left\{ \frac{2\mu^{s+2}}{(s+1)(s+2)} - \frac{\mu}{s+1} \left( \frac{1}{2^{s+1}} + 1 \right) + \frac{1}{s+2} \left( \frac{1}{2^{s+2}} + 1 \right) \right\} + \frac{|f'(b)|}{s+1} \left\{ \frac{2(1-\mu)^{s+2}}{s+2} - \frac{1}{2^{s+2}} \left( 1 + \frac{1}{s+2} \right) + \frac{\mu}{2^{s+1}} \right\}$$

Using above values of  $I_1$  and  $I_2$  in (2.3), we get the R. H. Sof (2.3).

$$= \frac{(b-a)}{(s+1)} \left[ \left\{ \frac{2\lambda^{s+2}}{(s+2)} - \frac{\lambda}{2^{s+1}} + \frac{2\mu^{s+2}}{(s+2)} - \mu \left( \frac{1}{2^{s+1}} + 1 \right) + \frac{s+1}{s+2} \left( \frac{1}{2^{s+1}} + 1 \right) \right\} |f'(a)| + \left\{ \frac{2(1-\lambda)^{s+2}}{s+2} + \lambda \left( 1 + \frac{1}{2^{s+1}} \right) - \frac{1}{2^{s+2}} \left( \frac{1}{s+2} + 1 \right) - \frac{1}{s+2} + \frac{2(1-\mu)^{s+2}}{s+1} - \frac{1}{2^{s+2}} \left( 1 + \frac{1}{s+2} \right) + \frac{\mu}{2^{s+1}} \right\} |f'(b)| \right] \text{ (Proved)}$$

**Corollary 3.1.1:** Let  $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a,$

$1 \geq \mu \geq 1/2 \geq \lambda \geq 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|$  is convex on  $[a, b]$ , then the following inequality holds.

$$\left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{24} \left\{ (10 - 3\lambda + 8\lambda^3 - 15\mu + 8\mu^3) |f'(a)| + (8 - 9\lambda + 24\lambda^2 - 8\lambda^3 - 21\mu + 24\mu^2 - 8\mu^3) |f'(b)| \right\}$$

**Proof:** Proof of this corollary is simple, by putting  $s = 1$  in theorem 3.1.

**Theorem 3.2:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a, 1 > \mu \geq 1/2 \geq \lambda > 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q > 1$  is  $s$ -convex on  $[a, b]$ , then

$$\left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b - a) \left( \frac{q-1}{2q-p-1} \right)^{1-\frac{1}{q}} \left[ \left( \lambda \frac{2q-p-1}{q-1} + \left( \frac{1}{2} - \lambda \right) \frac{2q-p-1}{q-1} \right)^{1-\frac{1}{q}} \left\{ |f'(a)|^q \lambda^{s+p+1} [B(s+1, p+1) + (-1)^{p+1} B\left(1 - \frac{1}{2\lambda}; p+1, s+1\right)] + |f'(b)|^q (1 - \lambda)^{s+p+1} \left[ (-1)^{p+1} B\left(\frac{\lambda}{\lambda-1}; p+1, s+1\right) + B\left(\frac{1-2\lambda}{2(1-\lambda)}; p+1, s+1\right) \right] \right\} + \left( \left( \mu - \frac{1}{2} \right) \frac{2q-p-1}{q-1} + (1 - \mu) \frac{2q-p-1}{q-1} \right)^{1-\frac{1}{q}} \left\{ |f'(a)|^q \mu^{s+p+1} [B(1 - 1/2\mu; p+1, s+1) + (-1)^{p+1} B\left(\frac{\mu-1}{\mu}; p+1, s+1\right)] + |f'(b)|^q (1 - \mu)^{s+p+1} \left[ (-1)^{p+1} B\left(\frac{1-2\mu}{2(1-\mu)}; p+1, s+1\right) + B(p+1, s+1) \right] \right\} \right]$$

Where

$$B(p, s) = \int_0^1 t^{p-1}(1-t)^{s-1} dt$$

$$\text{and } B(v; p, s) = \int_0^v t^{p-1}(1-t)^{s-1} dt$$

**Proof:**By lemma 2.1 and using the properties of modulus, we have

$$\begin{aligned} & \left| (1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \left\{ \int_0^{1/2} |\lambda-t| |f'(ta+(1-t)b)| dt + \int_{1/2}^1 |\mu-t| |f'(ta+(1-t)b)| dt \right\} \end{aligned}$$

Using Holder’s inequality,

$$\begin{aligned} & \leq (b-a) \left\{ \left( \int_0^{1/2} |\lambda-t|^{\frac{q-p}{q-1}} dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} |\lambda-t|^p |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_{1/2}^1 |\mu-t|^{\frac{q-p}{q-1}} dt \right)^{1-\frac{1}{q}} \left( \int_{1/2}^1 |\mu-t|^p |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\} \\ & = (b-a)[I_1 + I_2] \quad (2.4) \end{aligned}$$

By using the beta and incomplete beta function, we have

$$\begin{aligned} I_1 &= \left( \frac{q-1}{2q-p-1} \right)^{1-\frac{1}{q}} \left( \lambda^{\frac{2q-p-1}{q-1}} + \left( \frac{1}{2} - \lambda \right)^{\frac{2q-p-1}{q-1}} \right)^{1-\frac{1}{q}} \\ & \left\{ |f'(a)|^q \lambda^{s+p+1} \left[ B(s+1, p+1) + (-1)^{p+1} B\left(1 - \frac{1}{2\lambda}; p+1, s+1\right) \right] + \right. \\ & \left. |f'(b)|^q (1-\lambda)^{s+p+1} \left[ (-1)^{p+1} B\left(\frac{\lambda}{1-\lambda}; p+1, s+1\right) + B\left(\frac{1-2\lambda}{2(1-\lambda)}; p+1, s+1\right) \right] \right\}^{\frac{1}{q}} \quad (2.5) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left( \frac{q-1}{2q-p-1} \right)^{1-\frac{1}{q}} \left( \left( \mu - \frac{1}{2} \right)^{\frac{2q-p-1}{q-1}} + (1-\mu)^{\frac{2q-p-1}{q-1}} \right)^{1-\frac{1}{q}} \\ & \times \left\{ |f'(a)|^q \mu^{s+p+1} \left[ B\left(1 - \frac{1}{2\mu}; p+1, s+1\right) + (-1)^{p+1} B\left(\frac{\mu-1}{\mu}; p+1, s+1\right) \right] \right. \\ & \left. + |f'(b)|^q \left( 1 - \mu \right)^{s+p+1} \left[ (-1)^{p+1} B\left(\frac{1-2\mu}{2(1-\mu)}; p+1, s+1\right) + B(p+1, s+1) \right] \right\}^{\frac{1}{q}} \quad (2.6) \end{aligned}$$

Using (2.5) and (2.6) in (2.4). We get the desire result.

**Corollary 3.2.1:**Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a, 1 \geq \mu \geq 1/2 \geq \lambda \geq 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is  $s$ -convex on  $[a, b]$ , then

$$\begin{aligned} & \left| (1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{\frac{1}{q}} 2^{\frac{3(1-\frac{1}{q})}{q}}} \left[ (8\lambda^2 - 4\lambda + 1)^{1-\frac{1}{q}} \left\{ |f'(a)|^q \left[ \frac{\lambda^{s+2}}{s+2} + \frac{1-2\lambda}{2^{s+2}} + \frac{1}{s+2} \left( \lambda^{s+2} - \frac{1}{2^{s+2}} \right) \right] \right. \right. \\ & \quad \left. \left. + |f'(b)|^q \left[ \lambda - \frac{1}{s+2} \{ (1-\lambda)^{s+2} \} - \frac{1}{2^{s+1}} \left( \frac{1}{2} - \lambda \right) - \frac{1}{s+2} \left\{ \frac{1}{2^{s+2}} - (1-\lambda)^{s+2} \right\} \right] \right\} \right]^{\frac{1}{q}} + \\ & \quad (8\mu^2 - 12\mu + 5)^{1-\frac{1}{q}} \\ & \quad \left\{ |f'(a)|^q \left[ \frac{1-2\mu}{2^{s+2}} - \frac{1}{s+2} \left( \frac{1}{2^{s+2}} + 1 - 2\mu^{s+2} \right) + 1 - \mu \right] + \right. \\ & \quad \left. |f'(b)|^q \left[ \frac{1}{2^{s+1}} \left( \mu - \frac{1}{2} \right) - \frac{1}{s+2} \left\{ \frac{1}{2^{s+2}} - 2(1-\mu)^{s+2} \right\} \right] \right\}^{\frac{1}{q}} \end{aligned}$$

**Proof:** Putting  $p = 1$  in theorem 3.2.

**Corollary 3.2.2:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a, m > 0, m \geq 2l \geq 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is  $s$ -convex on  $[a, b]$ , then

$$\begin{aligned} & \left| \frac{l}{m} \{ f(a) + f(b) \} + \frac{m-2l}{m} f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{8} \left( \frac{1}{(s+1)(s+2)m^{s+2}2^{s-1}} \right)^{\frac{1}{q}} \left( \frac{8l^2-4ml+m^2}{m^2} \right)^{1-\frac{1}{q}} \times \\ & \quad \left[ \left\{ |f'(a)|^q [(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}] + |f'(b)|^q [2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}] \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \left\{ |f'(a)|^q [2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}] + |f'(b)|^q [(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}] \right\}^{\frac{1}{q}} \right] \end{aligned}$$

**Proof:**By putting  $\lambda = 1 - \mu = \frac{l}{m}$  in corollary 3.2.1.

**Corollary 3.2.3:**Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is  $s$ -convex on  $[a, b]$ , then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} \left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} \left[ \{(s+1)|f'(a)|^q + (2^{s+2} - 3 - s)|f'(b)|^q\}^{\frac{1}{q}} + \{(2^{s+2} - s - 3)|f'(a)|^q + (s+1)|f'(b)|^q\}^{\frac{1}{q}} \right] \tag{2.7}$$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} \left(\frac{1}{(s+1)(s+2)2^{s-1}}\right)^{\frac{1}{q}} \left[ \{|f'(a)|^q + (2^{s+1}s + 1)|f'(b)|^q\}^{\frac{1}{q}} + \{(2^{s+1}s + 1)|f'(a)|^q + |f'(b)|^q\}^{\frac{1}{q}} \right] \tag{2.8}$$

$$\left| \frac{1}{6} \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} \left[ \frac{1}{[5(s+1)(s+2)3^{2s-1}]^{\frac{1}{q}}} \left\{ [3^{s+1}(2s+1) + 2]|f'(a)|^q + [2 \cdot 5^{s+2} + 3^{s+1}\{2^{s+1}(s-4) - 2s - 7\}]|f'(b)|^q \right\}^{\frac{1}{q}} + \left\{ [2 \cdot 5^{s+2} + 3^{s+1}\{2^{s+1}(s-4) - 2s - 7\}]|f'(a)|^q + [3^{s+1}(2s+1) + 2]|f'(b)|^q \right\}^{\frac{1}{q}} \right] \tag{2.9}$$

**Proof:** By putting  $(l, m) = (0, 1), (1, 2)$  and (1,6) in corollary 3.2.2.

**Corollary 3.2.4:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|$  is  $s$ -convex on  $[a, b]$ , then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)(s+2)} \left(1 - \frac{1}{2^{s+1}}\right) [|f'(a)| + |f'(b)|] \tag{2.10}$$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)(s+2)2^{s+1}} (2^s s + 1) [|f'(a)| + |f'(b)|] \tag{2.11}$$

$$\left| \frac{1}{6} \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)(s+2)2^{s+2}3^{s+2}} [3^{s+1}\{2^{s+1}(s-4) - 6\} + 2(1 + 5^{s+2})] [|f'(a)| + |f'(b)|] \tag{2.12}$$

**Proof:** By putting  $q = 1$  in corollary 3.2.3.

**Remark:** By putting  $s = 1$  in corollary 3.2.4, we reform the inequality (1.2) and inequalities of [6] as follows.

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|$  is convex on  $[a, b]$ , then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} [|f'(a)| + |f'(b)|] \tag{2.13}$$

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} [|f'(a)| + |f'(b)|] \tag{2.14}$$

$$\left| \frac{1}{6} \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} [|f'(a)| + |f'(b)|] \tag{2.15}$$

**Theorem 2.3:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a$ ,  $1 > \mu \geq 1/2 \geq \lambda > 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q > 1$  is  $s$ -convex on  $[a, b]$ , then

$$\left| (1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \left(\frac{1}{2^{sp+1}(sp+1)}\right)^{\frac{1}{p}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[ |f'(a)| \left\{ (\lambda^{q+1} + \left(\frac{1}{2} - \lambda\right)^{q+1})^{\frac{1}{q}} + (2^{sp+1} - 1)^{\frac{1}{p}} \left(\mu - \frac{1}{2}\right)^{q+1} + (1-\mu)^{q+1} \right\}^{\frac{1}{q}} + |f'(b)| \left\{ (2^{sp+1} - 1)^{\frac{1}{p}} (\lambda^{q+1} + \left(\frac{1}{2} - \lambda\right)^{q+1})^{\frac{1}{q}} + \left(\left(\mu - \frac{1}{2}\right)^{q+1} + (1-\mu)^{q+1}\right)^{\frac{1}{q}} \right\} \right]$$

**Proof:** Using lemma 2.1, convexity of  $f'(x)$  on  $[a, b]$  and Holder's inequality

$$\left| (1-\mu)f(a) + \lambda f(b) + (\mu-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \left\{ \int_0^{1/2} |\lambda - t| |f'(ta + (1-t)b)| dt + \int_{1/2}^1 |\mu - t| |f'(ta + (1-t)b)| dt \right\} \leq (b-a) \left\{ \int_0^{1/2} |\lambda - t| \{t^s |f'(a)| + (1-t)^s |f'(b)|\} dt + \int_{1/2}^1 |\mu - t| \{t^s |f'(a)| + (1-t)^s |f'(b)|\} dt \right\} = (b-a) \{I_1 + I_2\} \tag{2.16}$$

Now

$$I_1 = |f'(a)| \int_0^{1/2} t^s |\lambda - t| dt + |f'(b)| \int_0^{1/2} (1-t)^s |\lambda - t| dt$$

Using Holder's inequality

$$I_1 \leq |f'(a)| \left[ \left( \int_0^{1/2} t^{sp} dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} |\lambda - t|^q dt \right)^{\frac{1}{q}} \right] + |f'(b)| \left[ \left( \frac{1}{2^{sp+1}(sp+1)} \right)^{\frac{1}{p}} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} (2^{sp+1} - 1)^{\frac{1}{p}} \left\{ \lambda^{q+1} + \left( \frac{1}{2} - \lambda \right)^{q+1} \right\}^{\frac{1}{q}} \right]$$

$$I_1 \leq \left( \frac{1}{2^{sp+1}(sp+1)} \right)^{\frac{1}{p}} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} \left[ |f'(a)| \left\{ \lambda^{q+1} + \left( \frac{1}{2} - \lambda \right)^{q+1} \right\}^{\frac{1}{q}} + |f'(b)| (2^{sp+1} - 1)^{\frac{1}{p}} \left\{ \lambda^{q+1} + \left( \frac{1}{2} - \lambda \right)^{q+1} \right\}^{\frac{1}{q}} \right]$$

Similarly we have

$$I_2 \leq \left( \frac{1}{2^{sp+1}(sp+1)} \right)^{\frac{1}{p}} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} \left[ |f'(a)| (2^{sp+1} - 1)^{\frac{1}{p}} \left\{ \left( \mu - \frac{1}{2} \right)^{q+1} + (1 - \mu)^{q+1} \right\}^{\frac{1}{q}} + |f'(b)| \left\{ \left( \mu - \frac{1}{2} \right)^{q+1} + (1 - \mu)^{q+1} \right\}^{\frac{1}{q}} \right]$$

Using above values of  $I_1$  and  $I_2$  in (2.16), the required result follows.

**Corollary 2.3.1:** Let  $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $b > a, m > 0, m \geq 2l \geq 0$  and  $f' \in L^1[a, b]$ . If  $|f'(x)|^q$  for  $q > 0$  is  $s$ -convex on  $[a, b]$ , then

$$\left| \frac{l[f(a)+f(b)]+(m-2l)f\left(\frac{a+b}{2}\right)}{m} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a) \left\{ 1 + (2^{sp+1}-1)^{\frac{1}{p}} \right\}}{\{2^{sp+1}(sp+1)\}^{\frac{1}{p}}} \left\{ \frac{2^{q+1}l^{q+1}+(m-2l)^{q+1}}{(q+1)2^{q+1}m^{q+1}} \right\}^{\frac{1}{q}} \left[ |f'(a)| + |f'(b)| \right]$$

**Proof:** The result is obtained, if we put  $\lambda = 1 - \mu = \frac{l}{m}$  in theorem 2.3.

### 3. APPLICATION TO SPECIAL MEAN.

For  $a > 0$  and  $b > 0$ , means are defined as:

Arithmetic Mean:  $A(a, b) = \frac{a+b}{2}$

Geometric Mean:  $G(a, b) = \sqrt{ab}$

Harmonic Mean:  $H(a, b) = \frac{2ab}{a+b}$  Logarithmic

Mean:  $I(a, b) = \begin{cases} \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}$

and

Generalized logarithmic mean

$$L_n(a, b) = \begin{cases} \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}}, & n \neq -1, 0 \\ \frac{b-a}{\ln b - \ln a}, & n = -1 \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, & n = 0 \end{cases}$$

Now, we deduce relations between means.

**Theorem 3.1:** Let  $b > a > 0, m > 0, m \geq 2l \geq 0$  and  $0 < s \leq 1, q \geq 1$ , then

$$\left| \frac{2lA(a^{s+1}, b^{s+1}) + (m-2l)A^{s+1}(a, b) - mL_{s+1}^{s+1}(a, b)}{m(s+1)} \right| \leq \frac{(b-a)}{8} \left( \frac{1}{(s+1)(s+2)m^{s+2}2^{s-1}} \right)^{\frac{1}{q}} \left( \frac{8l^2-4ml+m^2}{m^2} \right)^{1-\frac{1}{q}} \left[ \{a^{sq}[(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}] + b^{sq}[2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}]\}^{\frac{1}{q}} + \{a^{sq}[2^{s+3}(m-l)^{s+2} + 2l(s+2)(1+2^{s+1})m^{s+1} - (s+3+2^{s+2})m^{s+2}] + b^{sq}[(s+1)m^{s+2} - 2l(s+2)m^{s+1} + 2^{s+3}l^{s+2}]\}^{\frac{1}{q}} \right]$$

**Proof:** Take  $f(x) = \frac{x^{s+1}}{s+1}$ . As  $f'(x) = x^s$  is  $s$ -convex function. Applying  $f(x)$  on corollary 2.2.2. We get the desired result.

**Corollary 3.1.1:** Let  $b > a > 0, m > 0, m \geq 2l \geq 0$ , and  $1 \geq s > 0$  then

$$\left| \frac{2lA(a^{s+1}, b^{s+1}) + (m - 2l)A^{s+1}(a, b) - mL_{s+1}^{s+1}(a, b)}{m(s + 1)} \right|$$

$$\leq \frac{b - a}{(s + 1)(s + 2)2^s m^{s+2}} [2^{s+2}\{l^{s+2} + (m - l)^{s+2}\} + 2^{s+1}(s + 2)lm^{s+1} - (s + 2^{s+1})m^{s+2}]A(a^s, b^s)$$

**Proof:** Put  $q = 1$  in theorem 3.1.

**Corollary 3.1.2:** Let  $b > a > 0$ , and  $1 \geq s > 0$ ,  $q \geq 1$ , then

$$\left| \frac{A^{s+1}(a, b) - L_{s+1}^{s+1}(a, b)}{(s + 1)} \right|$$

$$\leq \frac{(b - a)}{8} \left( \frac{1}{(s + 1)(s + 2)2^{s-1}} \right)^{\frac{1}{q}} \left[ \{(s + 1)a^{sq} + (2^{s+2} - 3 - s)b^{sq}\}^{\frac{1}{q}} + \{(2^{s+2} - s - 3)a^{sq} + (s + 1)b^{sq}\}^{\frac{1}{q}} \right] \quad (3.1)$$

$$\left| \frac{A(a^{s+1}, b^{s+1}) - L_{s+1}^{s+1}(a, b)}{(s + 1)} \right| \leq \frac{(b - a)}{8} \left( \frac{1}{(s + 1)(s + 2)2^{s-1}} \right)^{\frac{1}{q}} \left[ \{a^{sq} + (2^{s+1}s + 1)b^{sq}\}^{\frac{1}{q}} + \{(2^{s+2}s + 1)a^{sq} + b^{sq}\}^{\frac{1}{q}} \right] \quad (3.2)$$

$$\left| \frac{2A(a^{s+1}, b^{s+1}) + A^{s+1}(a, b) - 3L_{s+1}^{s+1}(a, b)}{3(s + 1)} \right| \leq$$

$$\frac{5(b - a)}{72} \left[ \frac{1}{5(s + 1)(s + 2)3^s 2^{s-1}} \right]^{\frac{1}{q}} \left[ \{[3^{s+1}(s - 1) + 2^{s+3}]a^{sq} + [2^{2s+5} + 3^{s+1}\{2^{s+2}(s - 1) - s - 5\}]b^{sq}\}^{\frac{1}{q}} + \{[2^{2s+5} + 3^{s+1}\{2^{s+2}(s - 1) - s - 5\}]a^{sq} + [3^{s+1}(s - 1) + 2^{s+3}]b^{sq}\}^{\frac{1}{q}} \right] \quad (3.3)$$

$$\left| \frac{A(a^{s+1}, b^{s+1}) + A^{s+1}(a, b) - 2L_{s+1}^{s+1}(a, b)}{2(s + 1)} \right|$$

$$\leq \frac{(b - a)}{32} \left[ \frac{1}{(s + 1)(s + 2)2^{2(s-1)}} \right]^{\frac{1}{q}}$$

$$\left[ \{[2^s s + 1]a^{sq} + [3^{s+2} + 2^s\{2^{s+1}(s - 2) - s - 4\}]b^{sq}\}^{\frac{1}{q}} + \{[3^{s+2} + 2^s\{2^{s+1}(s - 2) - s - 4\}]a^{sq} + [2^s s + 1]b^{sq}\}^{\frac{1}{q}} \right] \quad (3.4)$$

$$\left| \frac{2A(a^{s+1}, b^{s+1}) + 3A^{s+1}(a, b) - 5L_{s+1}^{s+1}(a, b)}{5(s + 1)} \right| \leq$$

$$\frac{b - a}{200} \left[ \frac{1}{(s + 1)(s + 2)5^s 2^{s-1}} \right]^{\frac{1}{q}} \left[ \{[5^{s+1}(3s + 1) + 2^{s+3}]a^{sq} + [2^{3s+7} + 5^{s+1}\{2^{s+2}(s - 3) - 3s - 11\}]b^{sq}\}^{\frac{1}{q}} +$$

$$\{[2^{3s+7} + 5^{s+1}\{2^{s+2}(s - 3) - 3s - 11\}]a^{sq} + [5^{s+1}(3s + 1) + 2^{s+3}]b^{sq}\}^{\frac{1}{q}} \right] \quad (3.5)$$

$$\left| \frac{4A(a^{s+1}, b^{s+1}) + A^{s+1}(a, b) - 5L_{s+1}^{s+1}(a, b)}{5(s + 1)} \right| \leq$$

$$\frac{17(b - a)}{200} \left[ \frac{1}{17(s + 1)(s + 2)5^s 2^{s-1}} \right]^{\frac{1}{q}} \left[ \{[5^{s+1}(s - 3) + 2^{2s+5}]a^{sq} + [2^{s+3}3^{s+2} + 5^{s+1}\{2^{s+2}(2s - 1) - s - 7\}]b^{sq}\}^{\frac{1}{q}} + \{[2^{s+3}3^{s+2} + 5^{s+1}\{2^{s+2}(2s - 1) - s - 7\}]a^{sq} + [5^{s+1}(s - 3) + 2^{2s+5}]b^{sq}\}^{\frac{1}{q}} \right] \quad (3.6)$$

$$\left| \frac{A(a^{s+1}, b^{s+1}) + 2A^{s+1}(a, b) - 3L_{s+1}^{s+1}(a, b)}{3(s + 1)} \right| \leq$$

$$\frac{5(b - a)}{72} \left[ \frac{1}{5(s + 1)(s + 2)3^s 2^{s-1}} \right]^{\frac{1}{q}} \left[ \{[3^{s+1}(2s + 1) + 2]a^{sq} + [2 \cdot 5^{s+2} + 3^{s+1}\{2^{s+1}(s - 4) - 2s - 7\}]b^{sq}\}^{\frac{1}{q}} + \{[2 \cdot 5^{s+2} + 3^{s+1}\{2^{s+1}(s - 4) - 2s - 7\}]a^{sq} + [3^{s+1}(2s + 1) + 2]b^{sq}\}^{\frac{1}{q}} \right] \quad (3.7)$$

**Proof:** By putting  $(l, m) = (0, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 5)$  and  $(1, 6)$  in theorem 3.1.

**Theorem 3.2:** Let  $b > a > 0$ ,  $m > 0$ ,  $m \geq 2l \geq 0$  and  $q \geq 1$ , then

$$\left| \frac{1}{m} [2lA(\ln a, \ln b) + (m - 2l) \ln A(a, b)] - \ln I(a, b) \right| \leq$$

$$\frac{(b - a)}{24m^3 G^2(a, b)} \{3m(8l^2 - 4ml + m^2)\}^{1 - \frac{1}{q}} \left[ \{(2m^3 - 8l^3 + 24l^2m - 9lm^2)a^q + (m^3 + 8l^3 - 3lm^2)b^q\}^{\frac{1}{q}} + \{(m^3 + 8l^3 - 3lm^2)a^q + (2m^3 - 8l^3 + 24l^2m - 9lm^2)b^q\}^{\frac{1}{q}} \right]$$

**Proof:** Take  $s = 1$  and  $f(x) = \ln x$  in corollary 2.2.3. We got the desired result.

**Corollary 3.2.1:** Let  $b > a > 0$ ,  $m > 0$ , and  $m \geq 2l \geq 0$ , then

$$\left| \frac{1}{m} [2lA(\ln a, \ln b) + (m - 2l) \ln A(a, b)] - \ln I(a, b) \right| \leq$$

$$\frac{b - a}{4m^2 H(a, b)} (8l^2 - 4lm + m^2)$$

**Proof:** By putting  $q = 1$  in theorem 3.2.

**Theorem 3.3:** Let  $b > a > 0$ ,  $m > 0$ ,  $m \geq 2l \geq 0$ ,  $1 \geq s > 0$  and  $q > 1$ , then

$$\left| \frac{2lA(a^{s+1}, b^{s+1}) + (m - 2l)A^{s+1}(a, b) - mL_{s+1}^{s+1}(a, b)}{m(s + 1)} \right| \leq$$

$$\frac{2(b - a) \left\{ 1 + (2^{sp+1} - 1)^{\frac{1}{p}} \right\}}{\{2^{sp+1}(sp + 1)\}} \left\{ \frac{2^{q+1}l^{q+1} + (m - 2l)^{q+1}}{(q + 1)2^{q+1}m^{q+1}} \right\}^{\frac{1}{q}} A(a^s, b^s)$$

**Proof:** Take  $f(x) = \frac{x^{s+1}}{s+1}$ . As  $f'(x) = x^s$  is  $s$ -convex function. Applying  $f(x)$  on corollary 2.3.1. We get the desire result.

**Corollary 3.3.1:** Let  $b > a > 0, m > 0, m \geq 2l \geq 0, 1 \geq s > 0$  and  $q \geq 1$ , then

$$\left| \frac{2lA(a^{s+1}, b^{s+1}) + (m - 2l)A^{s+1}(a, b) - mL_{s+1}^{s+1}(a, b)}{m(s + 1)} \right| \leq \frac{2(b - a)}{s + 1} \left\{ \frac{2^{q+1}l^{q+1} + (m - 2l)^{q+1}}{(q + 1)2^{q+1}m^{q+1}} \right\}^{\frac{1}{q}} A(a^s, b^s)$$

**Proof:** By putting  $p = 1$  in theorem 3.3.

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