SOME DOUBLE INTEGRAL q-CHEBYCHEV GRUSS TYPE INEQUALITIES

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***ABSTRACT**. In this paper weighted double integral Montgomery identity is proved and there are some new double integral q-Chebychev Gruss type inequalities are derived.*

Keywords—weighted double integral Montgomery Identity, Peano Kernal,

1. INTRODUCTION AND PRELIMINARIES

The well-known integral inequality is the following one (see [4, 8]):

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\leq\frac{(P-p)(Q-q)}{4}$$

Provided that f and g are two integrable functions on [a, b] such that $p\leq f(x)\leq P,\ q\leq g(x)\leq Q,$ for $x\in[a,b]$, where p, q, P, Q are real constants. Over the years much efforts and time has been devoted to the improvement and generalization of this inequality. These includes, among others, the works in [2,3,6].B.G.Pachpatte proved Chebychev Gruss type inequality[8]. Inspired by the idea of Pachpatte, Wengui Yang [9] proved some weighted q-Chebychev-Gruss type inequalities. Cauciman [1] proved q-integral Gruss inequality, as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)d_qx-\left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)\left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)\leq\frac{(P-p)(Q-q)}{4}$$

Provided that $p\leq f(x)\leq P,\ q\leq g(x)\leq Q$ *for* $x\in[a,b]$, where p, q, P, Q are real constants. In the following we discuss some preliminaries about q-integral and differential calculus .In what follows q is a real constant such that 0 < q <1.

For an arbitrary function f, the q-differential is defined by

$$(d_q f)(x)= f(qx)-f(x),$$

And the q- derivative as (see [1])

$$(D_q f)(x)=\frac{(\mathrm{d_q} f)(x)}{\mathrm{d_q}x}=\frac{f(qx)-f(x)}{(q-1)x},$$

If $x\neq 0$

$$(D_q f)(x)= f'(0),$$

Provided $f'(0)$ exists. For differentiable function, it is very well clear that $\lim_{q-1}(D_q f)(x)=\frac{(\mathrm{d_q} f)(x)}{\mathrm{d_q}x}$.

For $0<a<b$, the definite q-integral is defined as (see [5])

$$\int_0^b f(x)d_qx=(1-\mathrm{q})\mathrm{b}\sum_{\mathrm{j}=0}^{\infty}q^j f(bq^j),$$

Provided the sum converges absolutely. The q-Jackson integral in an interval [a, b] is given by (see [5])

$$\int_a^b f(x)d_qx=\int_0^b f(x)d_qx-\int_0^a f(x)d_qx.$$

Motivated by the above ideas we define q-derivatives for function of two variables, that is, q partial derivatives, as

$$\frac{(\partial_q f)(x,y)}{\Delta_1 q}=\frac{f(qx,y)-f(x,y)}{(q-1)x},$$

If $x\neq 0$.

$$\frac{(\partial_q f)(x,y)}{\Delta_2 q}=\frac{f(x,qy)-f(x,y)}{(q-1)y},$$

If $y\neq 0$.

$$\frac{(\partial_q^2 f)(x,y)}{\Delta_1 q\Delta_2 q}=\frac{f(qx,qy)-f(x,qy)-f(qx,y)+f(x,y)}{(qx-x)(qy-y)},$$

If $x,y\neq 0$.

It is very well clear that for $\mathrm{q}\to 1^-$, these are nothing except the usual partial derivatives and the mixed derivative.

In this paper, by use of two variables q-integral Montgomery identity, some weighted double integral q-Chebychev-Gruss type inequalities are obtained.

2. MATERIALS AND METHODS

Assume that w is a non-negative weight function defined on compact interval $[a,b]$ and $[c,d]$ such that

$\int_a^b w(x)d_q x = \int_c^d w(y)d_q y = 1.$ Consider

$W(t) = \int_a^t w(x)d_q x;\ W(s) = \int_c^s w(y)d_q y$ for

$(t,s) \in [a,b]\times[c,d]$. $W(t) = W(s) = 0$ for $t<a,\ s<c$ and $W(t) = W(s) = 1$ for $t>b, s>d$.

The weighted q-integral Peano kernel $P_w(x,t)$ is defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x \\ W(t)-1, & x < t \le b. \end{cases} \quad (1)$$

For our presentation we use the following notations, for some function $f,g:[a,b]\times[c,d]\to\Re$

$$S(w,f,g) = f(qx,qy)g(qx,qy) - \frac{1}{2}\{f(qx,qy)\int_a^b\int_c^d w(t)w(s)g(qt,qs)d_q s d_q t + g(qx,qy)\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_q s d_q t\},$$

And

$$T(w,f,g) = \int_a^b\int_c^d w(x)w(y)f(qx,qy)g(qx,qy)d_q y d_q x - \left(\int_a^b\int_c^d w(x)w(y)f(qx,qy)d_q y d_q x\right)\left(\int_a^b\int_c^d w(x)w(y)g(qx,qy)d_q y d_q x\right).$$

And for $h \in C[a,b]$, the norm $\|.\|$ on h as

$\|h\| = \sup_{a\le t\le b}|h(t)|$.

For our main results consider the following one.

Lemma 1: [6] (Weighted q- Montgomery Identity)

Let $f:[a,b]\to\Re$ and $w:[a,b]\to[0,\infty)$ satisfying $\int_a^b w(x)d_q x = 1$, then

$$f(x) = \int_a^b w(t)f(qt)d_q t + \int_a^b P_w(x,t)(D_q f)(t)d_q t, \quad (2)$$

For $x\in[a,b]$, where the weighted q-intrgral Peano kernel P_w (x,t) is given by (1).

Lemma 2: Let $f:[a,b]\times[c,d]\to\Re$ be such that the partial derivatives $\frac{(\partial_q f)(t,s)}{\Delta_1 q}$, $\frac{(\partial_q f)(t,s)}{\Delta_2 q}$ and $\frac{(\partial_q^2 f)(t,s)}{\Delta_2\Delta_1 q}$ exist and are continuous on $[a,b]\times[c,d]$, then

$$f(x,y) = \int_a^b\int_c^d w(t)w(s)f(qt,qs)d_q s d_q t + \int_a^b\int_c^d w(t)P_w(y,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_q s d_q t + \int_a^b\int_c^d w(s)P_w(x,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_q s d_q t + \int_a^b\int_c^d P_w(x,t)P_w(y,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_q s d_q t, \quad (3)$$

For $(x,y)\in[a,b]\times[c,d]$.

Proof: According to the weighted q-integral Montgomery (2) for partal delta map $f(.\ y)$ we obtain

$$f(x,y)=\int_a^b w(t)f(qt,y)d_qt+\int_a^b P_w(x,t)\frac{(\partial_q f)(t,y)}{\Delta_1 q}d_qt, \quad (4)$$

For partial delta map f (qt,.)

$$f(qt,y)=\int_c^d w(s)f(qt,qs)d_qs+\int_c^d P_w(y,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qs, \quad (5)$$

Similarly, for partial delta map $\frac{(\partial_q f)(t,.)}{\Delta_1 q}$

$$\frac{(\partial_q f)(t,y)}{\Delta_1 q}=\int_c^d w(s)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qs+\int_c^d P_w(y,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qs, \quad (6)$$

By combining (4), (5) and the above equality, we obtain

$$\begin{aligned}f(x,y)=&\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt+\int_a^b\int_c^d w(t)P_w(y,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt\\&+\int_a^b\int_c^d w(s)P_w(x,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt\\&+\int_a^b\int_c^d P_w(x,t)P_w(y,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt,\end{aligned}$$

And this completes the proof.

3. RESULTS

Theorem 1:

Let $f,g:[a,b]\times[c,d]\to\Re$ .And w is a non-negative weight function as defined above, then

$$|T(w,f,g)|\le\int_a^b\int_c^d w(x)w(y)R(x,y)Q(x,y)d_qyd_qx, \quad (7)$$

Where,

$$R(x,y)=M_1G(qy)+M_2H(qx)+M_3G(qy)H(qx) \quad (8)$$

$$Q(x,y)=N_1G(qy)+N_2H(qx)+N_3G(qy)H(qx) \quad (9)$$

$$M_1=\sup_{c<t<d}\left|\frac{(\partial_q f)(t,s)}{\Delta_2 q}\right|, M_2=\sup_{a<t<b}\left|\frac{(\partial_q f)(t,s)}{\Delta_1 q}\right|,$$

$$M_3=\sup_{c<t<b:a<t<b}\left|\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}\right| \quad (10)$$

$$N_1=\sup_{c<s<d}\left|\frac{(\partial_q g)(t,s)}{\Delta_2 q}\right|, N_2=\sup_{a<t<b}\left|\frac{(\partial_q g)(t,s)}{\Delta_1 q}\right|,$$

$$N_3=\sup_{c<t<b:a<t<b}\left|\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}\right|, \quad (11)$$

and

$$H(x)=\int_a^b|P_w(x,t)|d_qt \text{ And } G(y)=\int_c^d|P_w(y,s)|d_qs \quad (12)$$

Proof: Since the function f and g satisfy the condition of Lemma 2 therefore by (3)

$$\begin{aligned}f(qx,qy)=&\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt+\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt+\\&\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt+\\&\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt, \quad (13)\end{aligned}$$

And

$$g(qx,qy) = \int_a^b\int_c^d w(t)w(s)g(qt,qs)d_qsd_qt +$$
$$\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_qsd_qt$$
$$+\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q}d_qsd_qt +$$
$$\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt, \quad (14)$$

$$f(qx,qy)g(qx,qy)\text{-}f(qx,qy)\int_a^b\int_c^d w(t)w(s)g(qt,qs)d_qsd_qt -$$
$$g(qx,qy)\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt$$
$$+(\int_a^b\int_c^d w(t)w(s)g(qt,qs)d_qsd_qt)$$
$$(\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt)$$
$$=[\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt +$$
$$\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt$$
$$+\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt]$$
$$[\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_qsd_qt$$
$$+\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q}d_qsd_qt$$
$$+\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt]. \quad (15)$$

Multiplying both sides by *w(x), w(y)*, then q-integrating the resulting identity over $(x,y)\in[a,b]\times[c,d]$.

$$T(w,f,g)=\int_a^b\int_c^d w(x)w(y)[\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt$$
$$+\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt$$
$$+\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt]$$
$$[\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_qsd_qt$$
$$+\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q}d_qsd_qt +$$
$$\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt,]d_qyd_qx. \quad (16)$$

By using the properties of modulus and (8)-(12) we have

$$\left|T(w,f,g)\right| \le \int_a^b\int_c^d w(x)w(y)R(x,y)Q(x,y)d_qyd_qx.$$

The proof of theTheorem 1 is complete.

Theorem 2:

Let $f,g:[a,b]\times[c,d]\to\Re$. And w is a non-negative weight function defined in Theorem 1, then

$$\left|T(w,f,g)\right| \le \int_a^b\int_c^d w(x)w(y)\{R(x,y)\left|g(qx,qy)\right|$$
$$+Q(x,y)\left|f(qx,qy)\right|\}d_qyd_qx.$$

Proof: The proof directly follows from (13) and (14) by multiplying both sides with *w(x), w(y)g(qx,qy)* and *w(x)w(y)f(qx,qy)* respectively, then adding

$$w(x)w(y)f(qx,qy)g(qx,qy)=\frac{1}{2}\{w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt + w(x)w(y)f(qx,qy)$$

$$\int_a^b\int_c^d w(t)w(s)g(qt,qs)d_qsd_qt\} +$$

$$\frac{1}{2}\{w(x)w(y)g(qx,qy)\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt$$

$$+ w(x)w(y)f(qx,qy)\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_qsd_qt\}$$

$$+\frac{1}{2}\{w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt +$$

$$w(x)w(y)f(qx,qy)\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt\}$$

$$+\frac{1}{2}\{w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d P_w(qx,t)P_{w(}(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt$$

$$+ w(x)w(y)f(qx,qy)$$

$$\int_a^b\int_c^d P_w(qx,t)P_{w(}(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt\}, \quad (17)$$

q- Integrating over $(x,y)\in[a,b]\times[c,d]$.

$$\mathrm{T}(w,f,g)=\frac{1}{2}\{\int_a^b\int_c^d w(x)w(y)g(qx,qy)$$

$$(\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt)d_qyd_qx$$

$$+\int_a^b\int_c^d w(x)w(y)f(qx,qy)$$

$$(\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_qsd_qt)d_qyd_qx\}$$

$$+\frac{1}{2}\{\int_a^b\int_c^d w(x)w(y)g(qx,qy)$$

$$(\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt)d_qyd_qx$$

$$+\int_a^b\int_c^d w(x)w(y)f(qx,qy)$$

$$(\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q}d_qsd_qt)d_qyd_qx\}$$

$$+\frac{1}{2}\{\int_a^b\int_c^d w(x)w(y)g(qx,qy)$$

$$(\int_a^b\int_c^d P_w(qx,t)P_w(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt)$$

$$d_qyd_qx+\int_a^b\int_c^d w(x)w(y)f(qx,qy)(\int_a^b\int_c^d P_w(qx,t)$$

$$P_{w(}(qy,s)\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt)d_qyd_qx\} \quad (18)$$

From the properties of modulus and (8)-(12) we have

$$|\mathrm{T}(w,f,g)|\le\int_a^b\int_c^d w(x)w(y)\{R(x,y)|g(qx,qy)|$$

$$+Q(x,y)|f(qx,qy)|\}d_qyd_qx.$$

The proof of the Theorem 2 is complete.

4. CONCLUSION

Theorem 3:

Let $f, g : [a,b]\times[c,d] \to \Re$. And *w* is non-negative weight function defined in Theorem 1, then

$$|S(w,f,g)| \le \{R(x,y)|g(qx,qy)| + Q(x,y)|f(qx,qy)|\}$$

and

$$|T(w,f,g)| \le \frac{\|f(qx,qy)\|}{2} \int_a^b\int_c^d w(x)w(y)Q(x,y)d_qyd_qx + \frac{\|g(qx,qy)\|}{2} \int_a^b\int_c^d w(x)w(y)R(x,y)d_qyd_qx.$$

Proof: The proof directly follows from (13) and (14) by multiplying both sides with *g(qx,qy)* and *f(qx,qy)* respectively ,then adding

$$f(qx,qy)g(qx,qy) = \frac{1}{2}\{ f(qx,qy) \int_a^b\int_c^d w(t)w(s)g(qt,qs)d_qsd_qt + g(qx,qy)\int_a^b\int_c^d w(t)w(s)f(qt,qs)d_qsd_qt\} +$$
$$\frac{1}{2}\{ f(qx,qy) \int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q} d_qsd_qt + g(qx,qy)\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_qsd_qt\}$$
$$+\frac{1}{2}\{ f(qx,qy) \int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q} d_qsd_qt + g(qx,qy)\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_qsd_qt\}$$
$$+\frac{1}{2}\{ f(qx,qy) \int_a^b\int_c^d P_w(qy,s)P_{w(}(qx,t)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q} d_qsd_qt + f(qx,qy)\int_a^b\int_c^d P_w(qx,t)P_{w(}(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_qsd_qt\}. \quad (19)$$

From the properties of modulus and (8)-(12) we have

$$|S(w,f,g)| \le \frac{1}{2}\{R(x,y)|g(qx,qy)| + Q(x,y)|f(qx,qy)|\}.$$

Multiplying (19) by *w(x)w(y)* then q-integrating over $(x,y)\in[a,b]\times[c,d]$

and re-arranging terms

$$\mathrm{T}(w,f,g) = \frac{1}{2}\{\int_a^b\int_c^d (w(x)w(y)f(qx,qy)$$

$$(\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q g)(qt,s)}{\Delta_2 q}d_q s d_q t)d_q y d_q x$$

$$+\int_a^b\int_c^d (w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_q s d_q t)d_q y d_q x\} +$$

$$\frac{1}{2}\{\int_a^b\int_c^d (w(x)w(y)f(qx,qy)$$

$$(\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q g)(t,qs)}{\Delta_1 q}d_q s d_q t)d_q y d_q x$$

$$+\int_a^b\int_c^d (w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d w(s)P_w(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_q s d_q t)d_q y d_q x\} +$$

$$\frac{1}{2}\{\int_a^b\int_c^d (w(x)w(y)f(qx,qy)$$

$$(\int_a^b\int_c^d P_w(qy,s)P_{w(}(qx,t)\frac{(\partial_q^2 g)(t,s)}{\Delta_2 q\Delta_1 q}d_q s d_q t)d_q y d_q x +$$

$$\int_a^b\int_c^d w(x)w(y)g(qx,qy)$$

$$\int_a^b\int_c^d P_w(qy,s)P_{w(}(qx,t)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_q s d_q t)d_q y d_q x\}. \quad (20)$$

From the properties of modulus and (8)-(12) we have

$$|\mathrm{T}(w,f,g)| \le \frac{\|f(qx,qy)\|}{2}\int_a^b\int_c^d w(x)w(y)Q(x,y)d_q y d_q x$$

$$+\frac{\|g(qx,qy)\|}{2}\int_a^b\int_c^d w(x)w(y)R(x,y)d_q y d_q x.$$

The proof of the Theorem 3 is complete.

Theorem 4:

Let $f,g:[a,b]\times[c,d]\to\Re$ .And w is non-negative weight function defined in Theorem 1,then

$$|\mathrm{T}(w,f,g)| \le \|g(qx,qy)\| R(x,y),$$

and

$$|\mathrm{T}(w,f,g)| \le \|f(qx,qy)\| Q(x,y).$$

Proof: Multiplying (13) by *w(x) w(y) g(qx,qy)*then , q-integrating over $(x,y)\in[a,b]\times[c,d]$ and rearranging terms

$$\mathrm{T}(w,f,g) = (\int_a^b\int_c^d w(x)w(y)g(qx,qy)\mathrm{d}_q y d_q x)$$

$$(\int_a^b\int_c^d w(t)P_w(qy,s)\frac{(\partial_q f)(qt,s)}{\Delta_2 q}d_q s d_q t)$$

$$+(\int_a^b\int_c^d w(x)w(y)g(qx,qy)\mathrm{d}_q y d_q x)(\int_a^b\int_c^d w(s)P_w$$

$$(qx,t)\frac{(\partial_q f)(t,qs)}{\Delta_1 q}d_q s d_q t) +$$

$$(\int_a^b\int_c^d w(x)w(y)g(qx,qy)\mathrm{d}_q y d_q x)(\int_a^b\int_c^d P_w(qx,t)$$

$$P_{w(}(qy,s)\frac{(\partial_q^2 f)(t,s)}{\Delta_2 q\Delta_1 q}d_q s d_q t).$$

From properties of modulus and (8), (10), (12) we have

$$|\mathrm{T}(w,f,g)| \le \|g(qx,qy)\| R(x,y).$$

Similarly we can prove that

$$|\mathrm{T}(w,f,g)| \le \|f(qx,qy)\| Q(x,y).$$

The proof of the Theorem 4 is complete.

REFERENCES:

1. H. Cauciman, Integral inequalities in q-Calculus, Compute. Math. Appl., 47(2-3)(2004), 281-300.
2. P. Ceron and S. S.Dragomir, A refinement of the Gruss inequality and applications, RGMIA Res. Rep. Collect., 5(2)(2002).
3. S. S.Dragomir and S. Wang, An inequality of Ostrowski-Gruss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Comput.Math. Appl., 33(11)(1997) 15-20.
4. G. Gruss, it Uber des maximum des absolutenBetrages von 1/(b-a)$\int_b^a f(x)\,dx$-1/(b-a)^2)$\int_b^a f(x)\,dx$)$\int_b^a f(g)\,dx$, Math.Z.,39(1935), no.1, 215-226.
5. F. H. Jackson, on q-definite integrals, Quaterly Journals of Pure and Applied Mathematics, 41(1910),193-203.
6. Z. Liu, notes on Gruss type inequality and its application, Veitnam J. Math., 35(2)(2007) 121-127.
7. D. S. Mitrinovic, J. E. Pecaric, A. M .Fink, Classical and new inequalities in Analysis, in: Mathematics and its Applications, Vol. 61, Kluwer Acadmic Publishers, Dordrecht, The Netherlands, 1993.
8. B. G. Pachpatte, On Chebyshev-Gruss type inequalities via Pecaricextention of Montgomery identity, J. Inequal. Pure Appl. Math.,7(1)(2006).
9.Wengui Yang, On weighted q-Chebyshev-Gruss type inequalities, Comput. Math. Appl., 61(2011) 1342-1347.