SOFT INTERSECTIONAL IDEALS IN TERNARY SEMIRINGS

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ABSTRACT: *In this paper, we introduce the notions of soft intersectional ternary subsemirings and soft intersectional ideals in ternary semirings. We also discuss some basic results associated with these notions. In the last part of the paper we characterize regular and weakly regular ternary semirings by their soft intersectional ideals.*

Key Words: Ternary semirings, regular ternary semirings, weakly regular ternary semirings, soft intersectional ideals in ternary semirings.

1. INTRODUCTION

The algebraic structure of semirings were first introduced by H. S. Vandiver [1]. Since then the concept of semiring have been deeply studied by the mathematician and proved very helpful in information sciences. Hemirings are the semirings with commutative addition and zero element. Semirings and hemirings are used to study graph theory, optimization theory, formal languages and automata theory [2, 3, 4]. Ideals in semiring play a vital role and are very useful for different purposes. J. Ahsan [5] introduced and characterized weakly regular semirings by the properties of their ideals. The concept of ternary rings was first introduced by D.H. Lehmer [6]. In his paper he also discussed certain algebraic structures called triplexes. Duta and Kar [7] generalized ternary rings and introduced the concept of ternary semirings. Bhambri et. al [8] introduced the concept of weakly regular ternary semirings.

To deal with the uncertainties L. A. Zadeh [9] introduced the concept of fuzzy sets. Further he generalized this concept in [10] and introduced the concept of interval valued fuzzy sets. Shabir et. al [11] used the concept of interval valued fuzzy sets and characterized regular and weakly regular semirings by using their interval valued fuzzy ideals.

Soft sets was first introduced by Molodtsov [12] aiming to deal with uncertainty or ambiguities using mathematical models. Soft sets got the importance as it was found that some problems that could not handled by existing tools like fuzzy sets and its generalizations can be handled by using soft sets. Maji [13] defined some new operations of soft sets and used it for decision making problems [14]. M. I. Ali et. al started working on soft sets and defined some new operations on soft sets [15]. Since then soft sets have been extensively used in many branches of Mathematics and information sciences. Feng Feng and Y. B. Jun [16] defined soft semirings and soft ideals in soft semirings. The study of soft groups by Aktas and Cagman [17], opened the doors of progress to use soft sets in algebric structure. This progress lead the researchers to the detailed study of soft rings [18], soft semigroup [19] and soft BCK/BCI algebra [20]. Song et. al [21] introduced the concept of soft intersectional ideals in semigroups. T. Mahmood and U. Tariq [22] carried out this concept and applied it on semirings. In this paper we introduce the notions of soft intersectional ternary subsemirings and soft intersectional ideals in ternary semirings. We also discuss some basic results associated with these notions. In the last part of the paper we characterize regular and weakly regular ternary semirings by their soft intersectional ideals.

2. Preliminaries

A set $S \neq \Phi$ with a binary operation addition "+" and a ternary multiplication "$\cdot$", denoted by juxtaposition, is said to be a ternary semiring S if it satisfies the following conditions:

$(i)(lmn)op = l(mno)p = lm(nop)$

$(ii)(l+m)no = \ln o + mno$

$(iii)l(m+n)o = lmo + \ln o$

$(iv)lm(n+o) = lmn + lmo,$ for all $l, m, n, o, p \in S.$

2.1 Remark

From now to onward, if otherwise stated, *S* will always denote a ternary semiring. Further for undefined terms and notions for *S* see [6].

If *U* is initial universe, *E* is a set of parameters and *A, B, C,…* are subsets of *E*. Then we have:

2.2 Definition [12]

A soft set (ε, A) over U means that ε is a mapping $\varepsilon: A \to P(U)$. Then we will write here (ε, A, U) instead of writing "(ε, A) is soft set over U", if otherwise stated.

2.3 Definition [21]

Let (ε, A, U) and $\gamma \in P(U)$. Then the set $i_A(\varepsilon;\gamma) = \{w \in A:$ $\gamma \subseteq \varepsilon_1(w)$ is called γ inclusive set of (ε, A).

3. Main Results

Here we take $E = S$, if otherwise stated.

3.1 Definition

For (ε_1, S, U) and (ε_2, S, U), the sum $(\varepsilon_1 + \varepsilon_2, S, U)$ is defned by

$$(\varepsilon_1 + \varepsilon_2)(u) = \bigcup_{u=l+m} \{\varepsilon_1(l) \cap \varepsilon_2(m)\}, \forall u \in S.$$

3.2 Definition

For (ε_1, S, U), (ε_2, S, U) and (ε_3, S, U), the product $(\varepsilon_1\varepsilon_2\varepsilon_3, S, U)$ is defined by

3.3 Definition

(ε, S, U) is called ternary soft intersectional subsemiring of S if $\forall\ l, m, r \in S$,

$(i)\ \ \varepsilon(l + m) \supseteq \varepsilon(l) \cap \varepsilon(m),$

$(ii)\ \ \varepsilon(lmr) \supseteq \varepsilon(l) \cap \varepsilon(m) \cap \varepsilon(r).$

It will be denoted by ε^t_{si-S}.

3.4 Definition

(ε, S, U) is called ternary soft intersectional left (right, lateral) ideal of S if $\forall\ l, m, r \in S$,

$(i)\ \ \varepsilon(l + m) \supseteq \varepsilon(l) \cap \varepsilon(m),$

$(ii)\ \ \varepsilon(lmr) \supseteq \varepsilon(r)$

$\big(\varepsilon(lmr) \supseteq \varepsilon(l), \varepsilon(lmr) \supseteq \varepsilon(m)\big)$. It will be denoted by $\varepsilon^t_{si-L_i}(\varepsilon^t_{si-R_i}, \varepsilon^t_{si-E_i})$.

(ε, S, U) is called ternary soft intersectional ideal, if it is soft intersectional left, right and lateral ideal of S at the same time. It will be denoted by ε^t_{si-I}.

3.5 Lemma

(ε, S, U) is ε^t_{si-S}. of S if and only if $\varepsilon + \varepsilon \subseteq \varepsilon$ and $\varepsilon^3 \subseteq \varepsilon$.

Proof. Let us assume (ε, S) is a ε^t_{si-S} of S. Then, $\forall$ $w \in S$

$$(\varepsilon+\varepsilon)(u) = \bigcup_{u=l+m} [\varepsilon(l) \cap \varepsilon(m)]$$
$$\subseteq \bigcup_{u=l+m} [\varepsilon(l+m)]$$
$$= \bigcup_{u=l+m} [\varepsilon(u)]$$
$$= \varepsilon(u).$$

Thus, $\varepsilon + \varepsilon \subseteq \varepsilon$.

Now, $\varepsilon^3(u) = (\varepsilon\varepsilon\varepsilon)(u)$

$$= \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i [\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)]\}$$
$$\subseteq \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i \varepsilon(l_i m_i r_i)\}$$
$$\subseteq \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\varepsilon(\textstyle\sum_{i=1}^n l_i m_i r_i)\}$$
$$= \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\varepsilon(u)\}$$
$$= \varepsilon(u).$$

Thus, $\varepsilon^3 \subseteq \varepsilon$.

Conversely, let us assume $\varepsilon + \varepsilon \subseteq \varepsilon$ and $\varepsilon^3 \subseteq \varepsilon$. Then, $\forall\, u, v, w \in S$,

$$\varepsilon(u+v) \supseteq (\varepsilon+\varepsilon)(u+v)$$
$$= \bigcup_{u+v=l+m} [\varepsilon(l) \cap \varepsilon(m)]$$
$$\supseteq \varepsilon(u) \cap \varepsilon(v).$$

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(u) = \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \begin{bmatrix} \bigcap_i \{\varepsilon_1(l_i) \\ \cap\, \varepsilon_2(m_i) \cap \varepsilon_3(r_i)\} \end{bmatrix},$$
$$\forall\, u \in S.$$

And

$$\varepsilon(uvw) \supseteq \varepsilon^3(uvw) = (\varepsilon\varepsilon\varepsilon)(uvw)$$
$$= \bigcup_{uvw=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i [\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)]\}$$
$$\supseteq [\varepsilon(u) \cap \varepsilon(v) \cap \varepsilon(w)]$$

Thus, $\varepsilon(u+v) \supseteq \varepsilon(u) \cap \varepsilon(v)$ and $\varepsilon(uvw) \supseteq \varepsilon(u) \cap \varepsilon(v) \cap \varepsilon(w)$, $\forall\, u, v, w \in S$.

Hence, (ε, S) is ε^t_{si-S} of S.

3.6 Lemma

(ε, S, U) is $\varepsilon^t_{si-L_i}(\varepsilon^t_{si-R_i}, \varepsilon^t_{si-E_i})$ of S if and only if $\varepsilon + \varepsilon \subseteq \varepsilon$ and $\chi_S\chi_S\varepsilon \subseteq \varepsilon$ ($\varepsilon\chi_S\chi_S \subseteq \varepsilon$, $\chi_S\varepsilon\chi_S \subseteq \varepsilon$).

Proof. Let us assume (ε, S) be a $\varepsilon^t_{si-L_i}$ of S. Then, $w \in S$

$$(\varepsilon+\varepsilon)(u) = \bigcup_{u=l+m} [\varepsilon(l) \cap \varepsilon(m)]$$
$$\subseteq \bigcup_{u=l+m} [\varepsilon(l+m)]$$
$$= \bigcup_{u=l+m} \varepsilon(u)$$
$$= \varepsilon(u).$$

Thus, $\varepsilon + \varepsilon \subseteq \varepsilon$.

Now, $(\chi_S\chi_S\varepsilon)(u)$

$$= \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i [\chi_S(l_i) \cap \chi_S(m_i) \cap \varepsilon(r_i)]\}$$
$$\subseteq \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i \{U \cap U \cap \varepsilon(r_i)]\}$$
$$= \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i \varepsilon(r_i)\}$$
$$\subseteq \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i \varepsilon(l_i m_i r_i)\}$$
$$\subseteq \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} [\varepsilon(\textstyle\sum_{i=1}^n l_i m_i r_i)]$$
$$= \bigcup_{u=\sum_{i=1}^n l_i m_i r_i} \varepsilon(u)$$
$$= \varepsilon(u).$$

Thus, $\chi_S\chi_S\varepsilon \subseteq \varepsilon$.

Conversely, let us assume $\varepsilon + \varepsilon \subseteq \varepsilon$ and $\chi_S\chi_S\varepsilon \subseteq \varepsilon$.

Then, for all $u, v, w \in S$

$$\varepsilon(u+v) \supseteq (\varepsilon+\varepsilon)(u+v)$$
$$= \bigcup_{u+v=l+m} [\varepsilon(l) \cap \varepsilon(m)]$$
$$\supseteq \varepsilon(l) \cap \varepsilon(m).$$

And,

$$\varepsilon(uvw) \supseteq (\chi_S\chi_S\varepsilon)(uvw)$$
$$= \bigcup_{uvw=\sum_{i=1}^n l_i m_i r_i} \{\textstyle\bigcap_i [\chi_S(l_i) \cap \chi_S(m_i) \cap \varepsilon(r_i)]\}$$
$$\supseteq [\chi_S(u) \cap \chi_S(v) \cap \varepsilon(w)]$$
$$= [U \cap U \cap \varepsilon(w)]$$
$$= \varepsilon(w).$$

Thus, $\varepsilon(u+v) \supseteq \varepsilon(u) \cap \varepsilon(v)$ and $\varepsilon(uvw) \supseteq \varepsilon(w)$, $\forall\, u, v, w \in S$. Hence, (ε, S, U) be a $\varepsilon^t_{si-L_i}$ of S.

3.7 Theorem

$\emptyset \neq W \subseteq S$. Then W is $\mp_{s-s}$ of S if and only if χ_W is

ε_{si-S}^{t} of S.

Proof. Suppose that, W is a $\mp_{s-s}$ of S and $u,..2v,w \in S$

Case (I) For $u,v,w \in W$, we have, $u+v, uvw \in W$. Then

$\chi_W(u+v)=U=U\cap U=\chi_W(u)\cap\chi_W(v)$ and

$$\chi_W(uvw)=U=U\cap U\cap U$$
$$=\chi_W(u)\cap\chi_W(v)\cap\chi_W(w).$$

Case (II) For at least one, say $v\notin W$, we have

$\chi_W(v)=\Phi$. Then

$$\chi_W(u+v)\supseteq\Phi=\chi_W(u)\cap\Phi$$
$$=\chi_W(u)\cap\chi_W(v),$$

and $\chi_W(uvw)\supseteq\Phi=\chi_W(u)\cap\Phi\cap\chi_W(w)$

$$=\chi_W(u)\cap\chi_W(v)\cap\chi_W(w).$$

By combining the both cases, we have

$\chi_W(u+v)\supseteq\chi_W(u)\cap\chi_W(v)$ and

$$\chi_W(uvw)\supseteq\chi_W(u)\cap\chi_W(v)\cap\chi_W(w).$$

Hence, χ_W is ternary ε_{si-S}^{t} of S.

Conversely, assume that χ_W is a ε_{si-S}^{t} of S and $u,v,w \in W$. Then

$$\chi_W(u+v)\supseteq\chi_W(u)\cap\chi_W(v)$$
$$=U\cap U=U,$$

and $\chi_W(uvw)\supseteq\chi_W(u)\cap\chi_W(v)\cap\chi_W(w)$

$$=U\cap U\cap U=U.$$

Thus, $u+v,\ uvw\epsilon W,\ \forall\, u,v,w \in W$

This shows that, W is a $\mp_{s-s}$ of S.

3.8 Theorem

Let $W\subseteq S$. Then W is $\mp_{L_i}$ $(\mp_{R_i},\mp_{E_i})$ of S if and only if the characteristic function χ_W is $\varepsilon_{si-L_i}^{t}(\varepsilon_{si-R_i}^{t},\varepsilon_{si-E_i}^{t})$ of S.

Proof. Suppose that, W is a $\mp_{L_i}$ of S and $u,v,w\epsilon$S.

Case (I) For $u,v,w\epsilon W$, we have, $u+v,\ uvw\epsilon W$. Then

$\chi_W(u+w)=U=U\cap U=\chi_W(u)\cap\chi_W(w)$ and

$$\chi_W(uvw)=U=\chi_W(w).$$

Case (II) For at least one, say $w\notin W$, we have,

$\chi_W(w)=\Phi$. Then

$$\chi_W(u+w)\supseteq\Phi=\chi_W(u)\cap\Phi$$
$$=\chi_W(u)\cap\chi_W(w),$$

and $\chi_W(uvw)\supseteq\Phi=\chi_W(u)\cap\chi_W(v)\cap\Phi$

$$=\chi_W(u)\cap\chi_W(v)\cap\chi_W(w).$$

By combining the both cases, we have

$\chi_W(u+w)\supseteq\chi_W(u)\cap\chi_W(w)$ and

$$\chi_W(uvw)\supseteq\chi_W(w).$$

Hence, χ_W is $\varepsilon_{si-L_i}^{t}$ of S.

Conversely, assume that χ_W is a $\varepsilon_{si-L_i}^{t}$ of S and $u,v,w \in W$. Then

$$\chi_W(u+v)\supseteq\chi_W(u)\cap\chi_W(w)$$
$$=U\cap U=U,$$

and $\chi_W(uvw)\supseteq\chi_W(u)\cap\chi_W(v)\cap\chi_W(w)$

$$=U\cap U\cap\chi_W(w)$$
$$=\chi_W(w).$$

Thus, $u+v,\ uvw\epsilon W,\ \forall\, u,v,w \in W$.

This shows that, S is a $\mp_{L_i}$ of S.

3.9 Theorem

If (ε_1,S,U), (ε_2,S,U) be two $\varepsilon_{si-L_i}^{t}{}'s$ $(\varepsilon_{si-R_i}^{t}{}'s$, $\varepsilon_{si-E_i}^{t}{}'s)$ of S, then their sum $(\varepsilon_1+\varepsilon_2,S,U)$ is also a $\varepsilon_{si-L_i}^{t}$ $(\varepsilon_{si-R_i}^{t}$, $\varepsilon_{si-E_i}^{t})$ of S.

Proof. To show that $(\varepsilon_1+\varepsilon_2,S,U)$ is a $\varepsilon_{si-L_i}^{t}$ of S, we will have to prove that

$(\varepsilon_1+\varepsilon_2)(u+v)\supseteq(\varepsilon_1+\varepsilon_2)(u)\cap(\varepsilon_1+\varepsilon_2)(v)$ and

$(\varepsilon_1+\varepsilon_2)(uvw)\supseteq(\varepsilon_1+\varepsilon_2)(w)$,

$\forall\, u,v,w \in S$.

Then,

$$(\varepsilon_1+\varepsilon_2)(u)\cap(\varepsilon_1+\varepsilon_2)(v)$$
$$=\{\bigcup_{u=l+m}[\varepsilon_1(l)\cap\varepsilon_2(m)]\}$$
$$\cap\{\bigcup_{v=l'+m'}[\varepsilon_1(l')\cap\varepsilon_2(m')]\}$$
$$=\bigcup_{u=l+m,\,v=l'+m'}[\varepsilon_1(l)\cap\varepsilon_2(m)\cap\varepsilon_1(l')\cap\varepsilon_2(m')]$$
$$\subseteq\bigcup_{u=l+m,\,v=l'+m'}[\varepsilon_1(l+l')\cap\varepsilon_2(m+m')]$$
$$\subseteq\bigcup_{u+v=x+y}[\varepsilon_1(x)\cap\varepsilon_2(y)]$$
$$=(\varepsilon_1+\varepsilon_2)(u+v).$$

And, $(\varepsilon_1+\varepsilon_2)(w)=\bigcup_{w=l+m}[\varepsilon_1(l)\cap\varepsilon_2(m)]$

$$\subseteq\bigcup_{w=u+v}[\varepsilon_1(uvl)\cap\varepsilon_2(uvm)]$$
$$\subseteq\bigcup_{uvw=x+y}[\varepsilon_1(x)\cap\varepsilon_2(y)]$$
$$=(\varepsilon_1+\varepsilon_2)(uvw).$$

Hence $(\varepsilon_1+\varepsilon_2,S)$ is a $\varepsilon_{si-L_i}^{t}$ of S.

3.10 Theorem

If (ε_1,S,U), (ε_2,S,U) and (ε_3,S,U) be three $\varepsilon_{si-L_i}^{t}{}'s$ $(\varepsilon_{si-R_i}^{t}{}'s$, $\varepsilon_{si-E_i}^{t}{}'s)$ of S, then $(\varepsilon_1\varepsilon_2\varepsilon_3,S,U)$ is also a $\varepsilon_{si-L_i}^{t}$ $(\varepsilon_{si-R_i}^{t}$, $\varepsilon_{si-E_i}^{t})$ of S.

Proof. Let $u,v,\in S$. Then

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(u)$$
$$=\bigcup_{u=\sum_{i=1}^{n}l_im_ir_i}\left\{\bigcap_i[\varepsilon_1(l_i)\cap\varepsilon_2(m_i)\cap\varepsilon_3(r_i)]\right\}$$

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(v) = \bigcup_{v=\sum_{j=1}^{n} l'_j m'_j r'_j} \{\textstyle\bigcap_j [\varepsilon_1(l'_j) \cap \varepsilon_2(m'_j) \cap \varepsilon_3(r'_j)]\}$$

Now

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(u) \cap (\varepsilon_1\varepsilon_2\varepsilon_3)(v)$$

$$= \left\{ \bigcup_{u=\sum_{i=1}^{n} l_i m_i r_i} \{{}^{i}[\varepsilon_1(l_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\} \right\} \cap \left\{ \bigcup_{v=\sum_{j=1}^{n} l'_j m'_j r'_j} \{\textstyle\bigcap_j [\varepsilon_1(l'_j) \cap \varepsilon_2(m'_j) \cap \varepsilon_3(r'_j)]\} \right\}$$

$$= \bigcup_{u=\sum_{i=1}^{n} l_i m_i r_i,\, v=\sum_{j=1}^{n} l'_j m'_j r'_j} \begin{bmatrix} \{\bigcap_i [\varepsilon_1(l_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\} \\ \{\bigcap_j [\varepsilon_1(l'_j) \cap \varepsilon_2(m'_j) \cap \varepsilon_3(r'_j)]\} \end{bmatrix}$$

$$= \bigcup_{u+v=\sum_{i=1}^{n} l_i m_i r_i + \sum_{j=1}^{n} l'_j m'_j r'_j} \begin{bmatrix} \{\bigcap_i [\varepsilon_1(l_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\} \\ \{\bigcap_j [\varepsilon_1(l'_j) \cap \varepsilon_2(m'_j) \cap \varepsilon_3(r'_j)]\} \end{bmatrix}$$

$$\subseteq \bigcup_{u+v=\sum_{l=1}^{n} x_k y_k z_k} \{\textstyle\bigcap_k [\varepsilon_1(x_k) \cap \varepsilon_2(y_k) \cap \varepsilon_3(z_k)]\}$$

$$= (\varepsilon_1\varepsilon_2\varepsilon_3)(u+v).$$

For $u, v, w \in S$

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(w) = \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} \{\textstyle\bigcap_i [\varepsilon_1(l_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\}$$

$$\subseteq \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} \{\textstyle\bigcap_i [\varepsilon_1(uvl_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\}$$

$$= \bigcup_{uvw=\sum_{i=1}^{n} uvl_i m_i r_i} \{\textstyle\bigcap_i [\varepsilon_1(uvl_i) \cap \varepsilon_2(m_i) \cap \varepsilon_3(r_i)]\}$$

$$\subseteq \bigcup_{uvw=\sum_{i=1}^{n} x_i y_i z_i} \{\textstyle\bigcap_i [\varepsilon_1(x_i) \cap \varepsilon_2(y_i) \cap \varepsilon_3(z_i)]\}$$

$$= (\varepsilon_1\varepsilon_2\varepsilon_3)(uvw).$$

Thus $(\varepsilon_1\varepsilon_2\varepsilon_3)(u+v) \supseteq (\varepsilon_1\varepsilon_2\varepsilon_3)(u) \cap (\varepsilon_1\varepsilon_2\varepsilon_3)(v)$ and $(\varepsilon_1\varepsilon_2\varepsilon_3)(uvw) \supseteq (\varepsilon_1\varepsilon_2\varepsilon_3)(w)$, $\forall\, u, v, w \in S$. Therefore, $(\varepsilon_1\varepsilon_2\varepsilon_3, S, U)$ is a $\varepsilon^{t}_{si\text{-}L_i}$ of S.

3.11 Theorem

For S the following are equivalent

(i) S is $\mp_{vN\text{-}r}$.

(ii) $E \cap F \cap G = EFG$, for any E, F and G as $\mp_{R_i}, \mp_{E_i}$ and $\mp_{L_i}$ of S, respectively.

(iii) $\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 = \varepsilon_1\varepsilon_2\varepsilon_3$, for any (ε_1, S), (ε_2, S) and (ε_3, S) as $\varepsilon^{t}_{si\text{-}R_i}$, $\varepsilon^{t}_{si\text{-}E_i}$ and $\varepsilon^{t}_{si\text{-}L_i}$ of S, respectively.

Proof. $(i) \Leftrightarrow (ii)$ is followed by Theorem r-4.

$(i) \Rightarrow (iii)$ Let (ε_1, S), (ε_2, S) and (ε_3, S) are $\varepsilon^{t}_{si\text{-}R_i}$, $\varepsilon^{t}_{si\text{-}E_i}$ and $\varepsilon^{t}_{si\text{-}L_i}$ of S, respectively. Then for any $u \in S$

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(u) = \bigcup_{u=\sum_{i=1}^{n} x_i y_i z_i} \{\textstyle\bigcap_i [\varepsilon_1(x_i) \cap \varepsilon_2(y_i) \cap \varepsilon_3(z_i)]\}$$

$$\subseteq \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \left\{ \begin{matrix} \bigcap_i [\varepsilon_1(x_i y_i z_i) \cap \\ \varepsilon_2(x_i y_i z_i) \cap \varepsilon_3(x_i y_i z_i)] \end{matrix} \right\}$$

$$= \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \begin{bmatrix} \{\bigcap_i \varepsilon_1(x_i y_i z_i)\} \\ \cap \{\bigcap_i \varepsilon_2(x_i y_i z_i)\} \\ \cap \{\bigcap_i \varepsilon_3(x_i y_i z_i)\} \end{bmatrix}$$

$$\subseteq \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \begin{bmatrix} \{\varepsilon_1(\sum_{i=1}^{n} x_i y_i z_i) \\ \cap \varepsilon_2(\sum_{i=1}^{n} x_i y_i z_i) \\ \cap \varepsilon_3(\sum_{i=1}^{n} x_i y_i z_i)\} \end{bmatrix}$$

$$= \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \{\varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)\}$$

$$= \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)$$

$$= (\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w)$$

Thus $\varepsilon_1\varepsilon_2\varepsilon_3 \subseteq \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3$ (1`)

Since, S is $\mp_{vN\text{-}r}$, so, for $w \in S$ there exist $l \in S$ such that $w = wlw$.

Now $(\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w) = \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)$

$$\subseteq \varepsilon_1(w) \cap \varepsilon_2(wlw) \cap \varepsilon_3(w)$$

$$(\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w)$$

Thus $\subseteq \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \{\bigcap_i [\varepsilon_1(x_i) \cap \varepsilon_2(y_i) \cap \varepsilon_3(z_i)]\}$

$$= (\varepsilon_1\varepsilon_2\varepsilon_3)(w)$$

This implies that

$$\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \subseteq \varepsilon_1\varepsilon_2\varepsilon_3 \qquad (2)$$

From (1) and (2)

$$\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3.$$

$(iii) \Rightarrow (i)$ Let E be a $\mp_{R_i}, F$ be a $\mp_{E_i}$ and G be a $\mp_{L_i}$ of S. Then the characteristic functions χ_E, χ_F and χ_G are $\varepsilon^{t}_{si\text{-}R_i}$, $\varepsilon^{t}_{si\text{-}E_i}$ and $\varepsilon^{t}_{si\text{-}L_i}$ of S, respectively, and by hypothesis

$$\chi_{EFG} = \chi_E \cdot \chi_F \cdot \chi_G = \chi_E \cap \chi_F \cap \chi_G$$
$$\Rightarrow \chi_{EFG} = \chi_{E\cap F\cap G}$$

Thus, $EFG = E \cap F \cap G$

Therefore, S is $\mp_{vN\text{-}r}$.

3.12 Theorem

For S with 1, the following are equivalent

(i) S is $\mp_{R_{w\text{-}r}}$.

(ii) All $\mp_{R_i}$'s of S are idempotent.

(iii) $EFG = E \cap F \cap G$, for any E, F and G as $\mp_{R_i}, \mp_{E_i}$ and $\mp_{L_i}$ of S, respectively.

(iv) All $\varepsilon^{t}_{si\text{-}R_i}$'s of S are fully idempotent.

(v) $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3$, for any $\varepsilon_1, \varepsilon_2$ and ε_3 as $\varepsilon^{t}_{si\text{-}R_i}$, $\varepsilon^{t}_{si\text{-}E_i}$ and $\varepsilon^{t}_{si\text{-}L_i}$ of S, respectively.

If S is commutative then $(i)-(v)$ are equivalent to

(vi) S is $\mp_{vN\text{-}r}$.

Proof $(i) \Rightarrow (iv)$ Let (ε, S) be a $\varepsilon^{t}_{si\text{-}R_i}$ of S and $w \in S$. Then

$$\varepsilon^3(w) = (\varepsilon\varepsilon\varepsilon)(w)$$
$$= \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} \left\{\bigcap_i \left[\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)\right]\right\}$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} \{\bigcap_i [\varepsilon(l_i m_i r_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)]\}$$
$$= \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} \left[\left\{\bigcap_i \varepsilon(l_i m_i r_i)\right\} \cap \{(\varepsilon(m_i) \cap \varepsilon(r_i))\}\right]$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} a_i b_i c_i} [\varepsilon(\textstyle\sum_{i=1}^{n} l_i m_i r_i) \cap \{\cap_i (\varepsilon(m_i) \cap \varepsilon(r_i))\}]$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} [\varepsilon(w) \cap \{\cap_i (\varepsilon(m_i) \cap \varepsilon(r_i))\}]$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} l_i m_i r_i} [\varepsilon(w)]$$
$$= \varepsilon(w).$$
$$\nearrow \quad \varepsilon^3 \subseteq \varepsilon.$$

Since, S is $\mp_{R_{w\text{-}r}}$. So, $w \in wSwSwSS$ and we can write $w = \sum_{i=1}^{n} wa_i wb_i wc_i d_i$ where a_i, b_i, c_i, $d_i \in S$ and $n \in \mathrm{N}$.

$$\varepsilon(w) \cap \varepsilon(w) \cap \varepsilon(w)$$
$$\subseteq \varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i), \text{ for all } i$$
$$\varepsilon(w) \subseteq \bigcap_i \left\{\varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i)\right\}$$
$$= \bigcap_i [\varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i)]$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} wa_i w(wb_i w) wc_i d_i} \left\{\begin{array}{l} \bigcap_i [\varepsilon(wa_i w) \\ \cap\, \varepsilon(wb_i w) \\ \cap\, \varepsilon(wc_i d_i)] \end{array}\right\}$$
$$\subseteq \bigcup_{w=\sum_{j=1}^{n} a_{j'} b_{j'} c_{j'}} \left\{\begin{array}{l} \bigcap_j [\varepsilon(a_{j'}) \cap \varepsilon(b_{j'}) \\ \cap\, \varepsilon(c_{j'})] \end{array}\right\}$$
$$= (\varepsilon\varepsilon\varepsilon)(w)$$
$$= \varepsilon^3.$$
$$\Rightarrow \varepsilon \subseteq \varepsilon^3.$$

Thus, $\varepsilon = \varepsilon^3$.

Hence, ε is fully idempotent.

$(iv) \Rightarrow (i)$ Let $w \in S$ and $A = wSS$ be a $\mp_{R_i}$ of S generated by w. Then $w \in A$ and the characteristic function χ_A of A is $\varepsilon^{t}_{si\text{-}R_i}$ of S and by hypothesis

$$\chi_A = \chi_A \cdot \chi_A \cdot \chi_A = \chi_{A^3}$$
$$\Rightarrow \quad A = A^3$$
$$\Rightarrow w \in A^3 = (wSS)^3$$
$$\Rightarrow w \in wSwSwSS$$

Thus, S is $\mp_{R_{w\text{-}r}}$.

$(i) \Rightarrow (v)$ Let ε_1, ε_2 and ε_3 are $\varepsilon^{t}_{si\text{-}R_i}$, $\varepsilon^{t}_{si\text{-}E_i}$ and $\varepsilon^{t}_{si\text{-}L_i}$ of S, respectively. Then for any $w \in S$

$$(\varepsilon_1\varepsilon_2\varepsilon_3)(w)$$
$$= \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \left\{\bigcap_i [\varepsilon_1(x_i) \cap \varepsilon_2(y_i) \cap \varepsilon_3(z_i)]\right\}$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \{\bigcap_i [\varepsilon_1(x_i y_i z_i) \cap \varepsilon_2(x_i y_i z_i) \cap \varepsilon_3(x_i y_i z_i)]\}$$
$$= \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \left[\begin{array}{c} \{\bigcap_i \varepsilon_1(x_i y_i z_i)\}\ \{\bigcap_i \varepsilon_2(x_i y_i z_i)\} \\ \{\bigcap_i \varepsilon_3(x_i y_i z_i)\} \end{array}\right]$$
$$\subseteq \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \left[\begin{array}{l} \{\varepsilon_1(\sum_{i=1}^{n} x_i y_i z_i) \cap \varepsilon_2(\sum_{i=1}^{n} x_i y_i z_i) \\ \cap\, \varepsilon_3(\sum_{i=1}^{n} x_i y_i z_i)\} \end{array}\right]$$
$$= \bigcup_{w=\sum_{i=1}^{n} x_i y_i z_i} \{\varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)\}$$
$$= \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)$$
$$= (\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w).$$
$$\nearrow \quad \varepsilon_1\varepsilon_2\varepsilon_3 \subseteq \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3.$$

Since, S is $\mp_{R_{w\text{-}r}}$. Then $w \in S$ can be written as $w = \sum_{i=1}^{n} wa_i wb_i wc_i d_i$, where $a_i, b_i, c_i, d_i \in S$,

$n \in \mathrm{N}$.

Now $(\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w) = \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)$

$\subseteq \varepsilon_1(wa_i w) \cap \varepsilon_2(wb_i w) \cap \varepsilon_3(wc_i d_i)$, $\forall \ i$.

Thus

$$(\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w)$$

$$\subseteq \bigcap_i \begin{bmatrix} \varepsilon_1(wa_i w) \cap \varepsilon_2(wb_i w) \cap \\ \varepsilon_3(wc_i d_i) \end{bmatrix}$$

$$\subseteq \bigcup_{w=\sum_{i=1}^{n} wa_i w(wb_i w) wc_i d_i} \begin{bmatrix} \{\bigcap_i [\varepsilon_1(wa_i w) \cap \\ \varepsilon_2(wb_i w) \cap \varepsilon_3(wc_i d_i)]\} \end{bmatrix}$$

$$\subseteq \bigcup_{w=\sum_{j=1}^{n} a_j' b_j' c_j'} \{\bigcap_j [\varepsilon_1(a_j') \cap \varepsilon_2(b_j') \cap \varepsilon_3(c_j')]\}$$

$$= (\varepsilon_1 \varepsilon_2 \varepsilon_3)(w).$$

⬈ $\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \subseteq \varepsilon_1 \varepsilon_2 \varepsilon_3$.

⬀ $\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 = \varepsilon_1 \varepsilon_2 \varepsilon_3$.

$(v) \Rightarrow (iii)$ Let E, F and G are $\mp_{R_i}$, $\mp_{E_i}$ and $\mp_{L_i}$ of S, respectively. Then the characteristic functions χ_E, χ_F and χ_G are $\varepsilon^t_{si\text{-}R_i}$, $\varepsilon^t_{si\text{-}E_i}$ and $\varepsilon^t_{si\text{-}L_i}$ of S, respectively, and by hypothesis

$$\chi_{EFG} = \chi_E \cdot \chi_F \cdot \chi_G = \chi_E \cap \chi_F \cap \chi_G$$

$$\Rightarrow \chi_{EFG} = \chi_{E \cap F \cap G}$$

$$\Rightarrow EFG = E \cap F \cap G.$$

$(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(i) \Rightarrow (vi)$ are straightforward.

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