SOFT INTERSECTIONAL IDEALS IN TERNARY SEMIRINGS

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ABSTRACT: In this paper, we introduce the notions of soft intersectional ternary subsemirings and soft intersectional ideals in ternary semirings. We also discuss some basic results associated with these notions. In the last part of the paper we characterize regular and weakly regular ternary semirings by their soft intersectional ideals.

Key Words: Ternary semirings, regular ternary semirings, weakly regular ternary semirings, soft intersectional ideals in ternary semirings.

1. INTRODUCTION

The algebraic structure of semirings were first introduced by H. S. Vandiver [1]. Since then the concept of semiring have been deeply studied by the mathematician and proved very helpful in information sciences. Hemirings are the semirings with commutative addition and zero element. Semirings and hemirings are used to study graph theory, optimization theory, formal languages and automata theory [2, 3, 4]. Ideals in semiring play a vital role and are very useful for different purposes. J. Ahsan [5] introduced and characterized weakly regular semirings by the properties of their ideals. The concept of ternary rings was first introduced by D.H. Lehmer [6]. In his paper he also discussed certain algebraic structures called tripexes. Dutta and Kar [7] generalized ternary rings and introduced the concept of ternary semirings. Bhambri et. al [8] introduced the concept of weakly regular ternary semirings.

To deal with the uncertainties L. A. Zadeh [9] introduced the concept of fuzzy sets. Further he generalized this concept in [10] and introduced the concept of interval valued fuzzy sets. Maji et. al [11] used the concept of interval valued fuzzy sets and characterized regular and weakly regular semirings by using their interval valued fuzzy ideals.

Soft sets was first introduced by Molodtsov [12] aiming to deal with uncertainty or ambiguities using mathematical models. Soft sets got the importance as it was found that some problems that could not handled by existing tools like fuzzy sets and its generalizations can be handled by using soft sets. Maji [13] defined some new operations of soft sets and used it for decision making problems [14]. M. I. Ali et. al started working on soft sets and defined some new operations on soft sets [15]. Since then soft sets have been extensively used in many branches of Mathematics and information sciences. Feng Feng and Y. B. Jun [16] defined soft semirings and soft ideals in soft semirings. The study of soft groups by Aktaş and Çağman [17], opened the doors of progress to use soft sets in algebraic structure. This progress lead the researchers to the detailed study of soft rings [18], soft semigroup [19] and soft BCK/BCI algebra [20]. Song et. al [21] introduced the concept of soft intersectional ideals in semigroups. T. Mahmood and U. Tariq [22] carried out this concept and applied it on semirings. In this paper we introduce the notions of soft intersectional ternary subsemirings and soft intersectional ideals in ternary semirings. We also discuss some basic results associated with these notions. In the last part of the paper we characterize regular and weakly regular ternary semirings by their soft intersectional ideals.

2. Preliminaries

A set \( S \neq \emptyset \) with a binary operation addition "+" and a ternary multiplication "\(*\)" denoted by juxtaposition, is said to be a ternary semiring \( S \) if it satisfies the following conditions:

\[ (i) \,(lmn)op = l(mno)p = lm(op) \]
\[ (ii) \,(l + m)n o = ln o + mno \]
\[ (iii) \,l(m + n) o = lmo + hno \]
\[ (iv) \,l(m + n) o = lmn + lmo, \,\text{for all} \, \, l, m, n, o, p \in S. \]

2.1 Remark

From now to onward, if otherwise stated, \( S \) will always denote a ternary semiring. Further for undefined terms and notions for \( S \) see [6].

If \( U \) is initial universe, \( E \) is a set of parameters and \( A, B, C, \ldots \) are subsets of \( E \). Then we have:

2.2 Definition [12]

A soft set \( (\mathcal{E}, A) \) over \( U \) means that \( \mathcal{E} \) is a mapping \( \mathcal{E}: A \to \mathcal{P}(U) \). Then we will write here \( (\mathcal{E}, A, U) \) instead of writing "\( (\mathcal{E}, A) \) is soft set over \( U \)", if otherwise stated.

2.3 Definition [21]

Let \( (\mathcal{E}, A, U) \) and \( \gamma \in \mathcal{P}(U) \). Then the set \( i_{\mathcal{E}}(\gamma; A) = \{ \gamma \subseteq \mathcal{E}(w) \mid w \in A \} \) is called \( \gamma \) inclusive set of \( (\mathcal{E}, A) \).

3. Main Results

Here we take \( E = S \), if otherwise stated.

3.1 Definition

For \( (e_1, S, U) \) and \( (e_2, S, U) \), the sum \( (e_1 + e_2, S, U) \) is defined by

\[ (e_1 + e_2)(u) = \bigcup_{u = l + m} \{ e_1(l) \cap e_2(m) \}, \forall u \in S. \]

3.2 Definition

For \( (e_3, S, U) \), \( (e_2, S, U) \) and \( (e_3, S, U) \), the product \( (e_1, e_2, e_3, S, U) \) is defined by

3.3 Definition

\( (\mathcal{E}, S, U) \) is called ternary soft intersectional subsemiring of \( S \) if \( \forall \, \, l, m, r \in S \),

\[ (i) \, \mathcal{E}(l + m) \supseteq \mathcal{E}(l) \cap \mathcal{E}(m) \],
\[ (ii) \, \mathcal{E}(lmr) \supseteq \mathcal{E}(l) \cap \mathcal{E}(m) \cap \mathcal{E}(r). \]

It will be denoted by \( \mathcal{E}\left(S_{i-S}\right) \).

3.4 Definition

\( (\mathcal{E}, S, U) \) is called ternary soft intersectional left (right, lateral) ideal of \( S \) if \( \forall \, \, l, m, r \in S \),

\[ (i) \, \mathcal{E}(l + m) \supseteq \mathcal{E}(l) \cap \mathcal{E}(m) \],
\[ (ii) \, \mathcal{E}(lmr) \supseteq \mathcal{E}(r) \].
\((\varepsilon(l \cdot m) \supseteq \varepsilon(l), \varepsilon(l \cdot m) \supseteq \varepsilon(m))\). It will be denoted by 
\(\varepsilon_{l \cdot m}^t(\varepsilon_{l \cdot R_t}^t \varepsilon_{l \cdot E_t}^t)\).

\((\varepsilon, S, U)\) is called ternary soft intersectional ideal, if it is soft intersectional left, right and lateral ideal of \(S\) at the same time. It will be denoted by \(\varepsilon_{S-l}^t\).

3.5 Lemma

\((\varepsilon, S, U)\) is \(\varepsilon_{S-l}^t\) of \(S\) if and only if \(\varepsilon + \varepsilon \subseteq \varepsilon\) and \(\varepsilon^3 \subseteq \varepsilon\).

Proof. Let us assume \((\varepsilon, S)\) is a \(\varepsilon_{S-l}^t\) of \(S\). Then, \(\forall w \in S\)

\[ (\varepsilon + \varepsilon)(u) = \bigcup_{u+l=m} [\varepsilon(l) \cap \varepsilon(m)] \]

\[ \subseteq \bigcup_{u+l=m} [\varepsilon(l + m)] \]

\[ = \bigcup_{u+l=m} [\varepsilon(u)] \]

\[ = \varepsilon(u). \]

Thus, \(\varepsilon + \varepsilon \subseteq \varepsilon\).

Now, \(\varepsilon^3(u) = (\varepsilon \cdot \varepsilon)(u)\)

\[ = \bigcup_{u+u+l+m} \{\bigcap_i [\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)]\} \]

\[ \subseteq \bigcup_{u+u+l+m} \{\bigcap_i \varepsilon(l_i, m_i, r_i)\} \]

\[ \subseteq \bigcup_{u+u+l+m} [\varepsilon(u)] \]

\[ = \varepsilon(u). \]

Thus, \(\varepsilon^3 \subseteq \varepsilon\).

Conversely, let us assume \(\varepsilon + \varepsilon \subseteq \varepsilon\) and \(\varepsilon^3 \subseteq \varepsilon\). Then, \(\forall u, v, w \in S, \varepsilon(l + w) \supseteq \varepsilon + \varepsilon(l + w)\)

\[ = \bigcup_{u+v+l+m} [\varepsilon(l) \cap \varepsilon(m)] \]

\[ \supseteq \varepsilon(u) \cap \varepsilon(v) \cap \varepsilon(w). \]

3.6 Lemma

\((\varepsilon, S, U)\) is \(\varepsilon_{S-l}^t(\varepsilon_{S-R_t}^t \varepsilon_{S-E_t}^t)\) of \(S\) if and only if \(\varepsilon + \varepsilon \subseteq \varepsilon\) and \(\chi_S \chi_S \varepsilon \subseteq \varepsilon\) (\(\chi_S \chi_S \subseteq \varepsilon\), \(\chi_S \chi_S \subseteq \varepsilon\)).

Proof. Let us assume \((\varepsilon, S)\) be a \(\varepsilon_{S-l}^t\) of \(S\). Then, \(w \in S\)

\[ (\varepsilon + \varepsilon)(u) = \bigcup_{u+l=m} [\varepsilon(l) \cap \varepsilon(m)] \]

\[ \subseteq \bigcup_{u+l=m} [\varepsilon(l + m)] \]

\[ = \bigcup_{u+l=m} [\varepsilon(u)] \]

\[ = \varepsilon(u). \]

Thus, \(\varepsilon + \varepsilon \subseteq \varepsilon\).

\(\chi_S \chi_S \varepsilon \subseteq \varepsilon\).

Conversely, let us assume \(\varepsilon + \varepsilon \subseteq \varepsilon\) and \(\chi_S \chi_S \varepsilon \subseteq \varepsilon\).

Then, for all \(u, v, w \in S\)

\[ \varepsilon(u + v) \supseteq \varepsilon + \varepsilon(u + v) \]

\[ = \bigcup_{u+v+l+m} [\varepsilon(l) \cap \varepsilon(m)] \]

\[ \supseteq \varepsilon(l) \cap \varepsilon(m). \]

And,

\[ \varepsilon(uvw) \supseteq \varepsilon^3(uvw) = (\varepsilon \cdot \varepsilon)(uvw) \]

\[ = \bigcup_{uvw+uvw+uvw+uvw} \{\bigcap_i [\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)]\} \]

\[ \supseteq \varepsilon(u) \cap \varepsilon(v) \cap \varepsilon(w). \]

Thus, \(\varepsilon(u + v) \supseteq \varepsilon(u) \cap \varepsilon(v)\) and \(\varepsilon(uvw) \supseteq \varepsilon(u) \cap \varepsilon(v) \cap \varepsilon(w). \forall u, v, w \in S.\)
$\varepsilon_{s-L}$ of $S$.

Proof. Suppose that, $W$ is a $\mathbb{F}_{s-S}$ of $S$ and $u, v, w \in S$.

Case (I) For $u, v, w \in W$, we have, $u + v, uvw \in W$. Then

$\mathcal{X}_W(u + v) = U = U \cap U = \mathcal{X}_W(u) \cap \mathcal{X}_W(v)$ and

$\mathcal{X}_W(uvw) = U = U \cap U \cap U$

$= \mathcal{X}_W(u) \cap \mathcal{X}_W(v) \cap \mathcal{X}_W(w)$.

Case (II) For at least one, say $v \notin W$, we have

$\mathcal{X}_W(u + v) = \mathcal{X}_W(u) \cap \mathcal{X}_W(v)$.

Then

$\mathcal{X}_W(u + v) \supseteq \mathcal{X}_W(u) \cap \mathcal{X}_W(v)$

$= U \cap U = U$.

and

$\mathcal{X}_W(uvw) \supseteq \mathcal{X}_W(u) \cap \mathcal{X}_W(v) \cap \mathcal{X}_W(w)$

Thus, $u + v, uvw \in W$, $\forall u, v, w \in W$. This shows that, $W$ is a $\mathbb{F}_{s-S}$ of $S$.

3.8 Theorem

Let $W \subseteq S$. Then $W$ is $\mathbb{F}_{s-L_1}(\mathbb{F}_{R_1}, \mathbb{F}_{E_1})$ of $S$ if and only if the characteristic function $\mathcal{X}_W$ is $\varepsilon_{s-L_1}^{t} (\varepsilon_{s-R_1}^{t}, \varepsilon_{s-E_1}^{t})$ of $S$.

Proof. Suppose that, $W$ is a $\mathbb{F}_{s-L_1}$ of $S$ and $u, v, w \in S$.

Case (I) For $u, v, w \in W$, we have, $u + v, uvw \in W$. Then

$\mathcal{X}_W(u + v) = U = U \cap U = \mathcal{X}_W(u) \cap \mathcal{X}_W(v)$ and

$\mathcal{X}_W(uvw) = U = \mathcal{X}_W(w)$.

Case (II) For at least one, say $w \notin W$, we have

$\mathcal{X}_W(w) = \Phi$.

Then

$\mathcal{X}_W(u + w) \supseteq \Phi = \mathcal{X}_W(u) \cap \Phi$

$= \mathcal{X}_W(u) \cap \mathcal{X}_W(w)$.

and

$\mathcal{X}_W(uvw) \supseteq \Phi = \mathcal{X}_W(u) \cap \mathcal{X}_W(v) \cap \Phi$

$= \mathcal{X}_W(u) \cap \mathcal{X}_W(v) \cap \mathcal{X}_W(w)$.

By combining the both cases, we have

$\mathcal{X}_W(u + w) \supseteq \mathcal{X}_W(u) \cap \mathcal{X}_W(w)$ and

$\mathcal{X}_W(uvw) \supseteq \mathcal{X}_W(w)$.

Hence, $\mathcal{X}_W$ is $\varepsilon_{s-L_1}^{t}$ of $S$.

Conversely, assume that $\mathcal{X}_W$ is a $\varepsilon_{s-L_1}^{t}$ of $S$ and $u, v, w \in W$. Then

$\mathcal{X}_W(u + v) \supseteq \mathcal{X}_W(u) \cap \mathcal{X}_W(w)$

$= U \cap U = U$.

and

$\mathcal{X}_W(uvw) \supseteq \mathcal{X}_W(u) \cap \mathcal{X}_W(v) \cap \mathcal{X}_W(w)$

$= U \cap U \cap U = U$.

Thus, $u + v, uvw \in W$, $\forall u, v, w \in W$. This shows that, $S$ is a $\mathbb{F}_{s-L_1}$ of $S$.

3.9 Theorem

If $(\varepsilon_{1}, S, U), (\varepsilon_{2}, S, U)$ be two $\varepsilon_{s-L_1}^{t}, (\varepsilon_{s-R_1}^{t}, \varepsilon_{s-E_1}^{t})$ of $S$, then their sum $(\varepsilon_{1} + \varepsilon_{2}, S, U)$ is also a $\varepsilon_{s-L_1}^{t}, (\varepsilon_{s-R_1}^{t}, \varepsilon_{s-E_1}^{t})$ of $S$.

Proof. To show that $(\varepsilon_{1} + \varepsilon_{2}, S, U)$ is a $\varepsilon_{s-L_1}^{t}$ of $S$, we will have to prove that

$(\varepsilon_{1} + \varepsilon_{2})(u + v) \supseteq (\varepsilon_{1} + \varepsilon_{2})(u) \cap (\varepsilon_{1} + \varepsilon_{2})(v)$ and

$(\varepsilon_{1} + \varepsilon_{2})(uvw) \supseteq (\varepsilon_{1} + \varepsilon_{2})(w)$.

$\forall u, v, w \in S$.

Then

$(\varepsilon_{1} + \varepsilon_{2})(u) \cap (\varepsilon_{1} + \varepsilon_{2})(v) = \bigcup_{l=m} \{ \varepsilon_{1}(l) \cap \varepsilon_{2}(m) \}$

$n \bigcup_{l=m} \{ \varepsilon_{1}(l) \cap \varepsilon_{2}(m) \}$

$= \bigcup_{l=m} \{ \varepsilon_{1}(l) \cap \varepsilon_{2}(m) \}$

$= \bigcup_{l=m} \{ \varepsilon_{1}(l + l') \cap \varepsilon_{2}(m + m') \}$

$= \bigcup_{l=m} \{ \varepsilon_{1}(l + l') \cap \varepsilon_{2}(m + m') \}$

$= (\varepsilon_{1} + \varepsilon_{2})(u + v)$.

And, $(\varepsilon_{1} + \varepsilon_{2})(uvl) \cap \varepsilon_{2}(uvm) = \bigcup_{w=m} \{ \varepsilon_{1}(uvl) \cap \varepsilon_{2}(uvm) \}$

$= \bigcup_{x+y} \{ \varepsilon_{1}(x) \cap \varepsilon_{2}(y) \}$

$= (\varepsilon_{1} + \varepsilon_{2})(uvw)$.

Hence $(\varepsilon_{1} + \varepsilon_{2}, S)$ is a $\varepsilon_{s-L_1}^{t}$ of $S$.

3.10 Theorem

If $(\varepsilon_{1}, S, U), (\varepsilon_{2}, S, U)$ and $(\varepsilon_{3}, S, U)$ be three $\varepsilon_{s-L_1}^{t}, (\varepsilon_{s-R_1}^{t}, \varepsilon_{s-E_1}^{t})$ of $S$, then $(\varepsilon_{1} \varepsilon_{2} \varepsilon_{3}, S, U)$ is also a $\varepsilon_{s-L_1}^{t}, (\varepsilon_{s-R_1}^{t}, \varepsilon_{s-E_1}^{t})$ of $S$.

Proof. Let $u, v \in S$. Then

$(\varepsilon_{1} \varepsilon_{2} \varepsilon_{3})(u) \cap \varepsilon_{3}(r_{i}) = \bigcup_{u=m} \{ \varepsilon_{1}(l_{i}) \cap \varepsilon_{2}(m_{i}) \cap \varepsilon_{3}(r_{i}) \}$

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\((e, e_2, e_3)(v)\)
\[
= \bigcup_{v = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
Now
\((e, e_2, e_3)(u) \cap (e, e_2, e_3)(v)\)
\[
= \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \left\{ \bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)] \right\}
\]
\[
= \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \left\{ \bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)] \right\}
\]
\[
\subseteq \bigcup_{u + v = \sum_{j=1}^{m} j(f_j)} \left\{ \bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)] \right\}
\]
\[
= (e, e_2, e_3)(u + v).
\]
For \(u, v, w \in S\)
\((e, e_2, e_3)(w)\)
\[
= \bigcup_{w = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
\subseteq \bigcup_{w = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
= \bigcup_{w = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
\subseteq \bigcup_{w = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
= (e, e_2, e_3)(uvw).
\]
Thus \((e, e_2, e_3)(u + v) \supseteq (e, e_2, e_3)(u) \cap (e, e_2, e_3)(v)\) and \((e, e_2, e_3)(uvw) \supseteq (e, e_2, e_3)(w)\), \(\forall u, v, w \in S\). Therefore, \((e, e_2, e_3, S, U)\) is a \(e_s\) of \(S\).

3.11 Theorem
For \(S\) the following are equivalent
\((i)\) \(S\) is \(\mathcal{F}_{\mathcal{N} = r}\).
\((ii)\) \(E \cap F \cap G = \mathcal{EFG}\), for any \(E, F, G\) as \(\mathcal{F}_{\mathcal{R}_l}, \mathcal{F}_{\mathcal{E}_l}\) and \(\mathcal{F}_{\mathcal{I}_l}\) of \(S\), respectively.
\((iii)\) \(e_1 \cap e_2 \cap e_3 = e_1 e_2 e_3\), for any \((e_1, S), (e_2, S)\) and \((e_3, S)\) as \(e'_{s_{-R}}, e'_{s_{-E}}\) and \(e'_{s_{-I}}\) of \(S\), respectively.

Proof. \((i) \iff (ii)\) is followed by Theorem r-4.
\((i) \implies (iii)\) Let \((e_1, S), (e_2, S)\) and \((e_3, S)\) are \(e'_{s_{-R}}, e'_{s_{-E}}\) and \(e'_{s_{-I}}\) of \(S\), respectively. Then for any \(u \in S\)
\((e, e_2, e_3)(u)\)
\[
= \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
\subseteq \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
= \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
\subseteq \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
= (e, e_2, e_3)(uvw).
\]
Thus \(e, e_2, e_3 \subseteq e_1 \cap e_2 \cap e_3\) \((I)\)
Since, \(S\) is \(\mathcal{F}_{\mathcal{N} = r}\), so, for \(w \in S\) there exist \(l \in S\) such that \(w = w_1 l_2\).
Now \((e_1 \cap e_2 \cap e_3)(w) = e_1(w) \cap e_2(w) \cap e_3(w)\)
\[
\subseteq e_1(w) \cap e_2(wh) \cap e_3(w)
\]
\[
(e_1 \cap e_2 \cap e_3)(w)
\]
Thus \(\subseteq \bigcup_{u = \sum_{j=1}^{m} j(f_j)} \{\bigcap_j [e_1(l_j) \cap e_2(m_j) \cap e_3(r_j)]\}
\]
\[
= (e, e_2, e_3)(w).
\]
This implies that
\(e_1 \cap e_2 \cap e_3 \subseteq e_1 e_2 e_3\) \((2)\)
From \((I)\) and \((2)\)
\((e_1 e_2 e_3) = e_1 \cap e_2 \cap e_3\).
\((iii) \implies (i)\) Let \(E\) be a \(\mathcal{F}_{\mathcal{R}_l}, F\) be a \(\mathcal{F}_{\mathcal{E}_l}\) and \(G\) be a \(\mathcal{F}_{\mathcal{I}_l}\) of \(S\). Then the characteristic functions \(\chi_E, \chi_F\) and \(\chi_G\) are \(e'_{s_{-R}}, e'_{s_{-E}}\) and \(e'_{s_{-I}}\) of \(S\), respectively, and by hypothesis

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\[ \chi_{EFG} = \chi_E \cdot \chi_F \cdot \chi_G = \chi_{E \cap F \cap G} \]

\[ \Rightarrow \chi_{EFG} = \chi_{E \cap F \cap G} \]

Thus, \( EFG = E \cap F \cap G \)

Therefore, \( S \) is \( \not\in V_{n-r} \).

**3.12 Theorem**

For \( S \) with 1, the following are equivalent

(i) \( S \) is \( \not\in R_{n-r} \).

(ii) All \( \not\in R_{n-r} \)'s of \( S \) are idempotent.

(iii) \( EFG = E \cap F \cap G \), for any \( E, F \) and \( G \) as \( \not\in R_{n-r} \) and \( \not\in R_{n-r} \) of \( S \), respectively.

(iv) All \( \not\in R_{n-r} \)'s of \( S \) are fully idempotent.

(v) \( \varepsilon_1, \varepsilon_2 \varepsilon_3 = \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \), for any \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) as \( \not\in R_{n-r} \), \( \not\in R_{n-r} \) and \( \not\in R_{n-r} \) of \( S \), respectively.

If \( S \) is commutative then (i) – (v) are equivalent to

(vi) \( S \) is \( \not\in V_{n-r} \).

Proof (i) \( \Rightarrow \) (iv) Let \( (\varepsilon, S) \) be a \( \not\in R_{n-r} \) of \( S \) and \( w \in S \). Then

\[ \varepsilon^3(w) = (\varepsilon \varepsilon \varepsilon)(w) = \bigcup_{w=\sum_{i} x_i y_i z_i \ v} \{ \bigcap_{i} [\varepsilon(l_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)] \} \]

\[ \subseteq \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \bigcap_{i} [\varepsilon(l_i m_i r_i) \cap \varepsilon(m_i) \cap \varepsilon(r_i)] \} \]

\[ = \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \varepsilon(\sum_{i} x_i y_i z_i) \cap \{ \varepsilon(m_i) \cap \varepsilon(r_i)] \} \}
\]

\[ \subseteq \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \varepsilon(w) \cap \{ \varepsilon(m_i) \cap \varepsilon(r_i)] \} \}
\]

\[ \subseteq \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \varepsilon(w) \} \]

\[ \Rightarrow \varepsilon^3 \subseteq \varepsilon^3 \]

Since, \( S \) is \( \not\in R_{n-r} \). So, \( w \in wSwSwS \) and we can write

\[ w = \sum_{n \in \mathbb{N}} x_i y_i z_i \]

\[ \varepsilon(w) \cap \varepsilon(w) \cap \varepsilon(w) \]

\[ \subseteq \varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i) \]

\[ \varepsilon(w) \subseteq \bigcap_{i} [\varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i)] \]

\[ = \bigcap_{i} [\varepsilon(wa_i w) \cap \varepsilon(wb_i w) \cap \varepsilon(wc_i d_i)] \]

\[ \Rightarrow \varepsilon^3 \subseteq \varepsilon^3 \]

Thus, \( \varepsilon^3 = \varepsilon^3 \).

Hence, \( \varepsilon \) is fully idempotent.

(v) \( \varepsilon_1, \varepsilon_2 \varepsilon_3 \subseteq \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \), for any \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) as \( \not\in R_{n-r} \), \( \not\in R_{n-r} \) and \( \not\in R_{n-r} \) of \( S \), respectively. Then for any \( w \in S \)

\[ (\varepsilon_1, \varepsilon_2, \varepsilon_3)(w) = \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \bigcap_{i} [\varepsilon_1(x_i y_i z_i) \cap \varepsilon_2(x_i y_i z_i) \cap \varepsilon_3(x_i y_i z_i)] \} \]

\[ \subseteq \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \bigcap_{i} [\varepsilon_1(x_i y_i z_i) \cap \varepsilon_2(x_i y_i z_i) \cap \varepsilon_3(x_i y_i z_i)] \} \]

\[ = \bigcup_{w=\sum_{i} x_i y_i z_i} \{ \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w) \} \]

\[ \Rightarrow \varepsilon_1 \varepsilon_2 \varepsilon_3 \subseteq \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \]

Since, \( S \) is \( \not\in R_{n-r} \). Then \( w \in S \) can be written as

\[ w = \sum_{i} x_i y_i z_i \]

\[ \Rightarrow \varepsilon_1 \varepsilon_2 \varepsilon_3 \subseteq \varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \]
Now \((\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w) = \varepsilon_1(w) \cap \varepsilon_2(w) \cap \varepsilon_3(w)\) 
\(\subseteq \varepsilon_1(wa_i, w) \cap \varepsilon_2(wb_i, w) \cap \varepsilon_3(wc_i, d_i), \ \forall \ i.\)
Thus
\((\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3)(w)\)
\[\subseteq \bigcup_{w=\sum_{a_i} a_i w(a_i w b_i w c_i d_i)} \left[ \left\{ \bigcap_{i} [\varepsilon_1(wa_i, w) \cap \varepsilon_2(wb_i, w) \cap \varepsilon_3(wc_i, d_i)] \right\} \right] \]
\[\subseteq \bigcup_{w=\sum_{a_i} a_i b_i c_i} \left[ \left\{ \bigcap_{i} [\varepsilon_1(a_i, b_i) \cap \varepsilon_2(b_i, c_i) \cap \varepsilon_3(c_i)] \right\} \right] \]
\[= (\varepsilon_1 \varepsilon_2 \varepsilon_3)(w).\]
\(\big\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 \subseteq \varepsilon_1 \varepsilon_2 \varepsilon_3.\)
\(\big\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 = \varepsilon_1 \varepsilon_2 \varepsilon_3.\)

(iii) Let \(E, F, G\) are \(\mathcal{T}_R, \mathcal{T}_E, \mathcal{T}_G\) of \(\mathcal{S}\), respectively. Then the characteristic functions \(\chi_E, \chi_F, \chi_G\) are \(\varepsilon'_{s_i, R}, \varepsilon'_{s_i, E}, \varepsilon'_{s_i, G}\) of \(\mathcal{S}\), respectively, and by hypothesis
\(\chi_{EFG} = \chi_{E} \cdot \chi_{F} \cdot \chi_{G} = \chi_{E} \cap \chi_{F} \cap \chi_{G}\) 
\(\Rightarrow \chi_{EFG} = \chi_{EFG} \cap \mathcal{S}\) 
\(\Rightarrow EFG = E \cap F \cap G.\)

The references are straightforward.

**REFERENCES**


