A BRIEF INTRODUCTION TO TOPOLOGICAL HYPERGROUP

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ABSTRACT: This paper deals with a hypergroup and a topology. As we cannot endow the power set with a topology derived from the topology of a hypergroup H, we define the notion of upper and lower semicontinuous mapping and then define a topological hypergroup as a hypergroup induced with a topology such that the hyperoperation must be compatible with the algebraic structure. We also study the quotient of a hypergroup by an equivalence relation and show that this quotient becomes a semigroup.

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1. PRELIMINARIES

First of all, we shall recall some basic definitions of hypergroup theory. Let H be a
Nonempty set and denote by c the set of all nonempty subsets of H.

Definition 1 *A hyperoperation on H is a mapping; $\circ: H \times H \rightarrow \mathcal{P}^*(H)$*

A nonempty set H endowed with a hyperoperation $\circ$ is said to be a hypergroupoid.

The image of the pair $(a,b) \in H \times H$ is usually denoted by $a \circ b$ and called the hyperproduct of a and b.

If A and B are nonempty subsets of H, then we define

$$A \circ B = \bigcup_{a \in A, b\in B} a \circ b$$

and call it the hyperproduct of A and B.

Also, for any, $a, b \in H$ we define

$a/_b = \{x \in H / \ a \in x \circ b\}$.

Definition 2 .

1. The hypergroupoid $(H, \circ)$ is called a semihypergroup if the hyperoperation $\circ$ is associative.
2. If $(H, \circ)$ satisfies the reproducibility law $a \circ H = H \circ a = H$; $\forall a \in H$; then we say that it is a quasihypergroup.
3. A hypergroup is a hypergroupoid which is both a semihypergroup and a quasihypergroup.
4. A hypergroup H is said to have a unity or to be unitary if there exists a unique element $1 \in H$ such that $1 \circ a = a \circ 1 = \{a\}$; $\forall a \in H$.

Definition 3 *A subset K of hypergroup $(H, \circ)$ is a subhypergroup if and only if the following conditions are satisfied:*

1. $a \circ b \subset K$, for all, $a; b \in K$.
2. $a \circ K = K \circ a = K$; for all $a \in K$.

Clearly, H is a subhypergroup of itself. Any other subhypergroup of H will be called a proper subhypergroup.

2 Main results

In the sequel $\mathcal{T}_m$ denotes a topology on a set where any element x is contained in a least (for the inclusion) open set $\mathcal{O}(x)$ and called a fundamental open set.

Remark 1 The discrete topology on a set is a topology having the desired property. The open set $\mathcal{O}(x)$ is just the singleton $\{x\}$.

Proposition 4 *If a set H is equipped with the topology $\mathcal{T}_m$ and $\circ$ is a hyperoperation on H defined by $x \circ y = \mathcal{O}(x) \cup \mathcal{O}(y)$; then the set $(H, \circ)$ is a hypergroup.*

The family $\{\mathcal{O}(x);\ x \in H\}$ is called in the sequel a fundamental family.

Proof. It is clear that $x \circ y \neq \emptyset$. Since the union is associative then $\circ$ is so. Let $a \in H$ then for all $x \in H ; x \in \mathcal{O}(a) \cup \mathcal{O}(x) = a \circ x$ so

$$H \subset \bigcup_{x \in H} (a \circ x) = a \circ H \subset H$$

and then it will be easy to get $a \circ H = H = H \circ a$.

Lemma 5 *If $x \in \mathcal{O}$; where $\mathcal{O}$ is an open set of the topological space $(H, \mathcal{T}_m)$; then*
$\mathcal{O}(x) \subset \mathcal{O}$.

Proof. Since $x \in \mathcal{O}(x) \cap \mathcal{O}$ then $\mathcal{O}(x) \cap \mathcal{O} \neq \emptyset$. Suppose that $\mathcal{O}(x)$ is not included in $\mathcal{O}$; then since $\mathcal{O}(x) \cap \mathcal{O} \subsetneq \mathcal{O}(x) \cap \mathcal{O}$ and $\mathcal{O}(x) \cap \mathcal{O}$ is also a nonempty open set containing x; the open set $\mathcal{O}(x)$ is not the least open set containing x which is a contradiction.

As it has been done for topological groups; we wish to introduce a topology on hypergroup in such a way that the hyperoperation becomes continue. But we cannot define on the power set $\mathcal{P}^*(H)$ a topology in narrow relation with the topology of the hypergroup H. We introduce the semicontinuity and then give an adequate definition of topological hypergroup.

Definition 6 *Let $(H, \mathcal{T}_1)$, $(K, \mathcal{T}_2)$ be two topological spaces and $\phi: H \longrightarrow \mathcal{P}(K)$ be a mapping.*

1. ϕ is said to be upper semicontinuous if for all $\mathcal{O} \in \mathcal{T}_2$, the set $\mathcal{O}^* = \{ x \in H : \phi(x) \subset \mathcal{O} \}$ is open in H.
2. ϕ is said to be lower semicontinuous if for all $\mathcal{O} \in \mathcal{T}_2$, the set $\mathcal{O}_* = \{ x \in H : \phi(x) \cap \mathcal{O} \neq \emptyset \}$ is open in H.
3. ϕ is said to be semicontinuous if it is upper semicontinuous and lower semicontinuous.

We can then adapt the above definition to a topological hypergroups.

Definition 7 *Let $(H,\circ)$ be a hypergroupoid and $(H, \mathcal{T})$ be a topological space, the cartesian product $H \times H$ will be equipped with the product topology. The hyperoperation $\circ$ is called:*

1. upper semcontinuous, if for every $\mathcal{O} \in \mathcal{T}$, the set $\mathcal{O}^* = \{ (x,y) \in H \times H : x \circ y \subset \mathcal{O} \}$ is open in $H \times H$.
2. lower semicontinuous, if for every $\mathcal{O} \in \mathcal{T}$, the set $\mathcal{O}_* = \{ (x,y) \in H \times H :$

$x \circ y \cap \mathcal{O} \neq \emptyset \}$ is open in $H \times H$.

3. Similarly, the hyperoperation $\circ$ is semicontinuous if it is upper and lower semicontinuous.

4. The hyperoperation $\circ$ is sp-continuous if and only if for any $\mathcal{O} \in \mathcal{T}$ and any pair $(x, y) \in H \times H$ such that $x \circ y = \mathcal{O}$ there exist $U, V \in \mathcal{T}, x \in U, y \in V$ such that $u \circ v = \mathcal{O}$ for any $u \in U$ and $v \in V$.

Proposition 8 *The hyperoperation of any hypergroup H endowed with the topology $\mathcal{T}_m$ is upper semicontinuous. More exactly for any open set $\mathcal{O}$; we have $\mathcal{O}^* = \mathcal{O} \times \mathcal{O}$.*

Proof. Let $\mathcal{O}$ be an open set of H. If $(x, y) \in \mathcal{O}^*$ then $\mathcal{O}(x) \cap \mathcal{O}(y) \subset \mathcal{O}$. Since $x \in \mathcal{O}(x) \subset \mathcal{O}$ and $y \in \mathcal{O}(y) \subset \mathcal{O}$ we get $(x, y) \in \mathcal{O} \times \mathcal{O}$.

Conversely; let $(x, y) \in \mathcal{O} \times \mathcal{O}$. By lemma 5; both the open sets $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are included in $\mathcal{O}$ so is their union and finally $(x, y) \in \mathcal{O}^*$.

Proposition 9 *The hyperoperation of any topological hypergroup $(H, \mathcal{T}_m)$ is lower semicontinuous if*
$\mathcal{O}(x) \cap \mathcal{O} \neq \emptyset \Longrightarrow \mathcal{O}(a) \cap \mathcal{O} \neq \emptyset;\ \forall a \in \mathcal{O}(x)$.

Proof. Let $\mathcal{O}$ be an open set of H. Since $\mathcal{O}(x)$ is a neighborhood of x for any $x \in H$. To prove that $\mathcal{O}_*$ is open; we will prove that for any $(x, y) \in \mathcal{O}_*$ there exists a neighborhood W of (x, y) such that $(x, y) \in W \subset \mathcal{O}_*$.

Let $(x, y) \in \mathcal{O}_*$ and set $W = \mathcal{O}(x) \times \mathcal{O}(y)$. This set is evidently an open set of $H \times H$ and then a neighborhood of (x, y).

The condition $(\mathcal{O}(x) \cup \mathcal{O}(y)) \cap \mathcal{O} \neq \emptyset$ implies that $\mathcal{O}(x) \cap \mathcal{O} \neq \emptyset$ or $\mathcal{O}(y) \cap \mathcal{O} \neq \emptyset$ For $(a, b) \in \mathcal{O}(x) \times \mathcal{O}(y)$ and from our condition we can deduce that $\mathcal{O}(a) \cap \mathcal{O} \neq \emptyset$ or $\mathcal{O}(b) \cap \mathcal{O} \neq \emptyset$ and so $(\mathcal{O}(a) \cup \mathcal{O}(b)) \neq \emptyset$ and $(a, b) \in \mathcal{O}_*$ Finally $(x, y) \in \mathcal{O}(x) \times \mathcal{O}(y) \subset \mathcal{O}_*$ and $\mathcal{O}_*$ is an open set.

Definition 10 *A topological hypergroup $(H, \circ)$ is a hypergroup endowed with a topology $\mathcal{T}$ such that the hyperoperation is semicontinue.*

Proposition 11 *If $(H, \circ)$ is a hypergroup and the topology $\mathcal{T}_m$ on H is such*
$\mathcal{O}(x) \cap \mathcal{O} \neq \emptyset \Longrightarrow \mathcal{O}(a) \cap \mathcal{O} \neq \emptyset;\ \forall a \in \mathcal{O}(x)$
for a fundamental saturated family $\{ \mathcal{O}(x); x \in H \}$; then H is a topological hypergroup.

Remark 2 It is trivial that the hyperoperation $\circ$ as defined above is commutative.

Example 12 *The discrete topology on $(H, \circ)$; $x \circ y = \{x, y\}$ has the required properties H and then H is a topological hypergroup.*

Proposition 13 *Let $\mathcal{T}_m$ be a topology on a set H as in proposition 4. If $\forall z \in \mathcal{O}(x) \cap \mathcal{O}(y)$ the open set $\mathcal{O}(z) \subset \mathcal{O}(x) \cap \mathcal{O}(y)$; then the family $\{ \mathcal{O}(x) \ / \ x \in H \}$ is a base of $\mathcal{T}$_m.*

Proof. The proof is trivial.

Proposition 14 *Any open set K of a hypergroup H endowed with the topology $\mathcal{T}_m$ is a subhypergroup of H.*

Proof.

1. If $x \in K$ then $x \in \mathcal{O}(x) \cap K$ which is an open set. Consequently $\mathcal{O}(x) \subset K$ and then $a \circ b \subset K$, for all $a, b \in K$.

2. By the definition of our topology, any $x \in K$ is such $x \in \mathcal{O}(x)$ and so $\forall a \in K;\ x \in \mathcal{O}(a) \cup \mathcal{O}(x)$ and then we get $K \subset K \circ a \subset K \Longrightarrow K = K \circ a$. The other equality can be obtained by a similar manner.

Clearly, H is a subhypergroup of itself. Any other subhypergroup of H will be called a proper subhypergroup.

Definition 15 *Let $\mathcal{O}(a)$ as defined in proposition 4. We say that $\mathcal{O}(a)$ is saturated if :*
$\forall x \in \mathcal{O}(a)$ we have $\mathcal{O}(x) = \mathcal{O}(a)$.

Proposition 16 *Let $(H, \mathcal{T}_m)$ as in proposition 4; if the elements of the family $\mathcal{F}_{\mathcal{T}_m} = \{ \mathcal{O}(x) \ / \ x \in H \}$ are saturated; then the family form a partition of H.*

Proof. It is clear that the family is a cover of H.

Suppose that $\mathcal{O}(a) \cap \mathcal{O}(x) \neq \emptyset$; so for $b \in \mathcal{O}(a) \cap \mathcal{O}(x)$ we have $\mathcal{O}(b) = \mathcal{O}(x)$ and $\mathcal{O}(b) \cap \mathcal{O}(a)$ so $\mathcal{O}(a) = \mathcal{O}(x)$.

Proposition 17 *Let $(H, \mathcal{T}_m)$ as above. The relation $\mathcal{R}_m$ defined on H by :*
$x \, \mathcal{R}_m \, y \Leftrightarrow \mathcal{O}(y) = \mathcal{O}(x)$;
is an equivalence relation.

If in addition the open sets $\mathcal{O}(x)$ are saturated; then the class of any element x is defined as:
$\overline{x} = \mathcal{O}(x)$.

Proof. The proof is trivial.

Proposition 18 *If the union of two fundamental open sets is a fundamental open set); the quotient set $(H / \mathcal{R}_m, \circ)$ where $\mathcal{O}(x) \circ \mathcal{O}(y) = \mathcal{O}(x) \cup \mathcal{O}(y)$ is a semigroup.*

Proof. From the condition on the union of two fundamental open sets; the operation $\circ$ on $H / \mathcal{R}_m$ is a binary operation. In the other from the associativity of the union of sets we get the associativity of the binary operation $\circ$ on $H / \mathcal{R}_m$.

Definition 19 *Let $(H, \circ)$ be a hypergroup an equivalence relation $\mathcal{R}$ on H is said to be compatible with the hyperoperation $\circ$ if :*
$a \, \mathcal{R} \, b \Longrightarrow a \circ c \, \widetilde{\mathcal{R}} \, b \circ c$ *and* $c \circ a \, \widetilde{\mathcal{R}} \, c \circ b$; $\forall c \in H$
where for $A, B \subset H$; $A \, \widetilde{\mathcal{R}} \, B \Leftrightarrow \forall a \in A, \exists \ b \in B$ such that $a \, \mathcal{R} \, b$ and conversely.

Proposition 20 *If $\mathcal{R}_m$ is the equivalence relation on the hypergroup $(H, \circ)$ endowed with the opology $\mathcal{T}_m$; the relation $\tilde{\mathcal{R}}_m$ is an equivalence relation on the family $\mathcal{F} = \{ \mathcal{O}(x);\ x \in H \}$.*

Proof. The proof is easy to establish.

Proposition 21 *The relation $\mathcal{R}_m$ defined in the hypergroup $(H, \circ)$ endowed with the topology $\mathcal{T}_m$ is compatible with $\circ$.*

Proof. Let $a \, \mathcal{R}_m b$ then $\mathcal{O}(a) = \mathcal{O}(b)$. For any $c \in H$; and since $\mathcal{O}(a) \cup \mathcal{O}(c) = \mathcal{O}(b) \cup \mathcal{O}(c)$; then $a \circ c = b \circ c$ and since $\tilde{\mathcal{R}}_m$ is reflexive; we have $a \circ c \, \tilde{\mathcal{R}}_m \, b \circ c$.

Proposition 22 *Let $\{ \mathcal{O}(x) \}$ be a fundamental family. For any open set $\mathcal{O}$ and saturated open set $\mathcal{O}(x)$; either $\mathcal{O}(x) \cap \mathcal{O} = \emptyset$ or $\mathcal{O}(x) \subset \mathcal{O}$.*

Proof. Suppose $\mathcal{O}(x) \cap \mathcal{O} \neq \emptyset$ then for any $a \in \mathcal{O}(x) \cap \mathcal{O}$ one has $\mathcal{O}(x) = \mathcal{O}(a)$ and $\mathcal{O}(a) \subset \mathcal{O}$ so $\mathcal{O}(x) \subset \mathcal{O}$.

Proposition 23 *Let $(H, \mathcal{T}_m)$ be a topological hypergroup and $\mathcal{R}$ be the equivalence relation defined as above. If the family $\{ \mathcal{O}(x) \}$ is saturated then $H / \mathcal{R}_m$ is a hypergroup for the hyperoperation :*
$\mathcal{O}(x) \circ \mathcal{O}(y) = \mathcal{O}(x) \cup \mathcal{O}(y)$.

Proof. The proof is trivial.

Definition 24 *For $a \in H$ the mapping*

$$f_a : \begin{array}{ccc} H \setminus \{a\} & \longrightarrow & \mathcal{P}^*(H) \\ x & \longmapsto & a \circ x \end{array}$$

is said

1. semi-injective if : $a \circ x = a \circ y \Longrightarrow \mathcal{O}(x) = \mathcal{O}(y)$.
2. semi-surjective if : $\forall \ \emptyset \neq A \subset H$; $\exists \ x \in H$ such that

$A \subset f_a(x) = a \circ x$.

Proposition 25 *Let* $\{ \mathcal{O}(x) \}$ *be a fundamental and saturated family, the mapping*

$$f_a : \begin{array}{ccc} H \setminus \{a\} & \longrightarrow & \mathcal{P}^*(H) \\ x & \longmapsto & a \circ x \end{array}$$

is semi-injective.

Proof. Suppose $\mathcal{O}(a) \cup \mathcal{O}(x) = \mathcal{O}(a) \cup \mathcal{O}(y)$ and $b \in \mathcal{O}(x)$

1. If $b \in \mathcal{O}(a)$ then $\mathcal{O}(a) = \mathcal{O}(x)$ and therefore $\mathcal{O}(x) = \mathcal{O}(x) \cup \mathcal{O}(y)$ which implies that $\mathcal{O}(y) \subset \mathcal{O}(x)$ and finally from the saturation since $y \in \mathcal{O}(y) \subset \mathcal{O}(x)$, we have $\mathcal{O}(y) = \mathcal{O}(x)$.
2. If $b \notin \mathcal{O}(a)$ but $b \in \mathcal{O}(a) \cup \mathcal{O}(x) = \mathcal{O}(a) \cup \mathcal{O}(y)$ then $b \in \mathcal{O}(y)$ and so $\mathcal{O}(x) \subset \mathcal{O}(y)$, which gives finally $\mathcal{O}(x) = \mathcal{O}(y)$.

Proposition 26 *Let* $\{ \mathcal{O}(x) \}$ *be a fundamental family where the union of two fundamental open sets is a fundamental open set, the mapping*

$$f_a : \begin{array}{ccc} H \setminus \{a\} & \longrightarrow & \mathcal{P}^*(H) \\ x & \longmapsto & a \circ x \end{array}$$

is then semi-surjective.

Proof. Let $A \in \mathcal{P}^*(H)$ then $A \subset \bigcup_{x \in A} \mathcal{O}(x)$ but from our condition there exists $b \in H$ such that $\mathcal{O}(b) \subset \bigcup_{x \in A} \mathcal{O}(x)$ and $A \subset \bigcup_{x \in A} \mathcal{O}(x) = \mathcal{O}(b) \subset \mathcal{O}(a) \cup \mathcal{O}(b)$.

Comment 1. *Next our goal is the study of the topology on the quotient semigroup* $H / \mathcal{R}_m$ *induced by the topology of hypergroup H. We think that the binary operation* $\circ$ *on* $H / \mathcal{R}_m$ *presents some nice continuity properties.*

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